A Model of Costly Interpretation of Asset Prices

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June 2016

Abstract

We propose a model in which investors have to spend effort to interpret the informational content of asset prices in financial markets. Investors do not fully understand the price function, but they still infer information from prices and choose the optimal trading strategies given their beliefs. We show that as investors' sophistication level increases, trading volume increases, while disagreement among investors can exhibit a hump-shape. In the limit, investors fully understand the price function, the price approaches to be fully revealing as in the standard rational-expectations equilibrium model, but trading volume diverges to infinity. Sophistication harms welfare through generating excessive speculative trading but benefits welfare through lowering the equilibrium return volatility. We finally allow investors to study market data to endogenize the sophistication level, and find that studying market data exhibits strategic complementarity, so that multiple equilibria can arise.

Key words: Investor sophistication, price informativeness, disagreement, trading volume, speculation, welfare, multiplicity

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1 Introduction

Data can be viewed as information only after it has been analyzed. Interpreting data is often *costly* in terms of time, effort, and other investor resources. This is particular true for market data given the complexity of modern financial markets. In the existing frameworks—such as the traditional rational-expectations equilibrium (REE) model (e.g., Grossman, 1976; Radner, 1979), and the more recent REE-disagreement hybrid models (e.g., Banerjee, 2011)—investors understand the price function and can *freely* invert the market price to uncover value-relevant information. In this paper, we propose a framework to explicitly capture the idea that it is costly for investors to interpret market data and examine how investor ability in interpreting the price affects equilibrium outcomes.

In our model, investors do not fully understand the price function but they still actively infer information from the price data. Each investor interprets the price as a private payoff-relevant signal in a form of "truth plus noise." The "truth" represents the best signal that a fully sophisticated investor could obtain (which will be endogenously determined in equilibrium), while the "noise" is negatively associated with the sophistication level of the investor in interpreting the price. After investors form their beliefs based on their personalized price signals, they behave as rational Bayesian and make optimal investments in response to their own beliefs. Through market clearing, investors' optimal asset investments in turn endogenously determine the equilibrium price function and hence the best price signal (i.e., the "truth" in investors' personalized signals extracted from the price data). To close the model, we endogenize investors' sophistication level using a learning technology: investors can spend resources to study market data, and the more resources they spend, the better can they read the price, and so the less noise is injected in the inference process.

We then use our framework to examine the behavior of asset prices, investors' beliefs, trading volume, and investors' welfare. Firstly, we show that investor sophistication improves price informativeness. In our economy, the price is a linear function of the asset fundamental and a noise term. The fundamental element comes from aggregating investors' private value-relevant information, which is also the root reason why the price contains information that investors care to learn. The noise term in the price arises from a common bias in investors' personalized price signals, which is meant to capture the notion that in processing the price data, investors may suffer a common cognition error (such as "sentiment") or technical error (such as a common bias in data processing algorithms). When investors become more sophisticated, they understand the true price signal better, and thus their trading brings less noise into the price, which makes the price more informative about the fundamental. As investor sophistication approaches to infinity, the asset price approaches to be fully revealing, corresponding to a standard rational-expectations equilibrium.

Secondly, investor sophistication can either spur or stifle disagreement across investors' expectations about the asset fundamental. On the one hand, investors interpret the price data differently, and so the more sophisticated they are, the higher weight they put to their diverse information extracted from the price in forecasting the asset fundamental, and thus the more likely they may end up with different understandings. On the other hand, investor sophistication improves price informativeness, which makes the price contain more precise information about the asset payoff. So, actively reading information from the price can also cause investors' beliefs to converge. The trade-off between these two counteracting forces determines the relation between disagreement and sophistication.

