

Games with strategic complementarities: New applications to industrial organization[☆]

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Abstract

This paper provides an introduction to the analysis of games with strategic complementarities and applications to industrial organization: oligopoly pricing, comparative statics and a taxonomy of strategic behavior in two-stage games.

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1. Introduction

Games of strategic complementarities are those in which the best response of any player is increasing in actions of the rivals. Many games usually studied in industrial organization display strategic complementarities including a large subset of those involving search, network externalities, oligopoly interaction, or patent races. Recently, there has been a surge of interest in the study of competition in the presence of complementarities in industries with a network component, such as credit cards, or where systems competition is important, like in software.

Supermodular games (Topkis, 1979; Vives, 1985a, 1990; Milgrom and Roberts, 1990) provide the appropriate framework to model strategic interaction in the

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presence of complementarities. The theory of supermodular games is based on a lattice-theoretic approach that exploits order and monotonicity properties. Both existence of equilibrium and comparative static properties are based on order and monotonicity properties in contrast to the usual box of tools based on convex analysis and calculus.

The approach is powerful and delivers strong results. In the class of supermodular games the existence of equilibrium in pure strategies is ensured without requiring quasiconcavity of payoffs; the equilibrium set has an order structure, having extremal elements that allows a global analysis of the set; there is an algorithm to compute extremal equilibria, which also bound the rationalizable set; and monotone comparative statics results are obtained with minimal assumptions.

The purpose of the paper is to provide an introduction to the class of supermodular games and provide some applications in industrial organization analysis. We obtain new results and new light is cast on old results by getting rid of unnecessary assumptions. In this way the role of the critical assumptions is highlighted. At the same time the range of application of the theory is extended beyond games of strategic complementarities, providing examples of results obtained in games displaying strategic substitutability. The reader is warned however that the approach, although useful in a very large class of cases, is not of universal applicability.

The plan of the paper is the following. Section 2 presents the basic results of the theory and some examples. Section 3 develops some applications to oligopoly pricing in homogenous and differentiated products environments as well as comparative static results. Section 4 extends the taxonomy of strategic behavior due to [Fudenberg and Tirole \(1984\)](#). Concluding remarks end the paper.

2. An introduction to games with strategic complementarities

This section contains basic definitions and some of the main results in the theory of supermodular games. The reader is referred to [Vives \(1999, in press\)](#) for a more thorough and general treatment of the theory, as well as proofs, and further references and applications.

The definition of a game with strategic complementarities is provided in a smooth context. This is done only to minimize the mathematical apparatus but it is not the most general way to define it. A game $(A_i, \pi_i; i \in N)$ is defined by the set of players $N, i = 1, \dots, n$, by the strategy set A_i and the payoff π_i of player $i \in N$ (the payoff is defined on the cross product of the strategy spaces of the players A). Let $a_i \in A_i$ and denote by a_{-i} the strategy profile (a_1, \dots, a_n) except the i th element. We have then $a_{-i} \in \prod_{j \neq i} A_j$. The game $(A_i, \pi_i; i \in N)$ is *smooth supermodular* if each A_i is a compact cube in Euclidean space, and $\pi_i(a_i, a_{-i})$ is twice continuously differentiable with

- (i) $\partial^2 \pi_i / \partial a_{ih} \partial a_{ik} \geq 0$ for all $k \neq h$ and
- (ii) $\partial^2 \pi_i / \partial a_{ih} \partial a_{jk} \geq 0$ for all $j \neq i$ and for all h and k ,

where a_{ih} denotes the h th component of the strategy a_i of player i .

Condition (i) is the strategic complementarity property in own strategies a_i . Condition (ii) is the strategic complementarity property in the strategies of rivals a_{-i} . The latter property in the general formulation involves the profit function π_i displaying increasing differences in (a_i, a_{-i}) : the marginal profit of action h of player i is increasing in any action of the rivals. Those conditions deliver monotone increasing best responses even when π_i is not quasiconcave in a_i .

