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# How Fast do Rational Agents Learn?

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A simple dynamic model of rational learning through market interaction by asymmetrically informed risk-neutral agents, uncertain about a valuation parameter but whose pooled information reveals it, is presented. The model is a variation of the classical partial equilibrium model of learning in rational expectations in which the market price is informative about the unknown parameter only through the actions of agents. It is found that learning from market prices and convergence to the rational expectations equilibrium is slow, at the rate $1/\sqrt{n^{1/3}}$ (where $n$ is the number periods of market interaction), whenever the average precision of private information in the market is finite. Convergence obtains at the standard rate $1/\sqrt{n}$ if there is a positive mass of *perfectly* informed agents. Comparative static results on more refined measures of the speed of convergence with respect to basic technological and informational parameters are also provided.

## 1. INTRODUCTION

In this paper I present a simple dynamic model of rational learning by asymmetrically informed risk-neutral agents, uncertain about a valuation parameter but whose pooled information reveals it, and examine what factors influence the *speed of learning* and the *rate of convergence to rational expectations* equilibria.

Rational expectations models have been widely used in many applications. A rational expectations equilibrium entails the solution to a fixed-point problem from beliefs to correct beliefs with the mediation of agents' actions. How do agents come to form rational expectations in the presence of unknown payoff-relevant parameters? The usual answer is that agents learn to form "correct" expectations through repeated observations of market data.

Taking for granted that a learning process converges to the rational expectations equilibrium it is of the utmost importance to know how fast is and what factors affect the speed of convergence. It is not much use for practical purposes to show convergence if it is not known whether this will happen quickly or will take a long time, when the underlying conditions of the economy will have changed and the parameters learned may well be irrelevant. In this sense "slow" convergence may mean no convergence. We would like to know also how structural market conditions, like the nature of uncertainty and its relation to market observables (namely, prices), the degree of asymmetric information, and the precision of private signals, affect the speed of convergence and in what direction.

The literature on learning and convergence to rational expectations splits naturally into rational, and "irrational" or boundedly rational learning models.[1] In the former,

1. Some of the papers in the first vein are Townsend (1978, 1983), Blume and Easley (1984), Feldman (1987) and Bray and Kreps (1988). Some of those in the second vein, Blume and Easley (1982), Bray (1982), Jordan (1985), Bray and Savin (1986), Fourgeaud, Gourieroux and Pradel (1986), Woodford (1990) and Marcet and Sargent (1988, 1989). Frydman (1982) considers both approaches.

agents have a correctly specified model of the economy and of the learning process and update their beliefs and take actions accordingly. In the latter, agents maintain incorrect hypotheses on the face of the evolution of the economy and use "reasonable" updating procedures, like least-squares estimation, for example. It has been found that rational learning tends to yield convergence to the rational expectations equilibrium (at least in terms of convergence of beliefs) while with bounded learning convergence is obtained typically only for certain regions of the parameter space. In any case there is a paucity of results on the rate of convergence to rational expectations equilibria (which may have implications for econometric work).[2]

Rate of convergence results are usually hard to obtain. Recent work by Jordan (1990) on a class of Bayesian myopic learning processes establishes an exponential rate of convergence to Nash equilibria for finite normal-form games. For boundedly rational learning models partial results, relying sometimes on simulations, have been obtained by Bray and Savin (1986), Fourgeaud *et al.* (1986) and Jordan (1992).

Assume that agents do not maintain an incorrect hypothesis about the evolution of the economy, so that convergence to the rational expectations equilibrium is not an issue. I study a simple learning problem by considering a variation of the classical learning in rational expectations model developed by Townsend (1978) and Feldman (1987) and extended in a number of ways by, for example, Bray and Savin (1986), Fourgeaud *et al.* (1986) and Frydman (1982). In this classical learning model competing firms are uncertain about a demand parameter and learn from prices. In the present paper they are uncertain about a cost parameter which is revealed only after all trading is completed. For example, firms pollute and are assessed a damage $\theta$ per unit of production after a certain number of periods ($n$) of testing.[3] The crucial difference is that the latter is a model where firms learn from prices the information that other firms may have on $\theta$. The market price depends on $\theta$ only *indirectly* through the actions of the agents. Financial markets provide another typical example where prices incorporate information about the valuation of assets only through the trades of agents. In the classical rational expectations model the price depends *directly* also on the unknown parameter, and prices are informative independently of the signals received by agents.

The central result is that information revelation through the price system can be slow precisely because it is successful. More precisely, it is found that, provided the shared information of the agents in the economy reveals the value of $\theta$, so that agents have the chance to learn its true value, agents eventually learn the value of $\theta$ and almost sure convergence to the (fully revealing) rational expectations equilibrium obtains. Nevertheless, if the average precision of private information in the market is finite, the rate of convergence is "slow": $1/\sqrt{n^{1/3}}$. This contrasts sharply with the standard rate of convergence $1/\sqrt{n}$ which obtains, for example, with i.i.d. noisy observations of $\theta$ or with classical OLS estimation. Indeed, agents need 1000 rounds of trade to obtain a certain precision $\tau$ in the estimation of the unknown parameter $\theta$ when 10 rounds would have been enough if learning were to be at the standard rate.

The basic idea that drives this slow convergence result is quite simple: the more informative prices are, as more periods accumulate, the less privately imperfectly informed agents will rely on their private signals, with the consequence that less information is

2. In this respect, according to Marcet and Sargent (1988), adaptive (least squares) learning makes "open and problematic" whether the learning system "can ever be expected to yield econometric models that can be applied".

3. Alternatively, with an infinite horizon, in each period there is a positive (small) probability that the tests conducted are definitive and that the pollution damage is assessed.

incorporated in current prices. In fact, since convergence to the rational expectations equilibrium obtains and prices end up being fully revealing, the average responsiveness of agents to their private information goes to zero as $n$ grows whenever there is no positive mass of agents perfectly informed.

If there is a positive mass of perfectly informed agents then convergence is at the standard rate $1/\sqrt{n}$. Perfectly informed agents have nothing to learn from prices and put a constant weight on their (perfect) signals. The result is that the informativeness of the new information in the current price is practically constant for $n$ large and learning is at the standard rate. Convergence would also be at this rate in the original Townsend-Feldman model since then prices depend directly also on the unknown parameter. Here again the new information in the current price does not decrease for $n$ large.

The conclusion that information revelation through prices can be a victim of its own success is reminiscent of the static rational expectations literature in financial markets, in which agents become less sensitive to their private information as prices become more informative (see, for example, Grossman (1976) and Hellwig (1980)). An exogenous change which raises the informativeness of prices (say a decrease in noise in the market) for a given responsiveness of agents to private information, is partially counterbalanced by the agents optimal updating giving less weight to their private signals. In the dynamic context the negative relationship between price informativeness and sensitivity to private information slows down the rate of convergence instead of weakening the comparative static effect of an exogenous change in price informativeness in the static model. In the extreme, we have the paradoxes associated with informationally efficient markets à la Grossman-Stiglitz (1980): if the market price is fully revealing then the agents response to private information has to be nil. How does the information get incorporated into the price in the first place? The Grossman-Stiglitz impossibility of an efficient market is resolved in the dynamic model with a slow revelation of the unknown $\theta$.

The results are obtained in the context of a linear-normal model with a continuum of agents facing a quadratic adjustment (production) cost per period. The presence of increasing marginal adjustment costs is crucial for there to be an issue of information revelation with a continuum of risk-neutral agents. Indeed, as it is well known from Palfrey (1985) and Vives (1988), in the constant returns to scale case *one-shot* market interaction already aggregates the dispersed information of agents, in the sense of replicating the outcome of an economy in which agents share their private information.

I show the existence of a *unique* Bayesian equilibrium (in an appropriate class of general strategies). The equilibrium and its convergence properties are characterized. I derive also the comparative static properties of the asymptotic precision of the optimal price statistic (which provides a refined measure of the speed of learning for a given convergence rate) with respect to the precision of the period-specific noise in the price function, the average precision of private information, the proportion of perfectly informed agents, and the steepness of the adjustment cost.

Two important issues are left for future research. First, welfare analysis. The presumption is that a social planner, facing similar informational and technological constraints to the decentralized market would be able to improve upon the market given that a competitive agent does not take into account the *informational externality* (benefit) that his trading has on the informational content of prices.[4] Second, the study of steady

4. A similar information externality has been studied by Rob (1991) in the context of a competitive entry model.

states in environments with time-varying parameters and their relationship with the convergence results in constant parameter situations such as the one studied in the present paper.

Section 2 presents the model. Section 3 develops some statistical estimation tools. Section 4 characterizes the dynamic equilibrium, and Section 5 studies its asymptotic properties and the speed of learning. Section 6 discusses briefly two extensions: a re-interpretation of the model in a context of limited information with linear prediction, and the case of correlated signals. The Appendix gathers together some of the proofs.

## 2. THE MODEL

In the classical model of learning in rational expectations (as developed by Townsend (1978) and Feldman (1987), for example) a continuum of risk-neutral firms, indexed in the unit interval $[0, 1]$, endowed with the Lebesgue measure, compete repeatedly in a homogeneous product market. Demand is stochastic and linear with an unknown permanent component of its intercept: $p_t = \phi + u_t - \eta x_t$, where $\phi$ is an unknown parameter, $u_t$ a period specific random shock, $\eta$ a positive constant, and $x_t$ the average (per capita) supply in period $t$, $x_t = \int_0^1 x_{it} di$. Firm $i$ has increasing (linear) marginal production costs $\theta x_{it} + \lambda x_{it}^2/2$ ($\theta \geqq 0$, $\lambda > 0$) and is endowed with a private signal $s_i$ about the unknown $\phi$. The joint distribution of the random variables, and the values of the parameters $\eta$, $\theta$, and $\lambda$, are common knowledge. Firm $i$ at period $t$ has to decide how much to produce estimating the period price $p_t$ on the basis of the information it has available: the private signal $s_i$ plus the (public) information contained in past prices. That is, at period $t$ the information set of agent $i$ is $\{s_i, p^{t-1}\}$, where $p^{t-1} = \{p_1, \ldots, p_{t-1}\}$. Beliefs on $\phi$ are updated in a Bayesian way. This basic model has been extended in a number of directions by, for example, Bray and Savin (1986), Fourgeaud *et al.* (1986) and Frydman (1982).