In general, when investors start with precise fundamental signals, the second negative effect always dominates so that disagreement monotonically decreases with investor sophistication. This is because when investors are endowed with precise information, the price signal is particularly accurate after aggregation, and thus the belief-convergence effect is particularly strong. By contrast, when investors are endowed with coarse private fundamental information, the positive effect can dominate. For instance, suppose that investors start with extremely coarse fundamental information and extremely low sophistication level, so that their expectations about the asset payoff are close to the prior distribution and thus almost homogeneous. Now if we increase investors' sophistication level, then they will start to read different information from the price, and so their expectations will diverge. Nonetheless, when investor sophistication level becomes sufficiently high, disagreement will decrease with sophistication again (i.e., the belief-convergence effect will eventually dominate), because as sophistication approaches to infinity, the asset price approaches to be fully revealing, and thus investor disagreement will vanish.

Thirdly, investor sophistication monotonically increases trading volume. In our setup, trading volume is determined by two factors. First, it is positively driven by investors' disagreement about fundamental expectations; that is, when investors disagree more, they trade more. Second, trading volume is negatively driven by investors' perceived risk in the process of trading. When investors perceive little risk, they trade aggressively, so that the aggregate trading volume is high.

As we discussed above, investor sophistication can either increase or decrease disagreement. Thus, through the disagreement channel, investor sophistication can either increase or decrease trading volume. In contrast, as investor sophistication increases, investors perceive lower trading risk for two reasons. First, they can directly read more information from the price, and so they perceive that they can predict the fundamental with less uncertainty. Second, as we discussed before, price informativeness increases with investor sophistication, which means that the price conveys more information to investors, which further reduces investors' trading risk. As a result, through the risk channel, investor sophistication tends to increase trading volume. We can show that this risk channel always dominates the disagreement channel, and thus overall, trading volume increases with investor sophistication.

In particular, when investors approach to be fully sophisticated, trading volume diverges to infinity (because investors study market data), although the price approaches to be fully revealing, which corresponds to the standard REE price. This result contrasts with the conventional wisdom that speculative trading is limited in REE settings without non-speculative trading motives. It suggests that if we view the standard REE as a limiting economy in which investor sophistication approaches infinity, trading volume can be very large. This view seems to well describe the modern financial market in which more real traders, such as high-frequency traders and hedge funds, have employed more sophisticated trading software/devices and trade more intensively in various trading venues.

In our setting, trading volume is purely speculative and it hurts investors' welfare. This is because the equilibrium holding of each investor is simply a linear combination of the error terms in their signals, which is a form of "winner's curse" (Biais, Bossaerts, and Spatt, 2010). This result is consistent with the existing empirical evidence documented in the literature (e.g., Odean, 1999; Barber and Odean, 2000). Through this volume channel, investor sophistication tends to reduce welfare.

However, investor sophistication also positively affects welfare through reducing the return volatility, which is a general-equilibrium effect. Given that trading is speculative, return volatility is negatively related to welfare: a higher return volatility means that the price deviates more from the fundamental, and thus the winner's curse hurts investors more. As we discussed before, investor sophistication improves price informativeness and hence reduces the equilibrium return volatility by causing the price to be closer to the fundamental. As a result, through this equilibrium return volatility channel, investor sophistication tends to improve welfare. Taken together, the overall welfare implication of sophistication is ambiguous due to the two offsetting forces.

Finally, when endogenizing investor sophistication, we find that the previous price-informativeness result leads to strategic complementarity in sophistication acquisition and the possibility of multiple equilibria. Specifically, when a representative investor spends more resources to become more sophisticated in reading prices, price informativeness increases and the price conveys more information, which increases the marginal value of attending to the price data. This in turn further strengthens investors' ex-ante incentives to study market data. This strategic complementarity implies that multiple sophistication levels can be sustained in equilibrium. Thus, when an exogenous parameter, for instance, the cost of achieving sophistication, changes, there can be jumps in equilibrium sophistication levels. This can correspond to waves of development of algorithmic trading in reality in response to exogenous shocks to the economy, say, some regulation changes.