Intuition can be gained from the one dimensional “classic” case where best responses are continuous functions. Let A_i be a compact interval. Suppose that the i th player best reply to a_{-i} is unique, interior, and equal to $r_i(a_{-i})$. We have then that $\frac{\partial \pi_i}{\partial a_i}(r_i(a_{-i}), a_{-i}) = 0$. Furthermore, if $\frac{\partial^2 \pi_i}{(\partial a_i)^2} < 0$, then r_i is continuously differentiable and $\frac{\partial r_i}{\partial a_j} = - \left(\frac{\partial^2 \pi_i}{\partial a_i \partial a_j} \right) / \left(\frac{\partial^2 \pi_i}{(\partial a_i)^2} \right)$, $j \neq i$. Therefore $\text{sign} \frac{\partial r_i}{\partial a_j} = \text{sign} \frac{\partial^2 \pi_i}{\partial a_i \partial a_j}$. Now, even when π_i is not quasiconcave, if $\frac{\partial^2 \pi_i}{\partial a_i \partial a_j} > 0$, $j \neq i$, then any selection from the best-reply correspondence of player i (which may have jumps) is increasing in the actions of the rivals. In summary, the positive cross-partial derivative of profits ensures that any best response of the firm is increasing even though it may have jumps; if so the jumps will be up and not down.

As an example think of a n -firm Bertrand oligopoly with differentiated substitutable products with each firm producing a different variety. The demand for variety i is given by $D_i(p_i, p_{-i})$ where p_i is the price of firm i and p_{-i} denotes the vector of the prices charged by the other firms. In this case we assume strategy sets are compact intervals of prices, and assumption (ii) means that the marginal profitability of an increase of the price of a firm is increasing in the prices of rivals (and, according to assumption (i), in the other prices charged by the same firm if it is a multiproduct firm). A linear demand system will satisfy the assumptions. In a dual way we could consider the case of a Cournot oligopoly with complementary products. In this case the strategy sets are compact intervals of quantities.

We say that the game is log-supermodular if π_i is nonnegative and $\log \pi_i$ fulfils conditions (i) and (ii). This provides a useful transformation that extends the range of application of the theory (because a monotone transformation of payoffs does not change the equilibrium set of the game). In the Bertrand oligopoly example, with constant marginal costs, c_i for firm i , the profit function of firm i , $\pi_i = (p_i - c_i)D_i(p_i, p_{-i})$, is log-supermodular in (p_i, p_{-i}) whenever $\frac{\partial^2 \log D_i}{\partial p_i \partial p_j} \geq 0$. This holds when η_i , the own-price elasticity of demand for firm i , is decreasing in p_{-i} as with constant elasticity, logit, or constant expenditure demand systems (see Chapter 6 in Vives, 1999).

The following results hold in a supermodular game.

2.1. Result 1: Existence and order structure

In a supermodular game there always exist extremal equilibria. That is, there is a largest \bar{a} and a smallest element \underline{a} of the equilibrium set.

The result is due to Topkis (1979) and is shown using Tarski’s fixed point theorem (which states, under the present restrictions, that an increasing function from a compact cube into itself has a fixed point) on the best-reply map, which is monotone because of the strategic complementarity assumptions. In fact, in a supermodular game any player has a largest and a smallest best response and those can be seen to determine the largest and smallest equilibria. It is worth noting that we do not need here quasiconcave payoffs and

convex strategy sets to deliver convex-valued best replies as required when showing existence using Kakutani's fixed point theorem.

For example, in the Bertrand oligopoly when the payoff is supermodular or log-supermodular extremal price equilibria do exist and the results can be extended to multiproduct firms with convex costs. We have thus a large class of Bertrand oligopoly cases where the [Roberts and Sonnenschein \(1977\)](#) non-existence of equilibrium problem does not arise. This is not say that all Bertrand games with product differentiation are supermodular games. See [Roberts and Sonnenschein \(1977\)](#), [Friedman \(1983\)](#), and Section 6.2 in [Vives \(1999\)](#) for examples, including games with avoidable fixed costs and the classical Hotelling model when firms are located close to each other. In those cases at some point best replies may jump down and a price equilibrium (in pure strategies) may fail to exist.¹

2.2. Result 2: Symmetric games

In a symmetric supermodular game (exchangeable against permutations of the players) symmetric equilibria exist (since the extremal equilibria \bar{a} and \underline{a} are symmetric). Therefore, if there is a unique symmetric equilibrium then the equilibrium is unique (since $\bar{a} = \underline{a}$). This result proves very useful to show uniqueness in symmetric supermodular games. As an example consider a symmetric version of the constant elasticity demand system Bertrand oligopoly with constant marginal costs. It is easy to check that there is a unique symmetric equilibrium and, since the game is log-supermodular, the equilibrium must be unique.