I will consider a variation of this classical model (inspired by an example provided by McKelvey and Page (1986)) where firms *know* the demand parameter $\phi$ but are uncertain about the intercept $\theta$ of their linear marginal cost function.

Suppose that the firms' output pollutes or is toxic but that the level of *toxicity* or long-term pollution effect of a unit of output produced is unknown (the product is a new chemical, for example). The pollution or toxicity damage is only assessed after $n$ periods of production.[5] Let $\theta$ be the assessed toxicity or pollution damage of a unit of output produced. Firms have private assessments of $\theta$ and are allowed to produce but they will have to pay the corresponding damages in proportion to total production after $n$ production periods.

At period $t$ firm $i$ produces $x_{it}$ obtaining a net revenue $p_t x_{it} - \lambda x_{it}^2/2$. After $n$ periods of production the damage per unit of output is ascertained at value $\theta$ and the firm has to pay $\theta(\sum_{t=1}^n x_{it})$. In summary, firms have to pay a proportional tax on production which is levied after $n$ production periods when the toxicity/pollution damage is evaluated. Profits of firm $i$ corresponding to period $t$ would be given by

$$\pi_{it} = (p_t - \theta)x_{it} - \lambda x_{it}^2/2,$$

although only the net revenue $p_t x_{it} - \lambda x_{it}^2/2$ is observable in period $t$. Firm $i$ tries to maximize the (expectation of the) undiscounted sum of period profits $\sum_{t=1}^n \pi_{it}$.

5. Definitive tests to determine the level of toxicity may need at least an $n$-sample, for example.

Alternatively, the model could be thought as having an *infinite horizon* supposing that at each period there is an independent (small) probability $m>0$ that $\theta$ is realized (that is, that the tests are definitive). In this case the probability of $\theta$ not being realized at period $n$, $(1-m)^n$, tends to zero as $n$ tends to infinity. As before when $\theta$ is realized the firm has to pay the damage of the accumulated production. At any period a firm, given $\theta$, expects a damage cost per unit of production[6] of exactly $\theta$, and the firm maximizes the (expected) discounted profits with discount factor $1-m$. It is easily seen (given that firms are risk neutral and negligible) that the equilibrium behaviour will be unchanged from the original model.

The model could be restated also in terms of competition among *buyers* of an asset of unknown ex-post return $\theta$. The asset could be a risky financial asset or a real asset like labour of unknown productivity, for example. Buyer $i$ would face a quadratic cost of adjustment of his position in period $t, x_{it}$, equal to $\lambda x_{it}^2/2, \lambda>0$. Inverse supply in period $t$ would be given by $p_t = u_t + \eta x_t$, with $x_t$ the average (per capita) quantity demanded in period $t$. Buyer $i$ would obtain benefit $\theta x_{it} - \lambda x_{it}^2/2$ (with profits $\pi_{it} = (\theta - p_t)x_{it} - \lambda x_{it}^2/2$) from the quantity $x_{it}$ demanded in period $t$.[7]

In the exposition I will keep to the convention that agents are sellers.

*Information structure*

Agent $i$ receives a signal $s_t = \theta + \varepsilon_t$, where $\varepsilon_t$ is an error term of mean zero; $\theta$ is randomly distributed with mean $\bar{\theta}(\phi > \bar{\theta} > 0)$ and finite variance $\sigma_\theta^2$; $\theta$ and $\varepsilon_t$ are uncorrelated, and errors are also uncorrelated across agents. The sequence $\{u_t\}_{t=1}^{\infty}$ is independently and identically distributed with zero mean and finite variance $\sigma_u^2$. The parameter $\theta$ and the sequence $\{u_t\}$ are uncorrelated. It is assumed that all random variables are normally distributed.[8]

The precision of the signals is given by a measurable function $T_\varepsilon : [0,1] \to R_+ \cup \{\infty\}$, where $T_{\varepsilon_t} = (\mathrm{Var}\ \varepsilon_t)^{-1}$, is the value of this function at $i \in [0,1]$. Denote $(\mathrm{Var}\ \varepsilon_t)^{-1}$ by $\tau_{\varepsilon_t}$.[9] It is assumed that there is a positive measure set of agents who receive signals of precision bounded away from zero. Let $\mu \geqq 0$ be the mass of agents with perfectly informative signals (that is, signals of infinite precision).

I will make the *convention* that the Strong Law of Large Numbers (SLLN) holds for this continuum economy. Suppose that $(q_t)_{t\in[0,1]}$ is a process of independent random variables with $Eq_t = 0$ for all $i$ and that variances $(\mathrm{Var}\ q_t)$ are uniformly bounded. Define $\int_0^1 q_t di = 0$ almost surely (a.s.). This convention will be used, taking as given the usual

6. That is, $(m+(1-m)m+(1-m)^2m+\cdots)\theta = m(m^{-1})\theta = \theta$.

7. In the financial context $\theta$ could be the unknown ex-post liquidation value of a (risky) asset, the inverse supply for which comes from the market clearing condition $u_t - p_t + x_t = 0$. The (random) demand of noise traders (which happens to be sensitive to price) would be $u_t - p_t$, and $x_t$, the demand of the informed traders. In this context $x_{it}$ is the *change* in the position of trader $i$ in period $t$ and $\lambda x_{it}^2/2$ is an adjustment cost associated with changes in the position (which can be thought as an, admittedly imperfect, proxy for risk aversion). The profits of trader $i$ corresponding to period $t$ are $\pi_{it}$. Total profits associated with the final position $\sum_{t=1}^n x_{it}$ are $\sum_{t=1}^n \pi_{it}$. Alternatively, in a real market context, we could think of firms buying labour of unknown productivity $\theta$, facing a random inverse linear labour supply and adjustment costs in the labour stock. Firm $i$ would buy $x_{it}$ in period $t$. The monetary return to the final labour stock $\sum_{t=1}^n x_{it}$ would be then $\theta(\sum_{t=1}^n x_{it})$.

8. The normality assumption will imply that prices and quantities may take negative values. Similarly, in the "real" interpretations of the model, labour of unknown productivity and polluting firms, $\theta$ may take negative values implying either negative productivity or a reward for polluting. Nevertheless, choosing appropriately the mean and variance of $\theta$ the probability of such events can be made small. In equilibrium, at any stage $n$, the variance of individual actions (quantities) can be made small controlling the variances of $\theta$, $u$ and $\varepsilon_t$. For $n$ large it is sufficient to control the variance of $\theta$.

9. Given a random variable $x$, $\tau_x$ will denote $(\mathrm{Var}\ x)^{-1}$.

linearity properties of integrals. For example, if signals have uniformly bounded variances, I will write

$$s=\int_0^1 s_i di=\int_0^1 (\theta+\varepsilon_i)di=\theta+\int_0^1 \varepsilon_i di=\theta \text{ (a.s.)},$$

using the linearity of the integral (which is being defined) and the convention (which implies that $\int_0^1 \varepsilon_i di=0$).[10]

These distributional assumptions, along with the parameters $\phi$, $\eta$ and $\lambda$, are *common knowledge* among the agents in the economy. In order to save on notation set $\eta=1$.

*Equilibria*

The equilibria of the dynamic game will be investigated. At stage $t$, a strategy for agent $i$ is a (measurable) function that maps her private information $s_i$ and the observed past prices $p^{t-1}$ into quantities. The market solution will be taken to be the Bayesian equilibria of the dynamic game. This embodies already the notion of rational learning.

The equilibria will necessarily involve a sequence of Bayesian equilibria of the one-shot game. Otherwise there would be a (positive measure) subset of agents that at some stage could, individually, improve their expected payoffs by reacting optimally to the average market action. The only difference between periods is that at period $t$ agent $i$ is endowed with the information $\{s_i, p^{t-1}\}$. The action of player $i$ in period $t$ does not affect the profits of other periods since the player is negligible (continuum of players) and actions are not observable (only prices are observable).[11]

*The rational expectations (shared information) benchmark and convergence*

If agents were to know $\theta$, the equilibrium action would be $X(\theta)=(\phi-\theta)/(1+\lambda)$, with associated price $P(\theta, u)=u+(\lambda\phi+\theta)/(1+\lambda)$. This can be taken to be the rational expectations (RE) benchmark (shared information). The issue I want to investigate is whether, and if so, how fast, agents learn the unknown parameter $\theta$ and convergence to the rational expectations equilibrium obtains with repeated market interaction.

After $n$ periods of trading the information contained in the price sequence $\{p^n\}$ will be summarized by a statistic $y_n$. I will say that agents *learn* the parameter $\theta$ if $y_n$ converges almost surely to $\theta$ as $n$ goes to infinity. If, for some $k>0$, $n^k(y_n-\theta)$ converges in distribution to a normal random variable with zero mean and positive finite variance, convergence is at the *rate* $n^{-k}$. The standard rate of convergence is $n^{-1/2}$. This obtains, for example, with i.i.d. noisy observations of $\theta$. We can define similarly convergence and convergence rates for (average) equilibrium outputs.