2 Related Literature

Our approach of modeling investors' understanding of market data shares similarity to the concept of "rationalizability" (Guesnerie, 1992; Jara-Moroni, 2012) and the "level-k" or "cognitive hierarchy" models (see Crawford, Costa-Gomes, and Iriberri (2013) for a survey). These existing studies make an effort to study whether and how rational expectations can be generated, starting from a more fundamental principle that investors are individually Bayesian rational and best respond to *some* beliefs. Similarly, under our approach, investors are fully rational at the individual level—more specifically, investors are subjective expected utility (SEU) maximizers (Savage, 1954) and they can perform perfect partial equilibrium analysis—but they do not perfectly understand the general structure of the economy and therefore may not necessarily have the best signal.

Our paper is also closely related to the recent literature on environment complexity that makes agents fail to account for the informational content of other players' actions in game settings. Eyster and Rabin (2005) develop the concept of "cursed equilibrium," which assumes that each player correctly predicts the distribution of other players' actions, but underestimates the degree to which these actions are correlated with other players' information. Esponda (2008) extends Eyster and Rabin's (2005) concept to "behavioral equilibrium" by endogenizing the beliefs of cursed players. Recently, Esponda and Pouzo (2016) propose the concept of "Berk-Nash equilibrium" to capture that people can have possibly misspecified view of their environment. In a Berk-Nash equilibrium, each player follows a strategy that is optimal given her belief, and her belief is restricted to be the best fit among the set of beliefs she considers possible, where the notion of best fit is formalized in terms of minimizing the Kullback-Leibler divergence. Although these models are cast in a game theoretical framework, the essential spirit of our financial market model is the same. In our model, investors' interactions are mediated by an asset price, which is sort of a summary statistics for all the other players' actions.

Eyster, Rabin, and Vayanos (2015) have applied the cursed equilibrium concept to a financial market setting and labeled the resulting equilibrium as the cursed-expectations equilibrium (CEE). In a CEE, an investor is a combination of a fully rational REE investor (who correctly reads information from the price) and a naive Walrasian investor (who totally neglects the information in the asset price). The investor in our economy is conceptually related to but different from a partially cursed investor; she does not understand the price function perfectly and has to spend an endogenous cost to infer information from the price. Eyster, Rabin, and Vayanos (2015) have also examined volume implications. Specifically, they show that as the *number* of investors goes to infinity, trading volume diverges. By contrast, we conduct a different analysis, that is, we show that as *investor sophistication* goes to infinity, trading volume explodes. Our analysis thus suggests that as the economy approaches to be fully rational, the equilibrium does not converge to standard REE in terms of volume behavior, which is different from Eyster, Rabin, and Vayanos (2015). In addition, we have conducted analysis on additional variables such as disagreement and welfare.

The recent finance literature, such as Banerjee, Kaniel, and Kremer (2009) and Banerjee (2011), have combined REE and disagreement frameworks to allow investors underestimate the precision of other investors' private information (and hence also labeled as "dismissiveness" models). A dismissive investor can be roughly viewed as a combination of fully rational and naive investors, and thus conceptually related to the investor in our economy. However, in the dismissiveness model, investors still perfectly understand the price function and they only disagree about the distribution of other investors' signals. Thus, at the conceptual level, our investors are fundamentally different. In addition, the volume implication of the dismissiveness model is different from our paper. Specifically, in the dismissiveness model, as investors' bias vanishes (and hence investors become fully sophisticated), volume would vanish as well, which is opposite of the prediction of our setting.

In the accounting literature, some researchers, Indjejikian (1991) and Kim and Verrecchia (1994) for instance, have considered settings in which investors have different interpretations about an *exogenous* public signal such as earnings announcements. In contrast, in our setting, investors have different interpretations about an *endogenous* public signal, which is the equilibrium price. In Ganguli and Yang (2009) and Manzano and Vives (2011), investors interpret the price information differently through acquiring information about the noise supply. Our setting differs from these supply-information models in two important ways. First, at the conceptual level, our investors lack the knowledge about the economy structure, while it is not the case in the supply-information models. Second, the supply-information models have focused on uniqueness versus multiplicity of equilibrium, while our analysis has broader implications for prices and volume.