2.3. Result 3: Welfare

In a supermodular game if the payoff to a player is increasing in the strategies of the other players then the largest equilibrium point is the Pareto best equilibrium and the smallest one is the Pareto worst ([Milgrom and Roberts, 1990](#); [Vives, 1990](#)). This simple result is at the base of the Pareto ranking of equilibria in many games with strategic complementarities. For instance, in Bertrand supermodular oligopoly the profits associated with the largest price equilibrium are also the highest for each firm.

2.4. Result 4: Stability

In a supermodular game with continuous payoffs, best-reply dynamics:

- (i) Approach the “box” $[a, \bar{a}]$ defined by the smallest and the largest equilibrium points of the game (which correspond to the largest and smallest serially undominated strategies). Therefore, if the equilibrium is unique, the game is dominance solvable and the equilibrium globally stable ([Vives, 1990](#); [Milgrom and Roberts, 1990](#)).
- (ii) Converge monotonically downwards to an equilibrium starting at any point in the intersection of the upper contour sets of the largest best replies of the players (A^+ in

¹ See, however, the modified Hotelling game in [Thisse and Vives \(1992\)](#) where best responses may be discontinuous but are increasing.

Fig. 1). Similarly, starting at any point in the intersection of the lower contour sets of the smallest best replies of the players (A^- in Fig. 1) converge monotonically upwards to an equilibrium (Vives, 1990).

In short, all relevant strategic action is happening in the box $[\underline{a}, \bar{a}]$ defined by the smallest and largest equilibrium points. Rationalizable outcomes (Bernheim, 1984) must lie in the box $[\underline{a}, \bar{a}]$. To show result (ii), start at any point in A^+ (see Fig. 1). Best reply dynamics define then a monotone decreasing sequence that converges to a point that, by continuity of payoffs, must be an equilibrium. And, indeed, starting at the largest (smallest) point of the cube that defines the strategy space A , best reply dynamics with the largest (smallest) best response will lead to the largest (smallest) equilibrium (Topkis, 1979). However, starting at an arbitrary point convergence is not ensured because a cycle is possible. For example, in Fig. 1 starting at $a^0 = (\underline{a}_1, \bar{a}_2)$ best reply dynamics cycle along the edges of the box $[\underline{a}, \bar{a}]$.

In the Bertrand oligopoly market with linear, constant elasticity, or logit demands the equilibrium is unique and therefore the game is dominance solvable and globally stable.

2.5. Result 5: Comparative statics

Consider a supermodular game with payoff for firm i , $\pi_i(a_i, a_{-i}; t)$, parameterized by t . If $\partial^2 \pi_i / \partial a_{ih} \partial t \geq 0$ for all h and i , then with an increase in t :

- (i) the largest and smallest equilibrium points increase, and
- (ii) starting from any equilibrium, best reply dynamics lead to a (weakly) larger equilibrium following the parameter change.

Result (ii) can be extended to a class of adaptive dynamics (including fictitious play and gradient dynamics); and continuous equilibrium selections that do not increase monotonically with t predict unstable equilibria (Echenique, 2002).

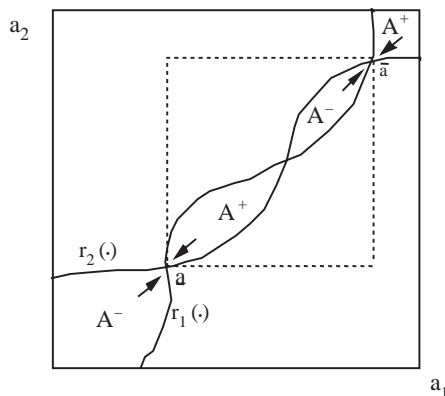


Fig. 1. Cournot tâtonnement in a supermodular game (with best reply functions $r_1(\cdot)$, $r_2(\cdot)$).

An heuristic argument for the result follows. The largest best reply of any player is increasing in t and this implies that the largest equilibrium point also increases with t . An increase in t leaves the old equilibrium in A^- and therefore sets in motion, via best reply (or more in general via adaptive dynamics), a monotone increasing sequence that necessarily converges to a larger equilibrium. See Milgrom and Shanon (1994) and Vives (1999) for detailed proofs and more results.