Before plunging into the analysis it may be worthwhile to pause and consider what can agents possibly learn in this model. Given that prices depend on $\theta$ only through the average market action, the maximum agents can learn about $\theta$ is given by the shared information of all agents. That is, an agent can learn at most the information other agents have about $\theta$. For example, if agents were to receive perfectly correlated signals then they would have the same information and there would be nothing to learn: one-shot

10. The convention is made to sidestep a well known *potential* technical (measure-theoretical) difficulty associated to the definition of $\int_0^1 q_i di$. (See, for example, Admati (1985)). The equilibria of the continuum model approximate those of large but finite economies as shown in the context of a financial market by Hellwig (1980) and in the context of a Cournot model by Vives (1988).

11. This holds for either the considered finite-horizon $n$ or the infinite-horizon reinterpretation with discounting with factor $1-m$. See Green (1980), Dubey and Kaneko (1984) and Massó and Rosenthal (1989) for a discussion of "anti-folk" theorems in dynamic games.

competition would yield the RE benchmark. Our assumption of (conditionally) independent signals, plus the fact that a positive measure of agents receive signals of precision bounded away from zero, together with our convention on the SLLN for i.i.d. processes, insures that if agents were to share information they would obtain $\theta$, and therefore, agents can, at least potentially, learn $\theta$.[12]

In order to prepare for the analysis which follows, some statistical results will be presented in the next section.

## 3. SUFFICIENT STATISTICS AND ASYMPTOTIC PROPERTIES OF OLS ESTIMATORS

Consider the following Bayesian statistical estimation problem. The unknown parameter $\theta$ is to be estimated from the observations $(s_i, z^n)$, $z^n \equiv (z_1, \ldots, z_n)$ where where $z_t = \theta a_t + u_t$, are derived form prices and $a_t$ are known constants, $t = 1, \ldots, n$. The joint distribution of the random variables $(\theta, s_i, u^n)$ $(u^n \equiv (u_1, \ldots, u_n))$ is as given in the last section. Denote by $y_n$ the OLS estimate of $\theta$ from regressing $z_t$ on $a_t$, that is $y_n = A_n^{-1} \sum_{t=1}^n a_t z_t$, $A_n = \sum_{t=1}^n a_t^2$, and by $\tau_n$ the precision of the price statistic $y_n$ in the estimation of $\theta$, that is, $\tau_n = (\text{Var}\, \theta | y_n))^{-1}$. In other words, $\tau_n$ is the *informativeness of the price statistic* $y_n$.

**Lemma 3.1.** *The random vector* $(s_i, y_n)$ *is sufficient in the estimation of* $\theta$ *based on* $(s_i, z^n)$. *Further,* $E(\theta | s_i, y_n) = \beta_{in}\bar{\theta} + \alpha_{in}s_i + \gamma_{in}y_n$, *where*

$$\beta_{in} = \tau_\theta/(\tau_{\varepsilon_i} + \tau_n)\alpha_{in} = \tau_{\varepsilon_i}/(\tau_{\varepsilon_i} + \tau_n), \qquad \gamma_{in} = \tau_u A_n/(\tau_{\varepsilon_i} + \tau_n) \quad \textit{and} \quad \tau_n = \tau_\theta + \tau_u A_n.$$

*Proof.* See Appendix. ||

*Remark* 3.1. The conditional expectation of $\theta$ given $(s_i, y_n)$ is a weighted average of $\bar{\theta}$, $s_i$ and $y_n$ according to the precisions of these variables in the estimation of $\theta$.

*Remark* 3.2. Under general distributional assumptions on $(\theta, s_i, z^n)$, it is well known that $E(\theta | s_i, z^n)$ is the unique best predictor of $\theta$ in the sense of minimizing the mean squared error. With finite second moments $(\sigma_\theta^2, \sigma_u^2, \sigma_{\varepsilon_i}^2)$ the unique best (mean squared error) linear predictor of $\theta$ based on $(s_i, z^n)$ (that is, the linear function $\delta(s_i, z^n) = \alpha_0 + \alpha_s s_i + \sum_{t=1}^n \alpha_t z_t$ which minimizes $E\{(\theta - \delta(s_i, z^n))^2\}$ over the $\alpha$ coefficients) is given by the same expression as the one for $E(\theta | s_i, z^n)$ in Lemma 3.1 under the normality assumption.[13]

We will say that the sequence $B_n$ is *of the order* $n^\nu$, with $\nu$ a real number, whenever $n^{-\nu}B_n \rightarrow_n k$, for some non-zero constant $k$. If $\tau_n$ is of the order $n^\nu$ define the *asymptotic precision*: $A\tau_\infty = \lim_{n\to\infty} n^{-\nu}\tau_n$. Denote by $\rightarrow_L$ convergence in law (distribution).

**Lemma 3.2.** *Assume that* $\tau_n$ *is of the order* $n^\nu$, *and that* $\{a_t\}$ *is a sequence of order* $t^{-\kappa}$ *for some constants* $\nu > 0$ *and* $\kappa \geqq 0$ *such that* $\nu + \kappa > 1/2$. *Then*

(i) $y_n$ *is strongly consistent (that is,* $y_n \rightarrow_n \theta$ *a.s.), and*

(ii) $\sqrt{n^\nu}\,(y_n - \theta) \rightarrow_L N(0, (A\tau_\infty)^{-1})$.

12. Just form the average of the signals of agents with precision bounded away from zero and put a zero weight on the signals of other agents.

13. These results should be clear since $E(\theta | s_i, z^n)$ is the *unique* best predictor of $\theta$ and under normality assumptions it depends only on first and second moments and it is linear.

*Proof.* See Appendix. ||

*Remark* 3.3. For the results of Lemma 3.2 to hold, normality of random variables is not needed. Those results are based on Strong Law of Large Numbers- and Central Limit Theorem-type arguments.

## 4. EQUILIBRIUM DYNAMICS

The strategy used to characterize the Bayesian equilibria of the dynamic game is as follows. First, attention is restricted to equilibria in linear strategies. The observational equivalence between the sequence of prices and a sequence of random variables which represent the new information contained in each price is then established. Linear equilibria are easily characterized. Finally, it is shown that only linear equilibria exist provided strategies belong to an appropriate class.

Let us postulate therefore the current price as a linear function of $\theta$, $u$, and the sequence of past prices: $p_n = a_n\theta + u_n + \varphi_n(p^{n-1})$, where $p^{n-1} = \{p_1, \ldots, p_{n-1}\}$ and $\varphi_n$ is a linear function of $p^{n-1}$.[14] Defining $z_n = a_n\theta + u_n$, $p_n$ equals $z_n + \varphi_n(p^{n-1})$, and it is not difficult to foresee the observational equivalence[15] of $p^n$ and the vector of random variables $z^n = \{z_1, \ldots, z_n\}$.

**Lemma 4.1.** *In a linear equilibrium, $p^n$ is observationally equivalent to $z^n$, where $z_t = a_t\theta + u_t$, $t = 1, \ldots, n$.*

*Proof.* Consider a linear equilibrium $(a_t, \varphi_t(\cdot))$, $t = 1, 2, \ldots$. I show the result by induction. For $n = 1$, $p_1 = z_1 + \varphi_1$. Since the constant $\varphi_1$ is known in equilibrium, $p_1$ is observationally equivalent to $z_1$. If $p^{n-1}$ is observationally equivalent to $z^{n-1}$, then, as before, $p_n$ is observationally equivalent to $z^n$, since $p_n = z_n + \varphi_n(p^{n-1})$. ||

The random variable $z_t$ represents the *new information* in the market price $p_t$.[16]

At stage $n$ agent $i$ has information $\{s_i, p^{n-1}\}$, which has been seen equivalent (Lemma 4.1) to $\{s_i, z^{n-1}\}$. According to Lemna 3.1 then agent $i$ needs to condition only on $s_i$ and the OLS sufficient statistic $y_{n-1}$ summarizing the information contained in the variables $z^{n-1}$. It is sufficient then to consider strategies of the form $X_{in}(s_i, y_{n-1})$.

Denote the population average of any parameter $\delta_{in}$, $\int_0^1 \delta_{in} di$, by $\delta_n$. Recall from Lemma 3.1. that $\beta_{in}$, $\alpha_{in}$, and $\gamma_{in}$ denote, respectively, the weights given to $\bar{\theta}$, $s_i$ and $y_n$ in the estimation of $\theta$ and let $y_0 = A_0 = 0$. The following proposition characterizes linear equilibria.

**Proposition 4.1.** *There is a unique equilibrium in which the price at period n is a linear function of $\theta$, $u_n$ and $p^{n-1}$. The price function is given by:*

$$p_n = a_n\theta + u_n + c_n y_{n-1} + b_n\bar{\theta} + \lambda\phi(1+\lambda)^{-1}$$

*where*

$$a_n = \alpha_{n-1}(\lambda + \alpha_{n-1})^{-1},\ c_n = \lambda\gamma_{n-1}(\lambda+1)^{-1}(\lambda + \alpha_{n-1})^{-1}$$

14. The coefficient of $u_n$ in the expression for $p_n$ is set equal to one without loss of generality.

15. The vector $p^n$ is observationally equivalent to the vector of random variables $z^n$ if $p^n$ can be inferred from $z^n$ and vice versa.

16. The variable $z_t = a_t\theta + u_t$ is closely associated with the *innovation* in the current price: $p_t - E(p_t | p^{t-1}) = a_t(\theta - E(\theta | p^{t-1})) + u_t$.