Finally, at the broad level, our paper contributes to the behavioral finance literature (see Shleifer (2000) and Barberis and Thaler (2003)). Our analysis highlights how investor sophistication and sentiment affect market efficiency and other market outcomes (such as disagreement, volume, and welfare) through the interpretation of asset prices. Recently, Gârleanu and Pedersen (2016) propose a model to show market efficiency is closely connected to the efficiency of asset management. In our model, market efficiency is determined by how investors (institutions or retail investors) interpret the asset price.

3 The Model

Environment We consider a one-period economy similar to Hellwig (1980). Two assets are traded in a competitive market: a risk-free asset and a risky asset. The risk-free asset has a constant value of 1 and is in unlimited supply. The risky asset is traded at an endogenous price $\tilde{p}$ and is in zero supply. It pays an uncertain cash flow at the end of the economy, denoted $\tilde{v}$. We assume that $\tilde{v}$ is normally distributed with a mean of 0 and a precision (reciprocal of variance) of τ_v—that is, $\tilde{v} \sim N(0, 1/\tau_v)$, with $\tau_v > 0$.

There is a continuum $[0, 1]$ of investors who have constant absolute risk aversion (CARA) utility with a risk aversion coefficient of $\gamma > 0$. Investors have fundamental information and trade on it. Specifically, investor i is endowed with the following private signal:

$$\tilde{s}_i = \tilde{v} + \tilde{\varepsilon}_i, \text{ with } \tilde{\varepsilon}_i \sim N(0, 1/\tau_\varepsilon) \text{ and } \tau_\varepsilon > 0, \tag{1}$$

where $\tilde{\varepsilon}_i$ is independent of $\tilde{v}$ and they are also independent of each other.

Belief specification The idea of our paper is to show that the financial market is so complex that traders cannot fully understand its structure so that they cannot perfectly interpret information in asset prices. In traditional REE models, traders look into the asset price to make inference about fundamentals, which is usually modeled as a statistical signal, $\tilde{s}_p$, about the asset fundamental $\tilde{v}$. The justification is that traders are sophisticated enough to understand the statistical properties of the price function that links the price $\tilde{p}$ to the fundamental $\tilde{v}$, and thus they can extract information about $\tilde{v}$ from seeing $\tilde{p}$.

In practice, it is arguable that the asset price in modern financial markets cannot be fully understood by market participants and a better understanding of the market structure needs more effort. To capture this idea, we adopt a reduced-form of belief specification, which takes the form "truth plus noise." That is, after seeing price $\tilde{p}$, investor i interprets it as a signal as follows:

$$\tilde{s}_{p,i} = \underbrace{\tilde{s}_p}_{\text{truth}} + \underbrace{\tilde{x}_i}_{\text{noise}}, \text{ with } \tilde{x}_i \sim N\left(0, \frac{1}{\tau_x}\right), \tag{2}$$

where $\tilde{s}_p$ is the true signal implied by the price, which is also the best signal that a fully sophisticated investor can obtain in a standard REE setting, and where $\tilde{x}_i$ is the noise in processing the price data, which can come from poor mental reasoning or from technology capacity. As standard in the literature, we assume that $\tilde{s}_p$ and $\tilde{x}_i$ are mutually independent. We do not model where equation (2) comes from and thus it is a reduced-form of belief formation. In standard REE models, investors fully understand the price function and can convert the price $\tilde{p}$ to the signal $\tilde{s}_p$, and in this case the noise $\tilde{x}_i$ degenerates to 0 (or $\tau_x = \infty$) in equation (2).