Samuelson's (1947) Correspondence Principle links unambiguous comparative statics with stable equilibria and is obtained with standard calculus methods applied to interior and stable one-dimensional models. The comparative statics result above can be understood as a multidimensional global version of the principle when the complementarity conditions hold. For example, in the (supermodular or log-supermodular) Bertrand oligopoly market there may be multiple equilibria but we know that extremal equilibrium price vectors are increasing in an excise tax t . This is so because $\pi_i = (p_i - t - c_i)D_i(p)$ and $\frac{\partial^2 \pi_i}{\partial p_i \partial t} = -\frac{\partial D_i}{\partial p_i} > 0$.

Another example is technology adoption. Suppose that each of n firms can adopt a new technology at any period $t=1, \dots, T$ (as in Farrell and Saloner, 1985). The larger the number of adopters the more profitable it is for a firm to switch to the new standard. Firms can be of different types, with larger types more likely to switch. This is a game of strategic complementarities with multiple equilibria, some displaying "excess inertia" where firms switch to the new technology only late in the game. If the cost of adopting the new technology is lowered, starting from an initial equilibrium, then adaptive dynamics will lead sequentially to increased levels of adoption of the new technology.

2.6. Result 6: Duopoly with strategic substitutability

For $n=2$ and with strategic complementarity in own strategies, $\partial^2 \pi_i / \partial a_{ih} \partial a_{ik} \geq 0$ for all $k \neq h$, and strategic substitutability in rivals' strategies, $\partial^2 \pi_i / \partial a_{ih} \partial a_{jk} \leq 0$ for all $j \neq i$ and for all h and k , we can transform the duopoly game into a (smooth) supermodular game. Indeed, let new strategies be $s_1 = a_1$ and $s_2 = -a_2$ is supermodular and note that Fig. 2 provides the mirror image of Fig. 1 with respect to the ordinate axis. We can conclude that all the stated results 1–5 apply to this duopoly game. However, the extension to the strategic substitutability case for n players does not hold since the trick does not work for $n > 2$ (Vives, 1990).

The interpretation of the welfare result is as follows. If for some players payoffs are increasing in the strategies of rivals, and for some others they are decreasing, then the largest equilibrium is best for the former and worst for the latter. This is the case in a strategic substitutes Cournot duopoly with the strategy transformation yielding a supermodular game. Indeed, the preferred equilibrium for firm 1 is the one in which its output is largest and the output of firm 2 lowest (see Fig. 2).

2.7. Remark 1

Comparative statics result 5 holds also if parameter t affects only the payoff of one firm. Let us formulate it in the duopoly case with actions of firms in a compact interval.

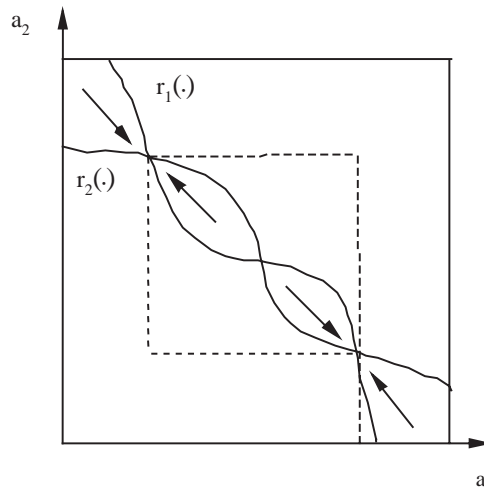


Fig. 2. A duopoly game with decreasing best replies.

Consider a duopoly supermodular game in which the payoff to player 1, parameterized by t , is $\pi_1(a_1, a_2; t)$, and to player 2 is $\pi_2(a_1, a_2)$. If $\partial^2 \pi_1 / \partial a_1 \partial t \geq 0$ then extremal equilibria are increasing in t . Note then that if the game is of strategic substitutes then extremal duopoly equilibrium strategies for firm 1 (2) are increasing (decreasing) in t if $\partial^2 \pi_1 / \partial a_1 \partial t \geq 0$. The results are reversed if $\partial^2 \pi_1 / \partial a_1 \partial t \leq 0$ (see Vives, 1999). A comparative statics result for n firms in a strategic substitutes game can be obtained provided that the payoffs of firms are symmetric (any firm does not care about the identity of the opponents, only about its action and payoff relevant parameters) and that $-\frac{\partial^2 \pi_i}{(\partial a_i)^2} > \left| \frac{\partial^2 \pi_i}{\partial a_i \partial a_j} \right|$, $i \neq j$ (see Athey and Schmutzler, 2001).