*and*

$$b_n = \lambda\beta_{n-1}(\lambda+1)^{-1}(\lambda+\alpha_{n-1})^{-1}.$$

*The departure from the RE equilibrium is*:

$$p_n - P(\theta, u_n) = c_n(y_{n-1}-\theta) + b_n(\bar{\theta}-\theta).$$

*Remark* 4.1. Strategies are given by:

$$X_{in}(s_i, y_{n-1}) = \phi(1+\lambda)^{-1} - b_{in}\bar{\theta} - a_{in}s_i - c_{in}y_{n-1},$$

where $a_{in} = \alpha_{in-1}(\lambda+\alpha_{n-1})^{-1}$, $c_{in} = ((\lambda+1)\gamma_{in-1} - \gamma_{n-1})(\lambda+1)^{-1}(\lambda+\alpha_{n-1})^{-1}$ and $b_{in} = ((\lambda+1)\beta_{in-1} - \beta_{n-1})(\lambda+1)^{-1}(\lambda+\alpha_{n-1})^{-1}$.

*Proof.* Consider a generic stage and drop the period subscripts (no confusion should result from that). The information available to agent $i$ is $\{s_i, y\}$, where the pair $(s_i, y)$ is jointly normally distributed with the characteristics given in Section 3 (in which case I will say that the information set is *conformable*). Expected profit maximization yields:

$$X_i(s_i, y) = (E\{(p-\theta) \mid s_i, y\})/\lambda$$

Positing a price function

$$p = a\theta + u + cy + b\bar{\theta} + \lambda\phi/(1+\lambda),$$

$$X_i(s_i, y) = ((a-1)E\{\theta \mid s_i, y\} + b\bar{\theta} + \lambda\phi(1+\lambda)^{-1} + cy)/\lambda.$$

Furthermore,

$$x = \int_0^1 X_i(s_i, y)di = ((a-1)(\beta\bar{\theta} + \alpha\theta + \gamma y) + b\bar{\theta} + \lambda\phi(1+\lambda)^{-1} + cy)/\lambda,$$

where $\beta$, $\alpha$ and $\gamma$ are, respectively, the averages of $\beta_i$, $\alpha_i$ and $\gamma_i$. This follows since $E\{\theta \mid s_i, y\} = \beta_i\bar{\theta} + \alpha_i s_i + \gamma_i y$ and $\int_0^1 \alpha_i s_i di = \alpha\theta + \int_0^1 \alpha_i\varepsilon_i di = \alpha\theta$, (a.s.). In order to check that $\int_0^1 \alpha_i\varepsilon_i di = 0$ (a.s.) note that $E\alpha_i\varepsilon_i = 0$, and that Var $\alpha_i\varepsilon_i = \tau_{\varepsilon_i}/(\tau_\theta + \tau_u A + \tau_{\varepsilon_i})^2$. The collection (Var $\alpha_i\varepsilon_i$) is uniformly bounded since Var $\alpha_i\varepsilon_i$ equals 0 for $\tau_{\varepsilon_i} = 0$, tends to zero when $\tau_{\varepsilon_i}$ tends to infinity, and is continuous with respect to $\tau_{\varepsilon_i}$ on $[0, \infty)$. Apply then our convention about the SLLN.

Then the price has to satisfy

$$\begin{aligned} p &= \phi + u - x \\ &= \phi + u + (1-a)\alpha\lambda^{-1}\theta + \lambda^{-1}((1-a)\gamma - c)y + \lambda^{-1}((1-a)\beta - b)\bar{\theta} - \phi(1+\lambda)^{-1}, \end{aligned}$$

and

$$p = a\theta + u + cy + b\bar{\theta} + \lambda\phi(1+\lambda)^{-1}.$$

The expressions for $a$, $c$ and $b$ follow immediately. The departure from the rational expectations price follows immediately by noting that $a+b+c = (1+\lambda)^{-1}$. The expressions for $a_i$, $c_i$, and $b_i$ follow then from $X_i(s_i, y) = (E\{(p-\theta) \mid s_i, y\})/\lambda$.

It is possible to conclude therefore that if the information sets of agents are conformable there is a unique linear equilibrium at every stage according to the statement of the proposition. Let us check now by induction that indeed information sets of agents are conformable. At stage 1 the information set of agent $i$ is $\{s_i\}$, which is clearly conformable, and therefore the price is as given in the proposition with $y_0 = 0$. If at stage $n$ the conformable information set of agent $i$ is $\{s_i, y_{n-1}\}$, then the equilibrium is as given in the proposition and the information set at stage $n+1$ for agent $i$ is $\{s_i, y_n\}$, which is again conformable. ||

*Remark* 4.2. The workings of linear equilibria are easy to understand. Agent $i$ in period $n$ has information $\{s_i, p^{n-1}\}$. With the sequence of past prices $\{p^{n-1}\}$ and the sequence of the coefficients of response to private information $\{a^{n-1}\}$ the agent forms the sufficient price statistic $y_{n-1}$. This price statistic is just the OLS estimate of $\theta$ based on the variables $\{z^{n-1}\}$. A linear predictor of $\theta$ is then easily derived from $s_i$ and $y_{n-1}$. As time evolves the weights given to the private information and to the price statistic change. It is worth emphasizing that agents are rational and understand the influence of the feed-back effect of the learning process on prices. That is, the response coefficients used by agents are equilibrium ones. In contrast, in the *boundedly rational learning* literature agents have a mis-specified model of the economy and assume a constant coefficient to be estimated while in fact the dynamics of learning induce a time-varying coefficient. The use of OLS in both cases is therefore different. In our rational learning model the OLS estimator is optimal and, obviously, it is applied to a well specified constant coefficient ($\theta$) regression model.

*Remark* 4.3. The market price departs from the rational expectations price according to the discrepancy of $\theta$ with prior information ($\bar{\theta}$) and with public information ($y_{n-1}$), weighted, respectively, by the average responsiveness of agents to prior ($b_n$) and public information ($c_n$).

*Remark* 4.4. The average responsiveness to information, $a_n$, will be always positive given our maintained assumption of a positive mass of agents with precisions bounded away from zero.[17]

*Remark* 4.5. When $\lambda = 0$, the constant returns to scale case, then $a_n = 1$, and $c_n = b_n = 0$, for any $n$. Consequently, the average output $x_n$ equals $\phi - \theta$, the rational expectations benchmark, at any stage. As we have pointed out before the one-shot market replicates the outcome of an economy in which agents share their private information (see Palfrey (1985) and Vives (1988)). This is due to the absence of strategic effects (continuum of traders) and friction (risk aversion or adjustment costs).

Two leading examples will help to understand Proposition 4.1:

*Example* 1. Suppose all agents have the same positive precision of information $\tau_\varepsilon$. Then the average responsiveness to private information is given by $a_n = [1+\lambda(1+\tau_{n-1}\tau_\varepsilon^{-1})]^{-1}$, where $\tau_n$ is the informativeness of the price statistic $y_n$ (or equivalently of the price sequence $p^n$), that is, the precision incorporated in prices in the estimation of $\theta$. As the price precision increases $a_n$ decreases as agents put more weight on the price statistic and less on their private information.

*Example* 2. Suppose agents are either perfectly informed (with proportion $\mu > 0$) or uninformed. Then the average weight to private information is given by $\alpha_n = \mu$, since the informed put weight 1 and the uninformed 0. Correspondingly, the response to private information is given by $a_{in} = (\lambda+\mu)^{-1}$ for the informed, and $a_{in} = 0$ for the uninformed, the average responsiveness being $a_n = \mu\ (\lambda+\mu)^{-1}$, for any $n$. The weight put by the the

17. This follows since $a_n$ will be positive whenever $\alpha_{n-1}$ is positive. Now, $\alpha_{n-1} = \int_0^1 \tau_{\varepsilon_i}(\tau_\theta + \tau_{\varepsilon_i} + \tau_u A_{n-1})^{-1} di$ and $\tau_{\varepsilon_i}(\tau_\theta + \tau_{\varepsilon_i} + \tau_u A_{n-1})^{-1}$ is bounded away from zero for a positive measure subset of agents.

informed on their (perfect) private information is constant since they have nothing to learn from prices.

What about non-linear equilibria?

The following proposition provides a *uniqueness* result showing that non-linear equilibria do not exist in our model provided strategies are restricted in an appropriate way.

**Proposition 4.2.** *The dynamic game has a unique equilibrium, as given in Proposition 4.1, in the class of strategies with bounded means and uniformly (across agents) bounded variances.*

*Proof.* See Appendix. ‖

## 5. THE SPEED OF LEARNING AND CONVERGENCE TO RATIONAL EXPECTATIONS

In order to study the properties of the equilibrium for $n$ large and to characterize the speed at which agents learn and convergence to the RE benchmark obtains I examine first the asymptotic behaviour of the parameters that define the equilibrium strategies of agents and the behaviour of the informativeness of the price statistic $y_n$, $\tau_n$. The large sample behaviour of $\tau_n$ is crucial in the determination of the speed of convergence.

**Lemma 5.1.** *For any (linear) equilibrium sequence:*

(i) *The average response to private information $a_n$ tends to $\mu(\lambda+\mu)^{-1}$ as $n$ goes to $\infty$.*

(ii) $c_n \to_n \lambda(1-\mu)(\lambda+\mu)^{-1}(\lambda+1)^{-1}$ *and* $b_n \to_n 0$.