Sophistication Investors can study market data to reduce their noise $\tilde{x}_i$ in (2), thereby bringing the price signal $\tilde{s}_{p,i}$ closer to the best signal $\tilde{s}_p$. We model this noise-reduction process as investors gleaning private information about $\tilde{x}_i$. Specifically, investor i can study the market and obtain the following signal:

$$\tilde{z}_i = \tilde{x}_i + \tilde{\eta}_i \text{ with } \tilde{z}_i \sim N\left(0, 1/\tau_{\eta_i}\right), \tag{3}$$

where $\tilde{\eta}_i$ is independent of all other random variables and independent of each other. Conditional on $\tilde{z}_i$, the noise in investor i's price signal $\tilde{s}_{p,i}$ has a posterior distribution

$$\tilde{x}_i | \tilde{z}_i \sim N\left(\frac{\tau_{\eta_i}\tilde{z}_i}{\tau_x + \tau_{\eta_i}}, \frac{1}{\tau_x + \tau_{\eta_i}}\right), \tag{4}$$

which indeed has a variance $\frac{1}{\tau_x+\tau_{\eta_i}}$ smaller than the prior variance $\frac{1}{\tau_x}$.

The precision τ_{η_i} captures investor i's ability or "sophistication" level in understanding the asset market. When $\tau_{\eta_i} = \infty$, investors fully understand the market, which reduces our economy to the traditional REE setup. When $\tau_x = \tau_{\eta_i} = 0$, investors cannot understand the price function at all and totally neglect the information in prices, which reduces our economy to the traditional Walrasian economy. Parameter τ_{η_i} is endogenous in the model and it comes from the intensity of studying market data. Specifically, being sophisticated is costly, which is reflected by a weakly increasing and convex cost function, $C\left(\tau_{\eta_i}\right)$. The cost can be monetary (e.g., Verrecchia, 1982) or represent costly attention (e.g., Veldkamp, 2011, ch 3; Myatt and Wallace, 2012; Pavan, 2014).[1] Investors choose τ_{η_i} to optimally balance the benefit from being more sophisticated against its cost.

Sentiment The noise term $\tilde{x}_i$ in (2) admits a factor structure as follows:

$$\tilde{x}_i = \tilde{u} + \tilde{e}_i, \text{ with } \tilde{u} \sim N(0, 1/\tau_u) \text{ and } \tilde{e}_i \sim N(0, 1/\tau_e), \tag{5}$$

where $\left(\tilde{u}, \{\tilde{e}_i\}_{i\in[0.1]}\right)$ is mutually independent and independent of all other random variables. Note that, by equations (2) and (5), we have $\frac{1}{\tau_x} = \frac{1}{\tau_u} + \frac{1}{\tau_e}$. In (5), the idiosyncratic noise $\tilde{e}_i$ is specific to investor i. The common noise $\tilde{u}$ in investors' price signals represents waves of optimism and pessimism, which is labeled as "sentiment" in the behavioral economics literature. That is, as in Angeletos and La'o (2013), the variable $\tilde{u}$ represents extrinsic shocks that have nothing to do with the fundamental $\tilde{v}$ but affect all agents' beliefs. This common error $\tilde{u}$ can also arise from a common bias in data-processing algorithms. As we will show shortly, the random variable $\tilde{u}$ will enter the price as an endogenous source of noise trading emphasized in the noisy REE literature (e.g., Grossman and Stiglitz, 1980).

4 Equilibrium Concept

The overall equilibrium in our model is composed of an equilibrium at the trading stage and an equilibrium at the sophistication determination stage. The financial market equilibrium at the trading stage determines the asset price $\tilde{p}$ and investors' demands for the assets, taking investors' sophistication level τ_{η_i} as given. The sophistication equilibrium determines investors' sophistication levels taking the behavior of asset prices as given.