3. Oligopoly

I present here some applications to oligopoly pricing: competition with differentiated products and comparative statics in Cournot markets. We will see how the approach, while guaranteeing the existence of equilibrium, allows equilibrium comparisons and delivers comparative static results with minimal assumptions.

3.1. Comparison of Cournot and Bertrand equilibria

Consider the same market as in the Bertrand oligopoly example with n firms competing in a differentiated product market with each firm producing a different variety. As before the demand for variety i is given by $D_i(p_i, p_{-i})$. In the Bertrand game firms compete in prices and in the Cournot game in quantities (to define the payoff then inverse demands are used). Bertrand equilibria are typically thought to be more competitive than Cournot equilibria. The lattice-theoretical approach makes precise in what sense this is true and what drives the result.

It can be shown that with gross substitute, or complementary, products if the price game is supermodular and quasiconcave (that is, π_i quasiconcave in p_i for all i) then at any interior Cournot equilibrium prices are higher than the smallest Bertrand equilibrium price vector. A dual result holds also. With gross substitute, or complementary, products, if the quantity game is supermodular and quasiconcave, then at any interior Bertrand equilibrium outputs are higher than the smallest Cournot equilibrium quantity vector (Vives, 1985b, 1990).

To show the result first note that Cournot prices p^C must lie in region A^+ (Fig. 1), that is, the region in price space defined by the intersection of the upper contour sets of the best replies of the firms in the Bertrand game. This is so because the perceived elasticity of demand for a firm is larger in price than in quantity competition and, consequently at Cournot price levels firms would have an incentive to cut prices if they were to compete in prices. Indeed, with quantity competition no market can be stolen from your competitor given their strategies. Then apply Result 4(ii) to the price game to conclude that starting at any Cournot price vector p^C best reply dynamics will lead the system to a Bertrand equilibrium with lower prices. A corollary is that starting at any interior Cournot equilibrium if firms were to compete in prices they will cut prices until the market stabilizes at a Bertrand equilibrium.

3.2. Comparative statics in Cournot markets

Consider a Cournot market in which the profit function of firm i is given by $\pi_i = P(Q)q_i - C_i(q_i)$, where P is the inverse demand, Q is total output, C_i is the cost function of the firm and q_i is its output level.

The standard approach (Dixit, 1986) assumes quasiconcavity of payoffs, downward sloping best replies, and that the equilibrium analyzed is unique and stable, to derive comparative static results. Are all those strong assumptions needed? What can we say if payoffs are not quasiconcave and/or there are multiple equilibria?

Let us review first the standard approach. Let P and C_i be smooth with $P' < 0$, $P' + q_i P'' \leq 0$ (implying that the game is of strategic substitutes), and $C_i'' - P' > 0$ for all i . Those conditions ensure uniqueness and local stability (with respect to continuous best reply dynamics). Parameterize the cost function of firm i by θ_i and let $C_i(q_i; \theta_i)$ be such that $\frac{\partial C_i}{\partial \theta_i} > 0$ and $\frac{\partial^2 C_i}{\partial \theta_i \partial q_i} > 0$. Then it can be shown, using the standard calculus apparatus with the implicit function theorem, that an increase in θ_i decreases q_i and π_i , and increases q_j and π_j , $j \neq i$.

The lattice-theoretical approach (Amir, 1996; Vives, 1999) makes minimal assumptions to obtain the same kinds of results. Let us consider two cases: a general (potentially asymmetric) oligopoly and a symmetric case.

In the general case I will consider a Cournot duopoly in which strategies are strategic substitutes and potentially multiple equilibria. A sufficient condition for best replies to be decreasing is that the inverse demand be log-concave (and costs strictly increasing in output for both firms).² Note that the strategic substitutes game is transformed into a

² Decreasing best replies in fact imply the existence of a Cournot equilibrium in a n -firm game (see Theorem 2.7 in Vives, 1999). Decreasing best replies are considered the normal case with Cournot competition but it is easy to generate examples with increasing or nonmonotone best replies (see Section 4.1 in Vives, 1999 for a discussion of the topic).

strategic complements game by changing the sign of the strategy space of one player. We know then that extremal equilibria exist (result 6) and that an increase in the parameter θ_i decreases q_i and π_i and increases q_j and π_j , $j \neq i$. The latter results follow from remark 1, where the parameter of interest only affects directly the payoff function of one player. This explains why the increase in θ_i decreases q_i and increases q_j . The result for profits follows immediately from $\frac{\partial C_i}{\partial \theta_i} > 0$ and $\frac{\partial \pi_i}{\partial q_j} < 0$, $j \neq i$. What if we are not at an extremal equilibrium? Similarly as in Result 5(ii), best reply dynamics lead to the comparative static result following the increase in θ_i starting at any equilibrium.