(iii) *The informativeness of the price statistic $y_n$, $\tau_n$, is of the order of $n^{\nu}$ and $a_n$ is of the order of $n^{-\kappa}$, where $2\kappa+\nu=1$, $\nu\in[1/3, 1]$, $\kappa \geqq 0$. Further, if the average precision $\tau_\varepsilon=\int_0^1 \tau_{\varepsilon_i} di$ is finite, $\nu=1/3$ and the asymptotic precision $A\tau_\infty$ is given by $(3\tau_u)^{1/3}(\tau_\varepsilon/\lambda)^{2/3}$. Otherwise, if $\mu>0$ then $\nu=1$ and $A\tau_\infty=\tau_u(\mu/(\lambda+\mu))^2$; if $\mu=0$ then $1>\nu>1/3>\kappa>0$.*

The proof of the Lemma is given in the Appendix. The fundamental steps of the proof are: (1) to establish that $a_n$ decreases weakly and that $\tau_n$ goes to infinity with $n$; and (2) to establish that the average weight given to private information $\alpha_n$ converges to the proportion of perfectly informed agents $\mu$ and that the average weight given to public information $\gamma_n$ converges to $1-\mu$ (obviously the average weight given to prior information $\beta_n$ goes to zero), then (i) and (ii) in the Lemma follow immediately. (3) If $\mu>0$ show that $\tau_n$ is of the order $n$ (immediate since $\tau_n=\tau_\theta+\tau_u\sum_{t=1}^n a_t^2$ and $a_n$ converges to a positive constant), and the value of $A\tau_\infty$ follows. (4) If $\mu=0$ show that $a_n\tau_n$ converges to $\tau_\varepsilon/\lambda$. Two cases are possible (both involving more elaborate proofs) according to whether $\tau_\varepsilon$ is finite or infinite. If $\tau_\varepsilon$ is finite establish that the order of $\tau_n$ is $n^{1/3}$ and compute its asymptotic value. If $\tau_\varepsilon$ is infinite characterize implicitly the order of $\tau_n$.

The intuition for the results should be clear. The informativeness of prices will never decrease and consequently the average responsiveness to private information $a_n$ will never increase with $n$. In fact, $\tau_n$ will tend to infinity and the price statistic will become fully revealing eventually. (The only possibility for $\tau_n$ to be bounded above is for $a_n$ to converge to zero (and fast) but this is self-contradictory: if $\tau_n$ does not go to infinity $a_n$ will not go to zero since agents will keep putting some weight to their private information.) If there is a positive mass of perfectly informed agents then $a_n$ will be bounded away from zero (these agents put constant weight on their private perfect information since they have nothing to learn from prices) and $\tau_n$ will go to infinity at a linear rate (each current price incorporating a roughly constant amount of new information). If there is no positive mass of perfectly informed agents then $a_n$ will tend to zero as the number of trading periods increases and prices become more informative. This is so since then agents put less and less weight on their private signal and more and more on the price statistic, which becomes increasingly informative and eventually fully revealing. The consequence will be that $\tau_n$ will go to infinity more slowly. Indeed, agents, by reacting on average less to their private information, will incorporate less of it to the current price, slowing down the convergence of the price statistic to $\theta$.

Result (iii) in the above Lemma is the key to characterizing the speed of learning and the convergence properties of the equilibrium. Once the order of magnitude and asymptotic value of the precision of the price statistic $y_n$ is known, its asymptotic distribution follows immediately from Lemma 3.2. The order of magnitude of $\tau_n$ depends on the information content of current prices (the new information) for $n$ large. This is given by the random variable $z_n = a_n\theta + u_n$. If $\mu > 0$ then $a_n$ tends to $a_\infty > 0$ and asymptotically $z_n$ looks like $a_\infty\theta + u_n$. This means that the information content of $z_n$ is asymptotically constant and the order of magnitude of $\tau_n$ is $n$, as in the standard linear regression model: the denominator of the OLS estimator $y_n$, $A_n$, is of the order of $n$. For $n$ large, $\tau_n$ is a linear function of $n$ with slope equal to $A\tau_\infty = \tau_u\ (\mu/(\lambda+\mu))^2$, $\tau_n \approx \tau_\theta + A\tau_\infty n$.[18] Notice that the asymptotic precision of $\tau_n$ (the "slope" of convergence) is larger, the larger is the proportion of perfectly informed agents $\mu$, and smaller, the larger is the slope of the adjustment cost $\lambda$.

If $\mu = 0$ then $a_n$ tends to 0 and asymptotically the *new information* is pure noise: $z_n$ looks like $u_n$ for $n$ large. This will not preclude convergence but it will slow it down. When the average precision of private information in the market $\tau_\varepsilon$ is finite the order of $\tau_n$ is the cubic root of $n$. For $n$ large $\tau_n$ is a strictly concave function of $n$: $\tau_n \approx \tau_\theta + A\tau_\infty n^{1/3}$, where $A\tau_\infty = (3\tau_u)^{1/3}(\tau_\varepsilon/\lambda)^{2/3}$. The asymptotic precision $A\tau_\infty$ is increasing with $\tau_u$ and with $\tau_\varepsilon$, and decreasing with the slope of the adjustment cost $\lambda$. Notice that in all cases an increase in the adjustment cost $\lambda$ decreases the asymptotic precision $A\tau_\infty$. An increase in $\lambda$ has a direct effect of restricting the response of agents to information, for a given price precision, and a countervailing indirect effect of decreasing the price precision, inducing agents to put more weight in their signals. The direct effect dominates eventually and the asymptotic price precision diminishes.

If, even when there is no positive mass of perfectly informed agents in the market, the average precision of private information is infinite, then the order $\nu$ of $\tau_n$ is strictly between 1 and 1/3.

The main results can be stated now. In the following propositions learning and its speed, and convergence to the rational expectations equilibrium and its speed are established formally.

**Proposition 5.1.** *Learning.*

(*i*) *The price statistic $y_n$ converges almost surely to $\theta$ as n goes to infinity.*

(ii) *If $\tau_\varepsilon$ is finite, $\sqrt{n^{1/3}}(y_n-\theta) \to_L N(0, (3\tau_u)^{-1/3}(\lambda/\tau_\varepsilon)^{2/3})$;*
*if $\mu > 0$, $\sqrt{n}(y_n-\theta) \to_L N(0, (\mu/(\lambda+\mu))^{-2}(\tau_u)^{-1})$;*
*if $\mu > 0$ and $\tau_\varepsilon$ is infinite, $\sqrt{n^\nu}(y_n-\theta) \to_L N(0, (A\tau_\infty)^{-1})$, for some $\nu \in (1/3, 1)$ and appropriate positive constant $A\tau_\infty$.*

18. The symbol $\approx$ will be taken to mean "equals approximately".

*Proof.*

(i) From Lemma 5.1 we know that $\nu+2\kappa=1$, $\nu\geqq\kappa\geqq 0$, where $n^{\nu}$ is the order of $\tau_n$. The result follows from Lemma 3.2. since necessarily $\nu+\kappa=1-\kappa>2/3>1/2$.

(ii) It follows from Lemma 3.2(ii) and Lemma 5.1. If $\tau_\varepsilon$ is finite, $n^{-1/3}A_n\to_n 3^{1/3}(\tau_\varepsilon/\lambda\tau_u)^{2/3}$ as $n\to\infty$. If $\mu>0$, since then $n^{-1}A_n\to_n(\mu/(\lambda+\mu))^2$. If $\mu=0$ and $\tau_\varepsilon$ is infinite, since then $n^{-\nu}A_n\to_n A\tau_\infty$ for some $\nu\in(1/3,1)$ and appropriate positive constant $A\tau_\infty$. ‖

**Proposition 5.2.** *Convergence to RE.*

(i) $p_n-P(\theta,u_n)$ *converges almost surely to zero as n goes to infinity.*

(ii) *If* $\tau_\varepsilon$ *is finite,* $\sqrt{n^{1/3}}(p_n-P(\theta,u_n))\to_L N(0,\lambda^{2/3}(1+\lambda)^{-2}(3\tau_u)^{-1/3}(\tau_\varepsilon)^{-2/3})$;
*if* $\mu>0$, $\sqrt{n}(p_n-P(\theta,u_n))\to_L N(0,(\lambda/(\lambda+1))^2((1-\mu)/\mu)^2(\tau_u)^{-1})$;
*if* $\mu>0$ *and* $\tau_\varepsilon$ *is infinite,* $\sqrt{n^{\nu}}(p_n-P(\theta,u_n))\to_L N(0,(1+\lambda)^{-2}(A\tau_\infty)^{-1})$, *for some* $\nu\in(1/3,1)$ *and appropriate positive constant* $A\tau_\infty$.

*Proof.*

(i) From Proposition 4.1 we have that $p_n-P(\theta,u_n)=c_n(y_{n-1}-\theta)+b_n(\bar\theta-\theta)$. From Lemma 5.1, $c_n\to_n c_\infty>0$ and $b_n\to_n 0$ as $n$ goes to $\infty$. The results follows since $(y_{n-1}-\theta)$ converges a.s. to zero.

(ii) If

$$\sqrt{n^{\nu}}(y_n-\theta)\to_L N(0,(A\tau_\infty)^{-1}),\ c_n\to_n c_\infty,$$

and $b_n\to_n 0$ then

$$\sqrt{n^{\nu}}(p_n-P(\theta,u_n))\to_L N(0,(c_\infty)^2(A\tau_\infty)^{-1}).$$

The result follows then from Proposition 5.1 and Lemma 5.1(ii). ‖

Agents eventually learn the unknown parameter $\theta$ since the price statistic $y_n$ is a (strongly) consistent estimator of $\theta$. Nevertheless the speed of learning depends crucially on the distribution of private information in the market. If a positive mass of agents is perfectly informed then convergence is at the standard rate $1/\sqrt{n}$. This speed of convergence is termed standard since it corresponds to the rate of convergence for an OLS estimator in a classical environment (and also to i.i.d. noisy observations of $\theta$). Otherwise, convergence is at a lower rate, reaching the lower bound $1/\sqrt{n^{1/3}}$ when the average precision of private information is finite.