At the trading stage, all investors are SEU maximizers and choose investments in assets to maximize their expected utilities conditional on their information sets. They are price takers

[1] Our information setup follows closely Pavan (2014). In the language of Pavan (2014), parameter τ_x measures the *accuracy* of the information source (which is the price in our context). Parameter τ_{η_i} can be thought of as the *time* investor i devotes to the information source and $C\left(\tau_{\eta_i}\right)$ denotes the attention cost incurred by the investor. Some studies in the rational inattention literature further adopt an entropy-based cost function (e.g., Myatt and Wallace, 2012). In these studies, the amount of information transmitted is captured by the concept of mutual information. The mutual information uses an agent's attention capacity and an agent can incur a cost to increase the attention capacity. In our context, the mutual information is given by $K \equiv \frac{1}{2}\log\frac{Var(\tilde{s}_p|\tilde{s}_{p,i})}{Var(\tilde{s}_p|\tilde{s}_{p,i},\tilde{z}_i)}$, which captures how much information is transmitted after the investor processes the price data. The investor incurs a cost $C(K)$ in order to process price information more accurately.

but still actively infer information from the price $\tilde{p}$. Specifically, investor i has information set $\{\tilde{p}, \tilde{s}_i, \tilde{z}_i\}$. When she makes forecast about fundamental $\tilde{v}$, she will interpret $\tilde{p}$ as a signal $\tilde{s}_{p,i}$ according to (2). Therefore, investor i chooses investment D_i in the risky asset to maximize

$$E_i\left(-e^{-\gamma\left[(\tilde{v}-\tilde{p})D_i - C\left(\tau_{\eta_i}\right)\right]}\middle|\, \tilde{p}, \tilde{s}_i, \tilde{z}_i\right) \tag{6}$$

where $E_i(\cdot)$ indicates that investor i takes expectation with respect to her own (subjective) belief.

The CARA-normal setting implies that investor i's demand for the risky asset is

$$D(\tilde{p}; \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i) = \frac{E(\tilde{v}|\, \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i) - \tilde{p}}{\gamma Var(\tilde{v}|\, \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i)}, \tag{7}$$

where $E(\tilde{v}|\, \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i)$ and $Var(\tilde{v}|\, \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i)$ are the conditional expectation and variance of $\tilde{v}$ given information $\{\tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i\}$. In (7), we have explicitly incorporated $\tilde{s}_{p,i}$ in the demand function to reflect the informational role of the price (i.e., the price helps to predict $\tilde{v}$) and used $\tilde{p}$ per se to capture the substitution role of the price (i.e., a higher price directly leads to a lower demand on the right-hand-side of (7)). Thus, the conditioning on the price in (7) is only used to gauge scarcity as with any other good but the learning on fundamentals is via the private signal $\tilde{s}_{p,i}$ or "price interpretation."

The financial market clears, i.e.,

$$\int_0^1 D(\tilde{p}; \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i)\, di = 0 \text{ almost surely.} \tag{8}$$

This market-clearing condition, together with the demand function (7), will determine an equilibrium price function

$$\tilde{p} = p(\tilde{v}, \tilde{u}), \tag{9}$$

where $\tilde{v}$ and $\tilde{u}$ come from the aggregation of signals $\tilde{s}_i$, $\tilde{s}_{p,i}$, and $\tilde{z}_i$. This equilibrium price function in turn endogenously determines the information content in the true signal $\tilde{s}_p$ given by equation (2).

Now let us formulate how investor sophistication is determined. Inserting the expression of $D(\tilde{p}; \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i)$ in (7) into the objective function $E_i\left(-e^{-\gamma\left[(\tilde{v}-\tilde{p})D_i - C\left(\tau_{\eta_i}\right)\right]}\middle|\, \tilde{p}, \tilde{s}_i, \tilde{z}_i\right)$ in (6), we can compute the indirect utility function of investor i, $E_i\left(-e^{-\gamma\left[(\tilde{v}-\tilde{p})D(\tilde{p};\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i) - C\left(\tau_{\eta_i}\right)\right]}\middle|\, \tilde{p}, \tilde{s}_i, \tilde{z}_i\right)$. In anticipation of this indirect utility, investor i determines the level of τ_{η_i} to maximize her expected utility before seeing the signal $\tilde{z}_i$. When computing this conditional expected utility, we assume that investors can condition on private fundamental information $\tilde{s}_i$ and the possible realizations of the price $\tilde{p}$, that is, the sophistication level τ_{η_i} is determined by