Restrict attention now to a symmetric Cournot oligopoly: $C_i = C$, $i = 1, \dots, n$. In the standard approach (Seade, 1980a,b) it is assumed that payoffs are quasiconcave and conditions are imposed ($((n+1)P'(nq) + nP''(nq)q < 0$ and $C''(q) - P'(nq) > 0$) so that there is a unique and locally stable symmetric equilibrium q^* . Let $\frac{\partial^2 C}{\partial \theta \partial q_i} \leq 0$. Then standard calculus techniques show that an increase in θ increases q^* and that total output increases and profits per firm decrease as n increases. The comparative statics of the output per firm with respect to the number of firms are ambiguous. The classical approach has several problems. First of all, it is silent about the potential existence of asymmetric equilibria. Second, it is restrictive and may be misleading. For example, if the uniqueness condition for symmetric equilibria does not hold and there are multiple symmetric equilibria, changing n either may make the equilibrium considered disappear or introduce more equilibria.

In the lattice-theoretic approach (Amir and Lambson, 2000; Vives, 1999) it is assumed only that $P' < 0$ and $C'' - P' > 0$. As will be shown below a symmetric equilibrium (and no asymmetric equilibrium) exists. Under the assumption that $\frac{\partial^2 C}{\partial \theta \partial q_i} \leq 0$, then at extremal (symmetric) Cournot equilibria individual outputs are increasing in θ , total output is increasing in n and profits per firm decrease with n . Furthermore, individual outputs decrease (increase) with n if demand is log-concave (log-convex and costs are zero). This approach does away with all the unnecessary assumptions of the standard approach and derives new results.

To illustrate the approach let us sketch the proof that, under the assumptions $P' < 0$ and $C'' - P' > 0$, a symmetric equilibrium (and no asymmetric equilibrium) Cournot exists; that individual outputs are increasing in θ and that total output is increasing in n . Let Ψ_i be the best reply map of firm i (identical for all i because of symmetry). Define the correspondence φ by assigning $(q_i + Q_{-i})(n-1)/n$, where $q_i \in \Psi_i(Q_{-i})$, to Q_{-i} . Symmetric equilibria are given by fixed points of this correspondence. Under the assumptions it can be checked that the Ψ_i have slopes larger than -1 .³ This implies that all selections from $\Psi_i(Q_{-i}) + Q_{-i}$ are (strictly) increasing and that no asymmetric equilibria can exist.⁴ Furthermore, all selections from the correspondence φ will be increasing. We can use then Tarski's fixed point theorem to show existence of extremal equilibria. Those extremal equilibria can be found using the extremal selections of φ (which are well-defined in our context). Similarly as in Result 5(i), individual outputs at those extremal equilibria will be increasing in θ because, from the assumption $\frac{\partial^2 C}{\partial \theta \partial q_i} \leq 0$,

³ That is, a segment joining any two points on the graph of the correspondence Ψ_i has a slope larger than -1 .

⁴ This is so because for any total output there is a unique output for every firm, identical for all firms because of symmetry, consistent with optimization behavior (see remark 17 in Section 2.3 in Vives, 1999).

extremal selections of φ (and Ψ_i) are increasing in θ . Let us see now that total output is increasing in n at extremal equilibria. First of all, it is easy to see that extremal selections of φ are increasing in n . This means that the total output of $(n-1)$ firms is increasing in n at any extremal equilibrium. It follows then that total output at extremal equilibria must be increasing in n because all selections from $\Psi_i(Q_{-i}) + Q_{-i}$ are (strictly) increasing in Q_{-i} .⁵ The results for profits and individual outputs in relation to n follow along similar lines.

4. Taxonomy of strategic behavior

Fudenberg and Tirole (1984) provided a taxonomy of strategic behavior in the context of a simple two-stage game between an incumbent and an entrant. At the first stage the incumbent (firm 1) can make an observable investment k yielding at the market stage $\pi_1(x_1, x_2; k)$, where x_i is the market action of firm i . The payoff of the entrant is $\pi_2(x_1, x_2)$. The point is that the incumbent can influence the market outcome (at the second stage) in its favor with an ex ante investment (at the first stage). He will do so by taking into account the effect of his investment on the equilibrium behavior of the rival at the market stage. The goal is to sign this strategic effect taking as benchmark “innocent” behavior where the incumbent when deciding about k only takes into account the direct effect of the investment on his payoff.