The asymptotic variance (AV) of the departure from rational expectations, $(p_n-P(\theta,u_n))$, equals $(c_\infty)^2(A\tau_\infty)^{-1}$. It is larger the larger the asymptotic variance of $y_n$ and the larger is the response of agents to public information. If $\tau_\varepsilon$ is finite an increase in $\lambda$ may increase or decrease AV according to whether $\lambda$ is small or large. An increase in $\lambda$ increases $(A\tau_\infty)^{-1}$ but decreases $c_\infty$, the overall effect depending on the size of $\lambda$. If $\mu$ is positive then AV increases in $\lambda$ always and decreases in $\mu$. Indeed, when $\mu=1$ and all agents are perfectly informed, AV equals zero. Comparative statics of AV with respect to $\tau_\varepsilon$ and $\tau_u$ are similar to comparative statics of $(A\tau_\infty)^{-1}$.

It is worth remarking that the slow convergence rates could not be obtained in markets where the unknown parameter was to enter the price function directly. Recall that in the learning problem considered, agents learn about $\theta$ through prices which depend on $\theta$ only through the aggregate actions in the market. Consider, for example, the classical learning in rational expectations model as stated at the beginning of Section 2 in which the demand intercept $\phi$ is unknown (and $\theta = 0$): $p_t = \phi + u_t - \eta x_t$. In this market firms would learn about $\phi$ even if they had no private information since prices are noisy signals of $\phi$ even if average output is constant. The result is that prices will contain asymptotically a constant amount of information and convergence will be at the standard rate. It is possible to show that now the new information in the price is related to the random variable $z_t = w_t\theta + u_t$, where $w_t = 1 - \eta a_t$. The point is that $w_t$ will converge to a positive constant even if $a_t$ converges to zero. The order of magnitude of $W_n = \sum_{t=1}^{n} w_t^2$ and of the informativeness of the price statistic will be $n$.

The results contrast with those of the least squares version of the bounded learning literature. While in the present model, convergence to the rational expectations equilibrium obtains for the whole domain of parameter values, typically, least squares estimation converges only for a range of underlying parameter values. Further, in the bounded learning case the rate of convergence of the OLS estimator may differ from the rates derived in the present paper (which are of the order of $A_n^{-1/2}$) and may depend on the parameter that governs convergence as pointed out by Fourgeaud *et al.* (1986). In the rational learning model considered, the OLS estimator is optimal and it is applied to a well-specified constant coefficient ($\theta$) regression model. In consequence its properties are standard although the rate of convergence may be slower than the usual $1/\sqrt{n}$ rate.

## 6. EXTENSIONS

### 6.1. *Rational agents with limited forecasting ability*

Up to now I have considered Bayesian agents who fully understand the workings of the market and behave accordingly. Consider a scenario where the characteristics of the market are as before except that the joint distribution of the random variables $(\theta, \varepsilon_i, u^n)$ is not known. Nevertheless their variance-covariance matrix is known (and it is given as in Section 2). Agents now have less prior knowledge on the random variables of the economy but still are assumed to be fully rational and risk neutral.

Restricting attention to linear strategies in a first instance, the market price will be a linear function of the unknown parameter $\theta$. At stage $n$, the information set of agent $i$ will be $I_t = \{s_t, p^{n-1}\}$ which will be observationally equivalent to $\{s_t, z^{n-1}\}$, as before, given that strategies are linear and that agents fully understand how the economy works. Agents will have to estimate $E(\theta \mid I_t)$. I will assume here that agents have limited forecasting ability and use *linear prediction* to estimate $E(\theta \mid I_t)$. Nevertheless, since they are rational they will use the *best linear predictor* of $E(\theta \mid I_t)$. As we know from Remark 3.2 this coincides with the Bayesian expression for $E(\theta \mid I_t)$, under the normality assumption, where $I_t = \{s_t, z^{n-1}\}$.

The implication is clear: neither the equilibria of the market (Propositions 4.1 and 4.2) nor their asymptotic properties (Proposition 5.1 and 5.2) will change. Propositions 5.1 and 5.2 will hold in this case since we know from Remark 3.3 that normality is not needed for the results to hold. I have shown therefore that risk-neutral agents with limited knowledge of the joint distribution of relevant random variables and limited forecasting ability (using linear prediction rules), who are otherwise rational, eventually learn the unknown parameter and their speed of learning can be characterized.[19]

### 6.2. *Correlated signals*

In presenting the analysis in Section 2 the effects of perfect signal correlation have been briefly discussed: in this case there is nothing to learn and one-shot competition is equivalent to the REE benchmark. The results obtained so far can be extended to the case where signals are imperfectly correlated. The basic difference is that now the rational expectations (shared information) benchmark will not in general fully reveal $\theta$. For example, under normality assumptions, if the error terms of the signals have the same precision and constant correlation $\rho$ in the open interval $(0, 1)$,[20] then the average signal $s$ is a sufficient statistic for all the information but is a noisy estimate of $\theta$. In this case it is easily seen that convergence to the rational expectations equilibrium and learning of $s$ (all that it is possible to learn) obtain at the "slow" rate $1/\sqrt{n^{1/3}}$. The correlation coefficient does not affect the convergence rate but does affect the asymptotic precision of the price statistic in the estimation of $s$, $A^*\tau_\infty$. The effect of $\rho$ on $A^*\tau_\infty$ is ambiguous. A larger $\rho$ on one hand tends to speed up learning (higher $A^*\tau_\infty$) since signals are more highly correlated, but on the other tends to slow it since it increases the variance of the variable agents are trying to predict: the average signal.[21] Whenever there is a positive mass of agents who do not have anything to learn from prices (for example, when a mass $\mu > 0$ of agents receives the same imperfect signal $s$) uninformed agents will learn at the standard rate for the same reason as in the independent signal case: informed agents put a constant weight on their signal.

In the classical case where $\phi$ is random (and $\theta = 0$) agents would eventually learn the true value $\phi$, even with correlated signals, and the speed of learning would not be affected by signal correlation. Take the perfect correlation case ($\rho = 1$) as a benchmark. Then there is no asymmetric information and an agent cannot learn anything from other agents. Nevertheless the agent can learn from market prices. The price in period $t$ is given by: $p_t = \phi + u_t - \eta x_t$, where $x_t$ is a function of the common signal received $s$ and of past prices $p^{t-1}$. The observation of $p_t$ is equivalent to the observation of $\phi + u_t$. Repeated sampling of prices will reveal the unknown parameter $\phi$ at the standard rate $1/\sqrt{n}$. If correlation is less than perfect the same result will hold since asymptotically the agents will place no weight on their signals.[22]

## APPENDIX

*Proof of Lemma* 3.1. Drop the $i$ subscript in $s$ and $\varepsilon$. Since all random variables are Normally distributed they are fully characterized by mean and variance and it is sufficient to show that (1) $E(\theta \mid s, z^n) = E(\theta \mid s, y_n)$ and (2) $\operatorname{Var}(\theta \mid s, z^n) = \operatorname{Var}(\theta \mid s, y_n)$. It is easy to see that (2) follows from (1) using the fact that $\theta - E(\theta \mid s, z^n)$ is independent of $(s, z^n)$ (Projection Theorem for normal random variables) and noticing that:

$$\operatorname{Var}(\theta \mid s, z^n) = E\{(\theta - E(\theta \mid s, z^n))^2 \mid s, z^n\} = E\{\theta - E(\theta \mid s, z^n)\}^2$$
$$= E(\theta - E(\theta \mid s, y_n))^2 = \operatorname{Var}(\theta \mid s, y_n).$$

19. The case where agents have a diffuse prior on the unknown parameter $\theta$, that is, if $\sigma_\theta^2 = \infty$, can also be accommodated by the analysis provided.

20. Make the convention (in accord with the finite dimensional version of the stochastic process of the error terms) that $\varepsilon = \int_0^1 \varepsilon_i di$ is a normal random variable with zero mean, and $\operatorname{Var} \varepsilon = \operatorname{Cov}(\varepsilon_i, \varepsilon) = \rho\sigma_\varepsilon^2$.

21. It can be shown that $A^*\tau_\infty = (3\tau_u)^{1/3}(\tau_\varepsilon/(\lambda(1-\rho)(1+\rho\tau_\theta/\tau_\varepsilon)))^{2/3}$. Recall also that $\operatorname{Var} s = \rho\sigma_\varepsilon^2 + \sigma_\theta^2$.

22. The response coefficient $a_t$ will converge to zero with $t$, and the $z_t$ variables which are observationally equivalent to prices will approximately equal $\phi + u_t$ for $t$ large: $z_t = (1 - \eta a_t)\phi - \eta a_t(\int_0^1 \varepsilon_i di) + u_t$.

The proof of (1), and of the associated expression for the conditional expectation, is omitted since it is standard. ||

*Proof of Lemma* 3.2.