$$\max_{\tau_{\eta_i}} E_i \left[E_i \left(-e^{-\gamma\left[(\tilde{v}-\tilde{p})D(\tilde{p};\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)-C\left(\tau_{\eta_i}\right)\right]} \middle| \tilde{p}, \tilde{s}_i, \tilde{z}_i \right) \middle| \tilde{p}, \tilde{s}_i \right], \tag{10}$$

where $E_i(\cdot)$ again indicates expectation under investor i's belief that interprets $\tilde{p}$ as a signal of $\tilde{s}_{p,i}$ in predicting $\tilde{v}$.

The assumption of conditioning on $\tilde{p}$ in (10) can be justified in two ways. First, REE makes sense only if implemented in demand functions, so we should consider strategies that are in the form of demand functions that condition on prices (see, for example, Vives, 2014). That is, when submitting her strategies, an investor should think through the effect of conditioning on different prices even without actually seeing them. Second, accessing to the prevailing price $\tilde{p}$ is a parsimonious way of capturing the notion of studying market data in reality: seeing the signal $\tilde{z}_i$ refers economically to studying the price data, and in our one-period model, the only price available is the prevailing price $\tilde{p}$.

The timeline of our economy is as follows:

1. Investors receive their private fundamental information $\tilde{s}_i$.

2. Investors choose simultaneously a sophistication level τ_{η_i} and a demand function $D(\tilde{p}; \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i)$. When choosing τ_{η_i}, investor i conditions on private information $\tilde{s}_i$ and the *possible* realizations of the price $\tilde{p}$.

3. The signal $\tilde{z}_i$ is realized according to the chosen sophistication level τ_{η_i}, the market clears according to the chosen demand function $D(\tilde{p}; \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i)$, and the price $\tilde{p}$ is realized.

4. Asset payoff $\tilde{v}$ is realized, and investors get paid and consume.

Definition 1 *An overall equilibrium is defined by the following two subequilibria:*
(a) Financial market equilibrium, which is characterized by a price function $p(\tilde{v}, \tilde{u})$ *and demand functions* $D(\tilde{p}; \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i)$*, such that: (a1)* $D(\tilde{p}; \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i)$ *is given by (7), which maximizes investors' conditional expected utilities given their beliefs; (a2) the market clears almost surely, i.e., equation (8) holds; and (a3) investors' beliefs are given by (2), (3), and*

(5), where $\tilde{s}_p$ in (2) is implied by the price function $p(\tilde{v}, \tilde{u})$ and where the sophistication levels $(\tau_{\eta_i})_{i\in[0,1]}$ are determined by the sophistication level equilibrium.
(b) Sophistication level equilibrium, which is characterized by sophistication levels $(\tau_{\eta_i})_{i\in[0,1]}$, such that τ_{η_i} solves (10), where investors' beliefs are given by (2), (3), and (5), with $\tilde{s}_p$ in (2) being determined by the price function $p(\tilde{v}, \tilde{u})$ in the financial market equilibrium.

5 Financial Market Equilibrium

5.1 Equilibrium Construction

We consider a linear financial market equilibrium in which the price function takes the following form:

$$\tilde{p} = a_v \tilde{v} + a_u \tilde{u}, \tag{11}$$

where the coefficients a's are endogenous.