The standard approach assumes that at the second stage there are well-defined best-response functions for both firms, and that there is a unique and (locally) stable Nash equilibrium that depends smoothly on k , $x^*(k)$. To obtain this it is assumed that $-\frac{\partial^2 \pi_i}{(\partial x_i)^2} > |\frac{\partial^2 \pi_i}{\partial x_i \partial x_j}|$, $i \neq j$, $i = 1, 2$. At a subgame-perfect equilibrium, we will have that $\frac{\partial \pi_1}{\partial k} + \frac{\partial \pi_1}{\partial x_2} \frac{\partial x_2^*}{\partial k} = 0$ where $S \equiv \frac{\partial \pi_1}{\partial x_2} \frac{\partial x_2^*}{\partial k}$ is the strategic effect. That is, the effect of the investment k in the equilibrium profits of the incumbent because of the modified market behavior of the entrant. Under the stated assumptions it follows, using standard calculus techniques, that $\text{sign} \frac{\partial x_2^*}{\partial k} = \text{sign} \left(\frac{\partial^2 \pi_2}{\partial x_1 \partial x_2} \frac{\partial^2 \pi_1}{\partial k \partial x_1} \right)$ and therefore $\text{sign} S = \frac{\partial \pi_1}{\partial x_2} \frac{\partial^2 \pi_1}{\partial k \partial x_1} \frac{\partial^2 \pi_2}{\partial x_1 \partial x_2}$. If $\frac{\partial \pi_1}{\partial x_2} \frac{\partial^2 \pi_1}{\partial k \partial x_1} < 0 (> 0)$, we say that the investment makes firm 1 tough (soft). Indeed, suppose that $\frac{\partial \pi_i}{\partial x_j} < 0$, $j \neq i$, so that an increase in the market action of firm j hurts firm i . Then if $\frac{\partial^2 \pi_1}{\partial k \partial x_1} > 0$ an increase in k will shift the best response function of firm 1 out and this will be an aggressive move, making firm 1 tough.

A taxonomy of strategic behavior (see Table 1) can be provided then depending on whether competition is of the strategic substitutes $\left(\frac{\partial^2 \pi_2}{\partial x_1 \partial x_2} < 0 \right)$ or complements $\left(\frac{\partial^2 \pi_2}{\partial x_1 \partial x_2} > 0 \right)$ variety and on whether investment makes firm 1 soft $\left(\frac{\partial \pi_1}{\partial x_2} \frac{\partial^2 \pi_1}{\partial k \partial x_1} > 0 \right)$ or tough $\left(\frac{\partial \pi_1}{\partial x_2} \frac{\partial^2 \pi_1}{\partial k \partial x_1} < 0 \right)$. If competition is of the strategic substitutes type and investment makes firm 1 tough then the incumbent wants to overinvest ($S > 0$) to push the entrant down his best response curve (see Fig. 3). This is the top dog strategy. Cournot competition and investment in cost reduction may be an example. If competition is of the strategic complements type and investment makes firm 1 tough then the incumbent wants to underinvest ($S < 0$) to move the entrant up his best response curve. This is the

⁵ See pp. 42–43, 93–96 and Section 4.3.1 in Vives (1999) for details.

Table 1
Taxonomy of strategic behavior

	Investment makes player 1	
	Tough	Soft
Strategic substitutes	Overinvest (top dog)	Underinvest (lean and hungry)
Strategic complements	Underinvest (puppy dog)	Overinvest (fat cat)

puppy dog strategy. Price competition with differentiated products and investment in cost reduction may be the example. Similarly, we can define the strategies “lean and hungry” and “fat cat”.