(i) Notice that $y_n - \theta = A_n^{-1}\sum_{t=1}^n a_t u_t$. $\{a_t u_t\}$ is a sequence of independent random variables, $A_n^{-1} \to_n 0$ and $\sum_{t=1}^\infty \operatorname{Var}(a_t u_t) A_t^{-2} = \sigma_u^2 \sum_{t=1}^\infty A_t^{-2} a_t^2 < \infty$. The last inequality holding since $A_t^{-2} = (\sum_{k=1}^t a_k^2)^{-2}$ is of the order $t^{-2\nu}$, and $a_t^2$ is of the order $t^{-2\kappa}$ therefore $A_t^{-2}a_t^2$ is of the order $t^{-2(\nu+\kappa)}$ and the series $\sum_{t=1}^\infty t^{-2(\nu+\kappa)}$ is convergent if $2(\nu+\kappa) > 1$. It is a well-known corollary of the SLLN then that $A_n^{-1}\sum_{t=1}^n a_t u_t \to_n 0$ a.s.[23]

(ii) From our assumptions it is clear that $\lim_{n\to\infty}(\sum_{t=1}^n a_t^2)^{-1}\max_{1\le t\le n} a_t^2 = 0$ since $\sum_{t=1}^n a_t^2 \to_n \infty$ and $a_t$ is bounded ($\{a_t\}$ is a sequence of order $t^{-\kappa}$ for some constant $\kappa \gtrless 0$). The result follows from Theorem 3.5.3 in Amemiya (1985): $\sigma_u^{-1}(\sum_{t=1}^n a_t^2)^{1/2}(y_n - \theta) \to_L Z$ where $Z \sim N(0, 1)$. By assumption $\lim_{n\to\infty} n^{-\nu}\tau_n = A\tau_\infty$, and therefore $n^{-\nu}\sum_{t=1}^n a_t^2 \to_n k$, with $k = (\tau_u)^{-1}A\tau_\infty$.

In consequence,

$$\left(\frac{1}{\sqrt{n^\nu}}\sigma_u^{-1}(\textstyle\sum_{t=1}^n a_t^2)^{1/2}\right)(\sqrt{n^\nu}(y_n - \theta)) \to_L Z.$$

The first factor converges to $\sqrt{(A\tau_\infty)^{-1}} > 0$. Therefore the second factor converges in law to $\sqrt{(A\tau_\infty)^{-1}}Z$ and $\sqrt{n^\nu}(y_n - \theta) \to_L N(0, (A\tau_\infty)^{-1})$. iigibbG

*Proof of Proposition* 4.2 *(Uniqueness).* It is sufficient to show that, at any stage, linear equilibria are the only possible equilibria. Suppose that the information available to agent $i$ at a certain stage is $I_i = \{s_i, y\}$ and assume that $(\theta, s_i, y)$ are jointly normally distributed. Time subscripts are dropped as before. Expected profit maximization yields $X_i(s_i, y) = \lambda^{-1}E\{(p-\theta)|s_i, y\}$. It follows that

$$X_i(s_i, y) = \lambda^{-1}(E(\phi - \theta|s_i, y) - E(x|s_i, y)), \tag{*}$$

where $x = \int_0^1 X_i(s_i, y)di$. The convention on the linearity properties of integrals and the assumption of bounded means of strategies implies that

$$\int_0^1 X_i(s_i, y)di = \int_0^1 (X_i(s_i, y) - E(X_i(s_i, y)|\theta, y))di + \int_0^1 E(X_i(s_i, y)|\theta, y)di.$$

The first term is zero according to our convention on the SLLN since $E\{X_i(s_i, y) - E(X_i(s_i, y)|\theta, y)\} = 0$ and $\operatorname{Var}(X_i(s_i, y)|\theta, y)$ are uniformly bounded by assumption. I conclude therefore that $\int_0^1 X_i(s_i, y)di = \int_0^1 E(X_i(s_i, y)|\theta, y)di$.

Let then $x(\theta, y) = \int_0^1 E(X_i(s_i, y)|\theta, y)di$. In order to ease notation, and without loss of generality, let $\bar\theta = 0$. Then $E(\theta|s_i, y) = \alpha_i s_i + \gamma_i y$. Define:

$$\Psi_i(s_i, y) = X_i(s_i, y) - (\phi(1+\lambda)^{-1} - a_i s_i - c_i y),$$

and

$$\Psi(\theta, y) = x(\theta, y) - (\phi(1+\lambda)^{-1} - a\theta - cy),$$

where $a_i$, $c_i$, and $c$ are the coefficients of linear equilibria with $\bar\theta = 0$ (in which case $b = b_i = 0$). Note that $\Psi(\theta, y) = \int_0^1 \Psi_i(s_i, y)di$ since $\int_0^1 a_i s_i di = a\theta$ and $c = \int_0^1 c_i di$.[24] It is immediate to check from (*) and the expressions for $a$, $a_i$, $c$, and $c_i$ that $\Psi_i(s_i, y) = -\lambda^{-1}E\{\Psi(\theta, y)|s_i, y\}$. Multiply by $\Psi_i(s_i, y)$ and taking expectations I obtain

$$E(\Psi_i(s_i, y))^2 = -\lambda^{-1}E\{\Psi(\theta, y)\Psi_i(s_i, y)\}.$$

Therefore,

$$\begin{aligned}\int_0^1 E\{\Psi(\theta, y)\Psi_i(s_i, y)\}di &= \int_0^1 E\{E\{\Psi(\theta, y)\Psi_i(s_i, y)|\theta, y\}\}di \\ &= \int_0^1 E\{\Psi(\theta, y)(E\{\Psi_i(s_i, y)|\theta, y\})\}di \\ &= E\left\{\Psi(\theta, y)\left(\int_0^1 E\{\Psi_i(s_i, y)|\theta, y\}di\right)\right\} \\ &= E\left\{\Psi(\theta, y)\left(\int_0^1 \Psi_i(s_i, y)di\right)\right\}.\end{aligned}$$

23. See White (1984) for related results.
24. $\int_0^1 a_i\varepsilon_i di = 0$ using our convention on the SLLN since $\operatorname{Var} a_i\varepsilon_i$ are uniformly bounded.

The last equality holding since

$$\int_0^1 \Psi_i(s_i, y)di = \int_0^1 E\{\Psi_i(s_i, y) \mid \theta, y\}di,$$

using the same reasoning as above to write $\int_0^1 X_i(s_i, y)di = \int_0^1 E(X_i(s_i, y) \mid \theta, y)di$.[25] Conclude that $\int_0^1 E(\Psi_i(s_i, y))^2 di = -\lambda^{-1}E(\Psi(\theta, y))^2$. Summarizing, I have shown that

$$0 \leqq \int_0^1 E(\Psi_i(s_i, y))^2 di = -\lambda^{-1}E(\Psi(\theta, y))^2 \leqq 0.$$

It follows that $E(\Psi(\theta, y))^2 = 0$, which implies $\Psi(\theta, y)) = 0$ (a.s.) or $x(\theta, y) = a\theta + cy$ (a.s.). This entails necessarily strategies as postulated in Proposition 4.1:

$$X_i(s_i, y) = a_i s_i + c_i y \quad \text{(a.s.)}$$

I check by induction now that indeed $(\theta, s_i, y_n)$ are jointly normally distributed for any $n$. If $(\theta, s_i, y_{n-1})$ are jointly normally distributed then at stage $n$ there is a unique linear equilibrium, prices are normally distributed, and $y_n$ will be normally distributed and $(\theta, s_i, y_n)$ jointly normally distributed. When $n = 1$, the information set of agent $i$ is $I_i = \{s_i\}$ and $(\theta, s_i)$ are jointly normally distributed by assumption. Therefore, $y_1$ is normally distributed and $(\theta, s_i, y_1)$ jointly normally distributed. ‖

*Proof of Lemma* 5.1.

(i) I show first that $a_n$ decreases (weakly) with $n$. From Proposition 4.1, $a_n = \alpha_{n-1}(\lambda + \alpha_{n-1})^{-1}$ and consequently it is increasing in $\alpha_{n-1}$; further $\alpha_{n-1} = \int_0^1 \alpha_{in-1}di = \int_0^1 \tau_{\varepsilon_i}(\tau_\theta + \tau_{\varepsilon_i} + \tau_u A_{n-1})^{-1}di$; when $n$ increases $A_{n-1}$ increases (weakly) and $\alpha_{n-1}$ decreases (weakly). I show now that $A_n$ goes to infinity with $n$.

Note first that $A_n = \sum_{t=1}^n a_t^2$ is a monotone-increasing sequence of $n$. If it does not tend to infinity it will converge to some constant $A_\infty > 0$ (the constant will be positive since $a_t > 0$). This implies that $a_n$ will converge to $a_\infty = \alpha_\infty(\lambda + \alpha_\infty)^{-1}$ where $\alpha_\infty = \int_0^1 \alpha_{i\infty}di$ and $\alpha_{i\infty} = \tau_{\varepsilon_i}(\tau_\theta + \tau_{\varepsilon_i} + \tau_u A_\infty)^{-1}$. The result follows form Lebesgue's Bounded Convergence Theorem (see, for example, Royden (1968, p. 81) since $\alpha_{it}$ is measurable and bounded above by 1, for all $t$ (and $i$), and $\alpha_{it}$ converges to $\alpha_{i\infty} = \tau_{\varepsilon_i}(\tau_\theta + \tau_{\varepsilon_i} + \tau_u A_\infty)^{-1}$ for all $i$; it follows then that $\int_0^1 \alpha_{it}di$ converges to $\int_0^1 \alpha_{i\infty}di$. This contradicts the hypothesis of convergence of $A_n$. Therefore, $A_n \to \infty$. I show that $a_n \to \mu(\lambda + \mu)^{-1}$. Note first that $\alpha_{i\infty} = 0$ if $\tau_{\varepsilon_i}$ is finite since $A_n \to \infty$, and that $\alpha_{i\infty} = 1$ otherwise since then $\alpha_{in} = 1$ for all $n$. Therefore $\alpha_\infty = \int_0^1 \alpha_{i\infty}di = \mu$.[26] Using the previous argument with the Bounded Convergence Theorem, $\alpha_\infty = \mu$ and $a_\infty = \mu(\lambda + \mu)^{-1}$.

(ii) From Proposition 4.1, $c_n = \lambda\gamma_{n-1}(\lambda + 1)^{-1}(\lambda + \alpha_{n-1})^{-1}$ and $b_n = \lambda\beta_{n-1}(\lambda + 1)^{-1}(\lambda + \alpha_{n-1})^{-1}$. Now, $\gamma_n = \int_0^1 \gamma_{in}di$ where $\gamma_{in} = \tau_u A_n(\tau_\theta + \tau_{\varepsilon_i} + \tau_u A_n)^{-1}$. Similar arguments as in the last part of (i) establish that $\gamma_n$ tends to $1 - \mu$ and $\beta_n$ to zero with $n$. This establishes the result.

(iii) Let $a_n$ be of order $n^{-\kappa}$ and $A_n$ be of order $n^\nu$. Since $a_n \to \mu(\lambda + \mu)^{-1}$ and $A_n \to \infty$ necessarily $\kappa \geqq 0$ and $\nu > 0$. Recall that $\tau_n = \tau_\theta + \tau_u A_n$ and therefore the asymptotic precision will be given by $\lim_{n\to\infty} n^{-\nu}\tau_n = \tau_u(\lim_{n\to\infty} n^{-\nu}A_n)$.

If $\mu > 0$ then $a_\infty = \mu(\lambda + \mu)^{-1} > 0$ and therefore $\kappa = 0$ and $\nu = 1$. It follows that $n^{-1}A_n \to_n (\mu/(\lambda + \mu))^2$. Otherwise ($\mu = 0$), I argue in two steps:

**Claim 1.** *$a_n A_{n-1}$ converges to $\tau_\varepsilon(\lambda\tau_u)^{-1}$, where $\tau_\varepsilon = \int_0^1 \tau_{\varepsilon_i}di$, if $\tau_\varepsilon$ is finite and to $\infty$ otherwise.*

*Proof.* From the expression for $a_n$:

$$a_n A_{n-1} = ((A_{n-1})^{-1} + \lambda(A_{n-1}\alpha_{n-1})^{-1})^{-1}.$$

Now,

$$A_{n-1}\alpha_{n-1} = \int_0^1 A_{n-1}\alpha_{in-1}di = \int_0^1 \tau_{\varepsilon_i}((\tau_\theta + \tau_{\varepsilon_i})(A_{n-1})^{-1} + \tau_u)^{-1}di.$$

25. By assumption strategies have bounded means and uniformly bounded variances, the linear equilibrium satisfies this assumption, and therefore $E\{\Psi_i(s_i, y) \mid \theta, y\}$ is bounded for all $i$ and $\mathrm{Var}\{\Psi_i(s_i, y) \mid \theta, y\}$ are uniformly bounded.

26. Notice that the set $\{i \in [0, 1] : \tau_{\varepsilon_i} = \infty\}$ is measurable, with measure $\mu$, and consequently the complement $\{i \in [0, 1] : \tau_{\varepsilon_i} < \infty\}$ is also measurable.

Observing that $A_{n-1}\alpha_{i,n-1}$ is a monotone increasing sequence of nonnegative measurable functions of $i$ converging almost everywhere to $\tau_{\varepsilon_i}/\tau_u$, it is possible to conclude (Lebesgue Monotone Convergence Theorem; see, for example, Royden (1968, p. 227)) that $\int_0^1 A_{n-1}\alpha_{i,n-1}di$ converges to $(\tau_u)^{-1}\int_0^1 \tau_{\varepsilon_i}di$ and the result follows. ‖

**Claim 2.** *If $\tau_\varepsilon$ is finite then $n^{-1/3}A_n \to_n 3^{1/3}(\tau_\varepsilon/\lambda\tau_u)^{2/3}$, as $n\to\infty$. If $\tau_\varepsilon$ is infinite (and $\mu=0$) then $1>\nu>1/3>\kappa>0$.*

*Proof.* If $\mu=0$ then $a_n\to_n 0$. If $\tau_\varepsilon$ is finite the result follows from Claim 1 and Lemma A1 below. If $\tau_\varepsilon$ is infinite then $a_nA_n$ tends to infinity with $n$. Further, $n^{-1}A_n\to_n 0$ (since $a_n\to_n 0$) and $n^{-1/3}A_n\to_n\infty$ (using the fact that $A_n\geqq n(a_n)^2$, $(a_nA_n)^{-1}\geqq(n^{-1/3}A_n)^{-3/2}$, from which the result follows since $a_nA_n\to\infty$). It follows then from Lemma A2 below (with $\nu>\kappa>0$) that $\nu+2\kappa=1$ and $1>\nu>1/3$.

Note that in all cases $\nu+2\kappa=1$ and $1\geqq\nu\geqq 1/3$. ‖

**Lemma A1.** *Assume that $a_nA_n\to_n k>0$, then $n^{-1/3}A_n\to_n 3(k/3)^{2/3}$, and $n^{1/3}a_n\to_n(k/3)^{1/3}$.*

*Proof.* Note first that $a_nA_1\geqq n(a_n)^3$ since $a_n$ is decreasing. It follows then that the sequence $\{n(a_n)^3\}$ is eventually bounded above by $k$ and it is always bounded below by zero. It is possible to establish that it converges (omitted). I show now that $\{n(a_n)^3\}$ converges to $k/3$. Suppose $\{n(a_n)^3\}$ converges to $h$. Fix $\varepsilon>0$. There is $T(\varepsilon)$ such that for $t\geqq T(\varepsilon)$: $t(a_t)^3\in[h-\varepsilon,h+\varepsilon]$. Therefore, for $t\geqq T(\varepsilon)$, $(h+\varepsilon)^{2/3}t^{-2/3}\geqq a_t^2\geqq(h-\varepsilon)^{2/3}t^{-2/3}$ and

$$\begin{aligned}3(h+\varepsilon)^{2/3}(n^{1/3}-(T(\varepsilon)-1)^{1/3}&=(h+\varepsilon)^{2/3}\int_{T(\varepsilon)-1}^{n}t^{-2/3}dt\geqq(h+\varepsilon)^{2/3}\textstyle\sum_{t=T(\varepsilon)+1}^{n}t^{-2/3}\\&\geqq\textstyle\sum_{t=T(\varepsilon)+1}^{n}a_t^2\geqq(h-\varepsilon)^{2/3}\sum_{t=T(\varepsilon)+1}^{n}t^{-2/3}\\&\geqq(h-\varepsilon)^{2/3}\int_{T(\varepsilon)+1}^{n+1}t^{-2/3}dt=3(h-\varepsilon)^{2/3}((n+1)^{1/3}-(T(\varepsilon)+1)^{1/3}).\end{aligned}$$

Since this holds for all $\varepsilon>0$, multiplying by $n^{-1/3}$ and taking limits I obtain $n^{-1/3}A_n\to_n 3h^{2/3}$. Consequently, $(n^{1/3}a_n)(n^{-1/3}A_n)\to_n(h^{1/3})(3h^{2/3})=3h$, and $a_nA_n$ converges to $k$. Conclude that $h=k/3$. ‖

**Lemma A2.** *Assume that $n^{-\nu}\sum_{t=1}^n a_t^2\to_n k$, where $k>0$, and that $a_t$ is of the order $t^{-\kappa}$, where $\nu\geqq\kappa>0$. Then $\nu+2\kappa=1$ and $1>\nu\geqq1/3$.*

*Proof.* It follows that $n^{-\nu}\sum_{t=1}^n t^{-2\kappa}$ must be bounded above and away from zero. I check first that $\kappa\leqq1/2$. Let $A_n=\sum_{t=1}^n a_t^2$.

Suppose $\kappa>1/2$, we have

$$\int_1^n t^{-2\kappa}dt\geqq\textstyle\sum_{t=2}^n t^{-2\kappa}\quad\text{and}\quad\displaystyle\int_1^n t^{-2\kappa}dt=\frac{1-n^{1-2\kappa}}{2\kappa-1}. \tag{$*$}$$

For $\kappa>1/2$ this integral goes to $1/(2\kappa-1)$ with $n$ but $A_n\to\infty$, a contradiction.

Suppose now $\kappa=1/2$, we have then $n^{-\nu}\ln n=n^{-\nu}\int_1^n(1/t)dt\geqq n^{-\nu}\sum_{t=2}^n t^{-1}$. The last term must be bounded away from zero but $n^{-\nu}\ln n\to0$ as $n\to\infty$ if $\nu\geqq\kappa>1/2$, a contradiction. I conclude that $\kappa<1/2$.

From $(*)$ it follows that

$$\frac{1}{n^\nu}\frac{n^{1-2\kappa}}{1-2\kappa}\geqq\frac{1}{n^\nu}\left(\frac{1}{1-2\kappa}+\textstyle\sum_{t=2}^n t^{-2\kappa}\right)\geqq\dfrac{1}{n^\nu}\textstyle\sum_{t=1}^n t^{-2\kappa}.$$

The last term must be bounded away from zero, and therefore, $1-2\kappa-\nu\geqq0$.

Furthermore,

$$\frac{1}{n^\nu}\textstyle\sum_{t=1}^{n-1}t^{-2\kappa}\geqq\dfrac{1}{n^\nu}\left[\displaystyle\int_1^n t^{-2\kappa}dt\right]=\frac{1}{n^\nu}\left[\frac{n^{1-2\kappa}-1}{1-2\kappa}\right].$$

The first term must be bounded above. Therefore, $1-2\kappa-\nu\leqq0$. I conclude that $1=2\kappa+\nu$. Consequently, $1>\nu\geqq1/3$ since $\nu\geqq\kappa>0$. ‖

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