In the lattice theoretic version of the result (Section 7.4.3, Vives, 1999) the taxonomy follows from minimal assumptions (the character of competition and investment) as applied to extremal equilibria. There is no need for the strong restrictions imposed above to obtain a unique and stable equilibrium at the market stage. Indeed, from remark 1 if the market game is supermodular ($\frac{\partial^2 \pi_2}{\partial x_1 \partial x_2} \geq 0$) and $\frac{\partial^2 \pi_1}{\partial k \partial x_1} \geq (\leq) 0$ then extremal equilibria are increasing (decreasing) in k . If the market game is of the strategic substitutes variety ($\frac{\partial^2 \pi_2}{\partial x_1 \partial x_2} \leq 0$) then changing signs in the strategy space of one player the game becomes a supermodular game and extremal equilibrium strategies for player 1 (2) are increasing (decreasing) in k if $\frac{\partial^2 \pi_1}{\partial k \partial x_1} \geq 0$ and the result is reversed if $\frac{\partial^2 \pi_1}{\partial k \partial x_1} \leq 0$. Therefore, $(\text{sign} \frac{\partial x_k^*}{\partial k} = \text{sign}(\frac{\partial^2 \pi_2}{\partial x_1 \partial x_2} \frac{\partial^2 \pi_1}{\partial k \partial x_1}))$ when x_2^* is an extremal equilibrium and the taxonomy follows for extremal equilibria.

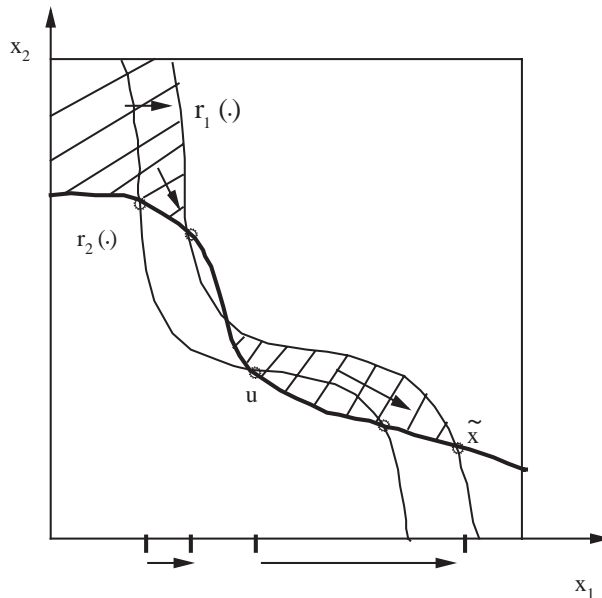


Fig. 3. Effect of k increase in unstable equilibrium: $u \rightarrow \tilde{x}$.

What if at the market stage firms are sitting on a non-extremal equilibrium (for instance at the unstable equilibrium u in Fig. 3)? Then if out-of-equilibrium dynamics are governed by best reply dynamics the sign of the impact of a change in k is as with an extremal equilibrium. In Fig. 3, depicting strategic substitutes competition and investment that makes the incumbent tough, an increase in k will generate an adjustment process that will lead to the new equilibrium \tilde{x} with $u_2 > \tilde{x}_2$.

In summary, the taxonomy of strategic behavior can be obtained with just the crucial assumptions on monotonicity of marginal payoffs without any need of quasiconcavity of payoffs and the requirement of a unique and stable market equilibrium. The method can be extended to study in what situations leaders or laggards in an industry have more incentive to invest, in cost reduction or quality enhancement, and whether this leads to increasing or decreasing dominance (see Athey and Schmutzler, 2001).

5. Concluding remarks

In this paper I have surveyed briefly the theory and some applications to industrial organization of supermodular games. The survey has not been, by any means, exhaustive. For example, dynamics have only been considered in the simple format of two-stage games in Section 4. However, full-fledged dynamic games can be analyzed with the lattice-theoretic techniques. For example, Jun and Vives (2004) and Sleet (2001) analyze differential games, and Hoppe and Lehmann-Grube (2002) develop applications to innovation timing games. Cabral and Villas-Boas (2002) consider multimarket oligopoly applications and Anderson and Schmitt (2003) a quota game in international trade. Coordination failures in macroeconomics and financial markets as well as cumulative processes in the presence of complementarities and monopolistic competition provide more examples outside the realm of Industrial Organization.⁶ Bayesian games provide another fertile ground of applications of the method, advancing the frontier in auction theory and global games and equilibrium selection. See Vives (1999, *in press*) for some of the mentioned extensions. Finally, recent work has tested for complementarities in innovation and organizational design using the tools presented in this article.⁷

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⁶ See Cooper and John (1988), Diamond and Dybvig (1983), and Matsuyama (1995).

⁷ See Athey and Stern (1998), Miravete and Pernias (2000), and Mohnen and Roeller (2000).

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