By equation (11), the price $\tilde{p}$ is equivalent to the following signal in predicting the asset fundamental $\tilde{v}$:

$$\tilde{s}_p = \tilde{v} + \alpha \tilde{u} \text{ with } \alpha \equiv \frac{a_u}{a_v}, \tag{12}$$

which would be the best signal that a fully sophisticated investor can achieve. However, as we mentioned in Section 3, investor i cannot fully understand the price and she can only extract limited information from the price to the extent that she reads a coarser signal as follows:

$$\tilde{s}_{p,i} = \tilde{s}_p + \tilde{x}_i = (\tilde{v} + \alpha \tilde{u}) + (\tilde{u} + \tilde{e}_i) = \tilde{v} + (\alpha + 1)\, \tilde{u} + \tilde{e}_i, \tag{13}$$

where the second equality follows from equations (2) and (5). In other words, our investors add noise to the best signal that a fully sophisticated trader could obtain; that is, it adds noise in the inference process.

Using Bayes' rule, we can compute

$$E(\tilde{v}|\tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i) = \frac{\tau_\varepsilon \tilde{s}_i + \frac{\tau_e\tau_u + \tau_e\tau_{\eta_i} + \tau_u\tau_{\eta_i}}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_{\eta_i}} \tilde{s}_{p,i} - \frac{\tau_{\eta_i}(\tau_u + \tau_e + \alpha\tau_e)}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_{\eta_i}} \tilde{z}_i}{\tau_v + \tau_\varepsilon + \frac{\tau_e\tau_u + \tau_e\tau_{\eta_i} + \tau_u\tau_{\eta_i}}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_{\eta_i}}}, \tag{14}$$

$$Var\left(\tilde{v}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)=\frac{1}{\tau_v+\tau_\varepsilon+\frac{\tau_e\tau_u+\tau_e\tau_{\eta_i}+\tau_u\tau_{\eta_i}}{\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_{\eta_i}}}. \tag{15}$$

Inserting these two expressions into (7), we can compute the expression of $D\left(\tilde{p};\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)$, which is in turn inserted into (8), yielding the following equilibrium price:

$$\tilde{p}=\frac{\tau_\varepsilon+\int_0^1\frac{\tau_e\tau_u+\tau_e\tau_{\eta_i}+\tau_u\tau_{\eta_i}}{\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_{\eta_i}}di}{\tau_v+\tau_\varepsilon+\int_0^1\frac{\tau_e\tau_u+\tau_e\tau_{\eta_i}+\tau_u\tau_{\eta_i}}{\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_{\eta_i}}di}\tilde{v}+\frac{(1+\alpha)\int_0^1\frac{\tau_e\tau_u+\tau_e\tau_{\eta_i}+\tau_u\tau_{\eta_i}}{\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_{\eta_i}}di-\int_0^1\frac{\tau_{\eta_i}(\tau_u+\tau_e+\alpha\tau_e)}{\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_{\eta_i}}di}{\tau_v+\tau_\varepsilon+\int_0^1\frac{\tau_e\tau_u+\tau_e\tau_{\eta_i}+\tau_u\tau_{\eta_i}}{\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_{\eta_i}}di}\tilde{u}. \tag{16}$$

Comparing with (11), we thus have

$$\alpha=\frac{(1+\alpha)\int_0^1\frac{\tau_e\tau_u+\tau_e\tau_{\eta_i}+\tau_u\tau_{\eta_i}}{\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_{\eta_i}}di-\int_0^1\frac{\tau_{\eta_i}(\tau_u+\tau_e+\alpha\tau_e)}{\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_{\eta_i}}di}{\tau_\varepsilon+\int_0^1\frac{\tau_e\tau_u+\tau_e\tau_{\eta_i}+\tau_u\tau_{\eta_i}}{\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_{\eta_i}}di}, \tag{17}$$

which determines the single unknown α.

Analyzing equation (17), we have the following theorem that characterizes the financial market equilibrium.

Theorem 1 *There exists a linear equilibrium price function*

$$\tilde{p}=a_v\tilde{v}+a_u\tilde{u},$$

where

$$a_v=\frac{\tau_\varepsilon+\int_0^1\frac{\tau_e\tau_u+\tau_e\tau_{\eta_i}+\tau_u\tau_{\eta_i}}{\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_{\eta_i}}di}{\tau_v+\tau_\varepsilon+\int_0^1\frac{\tau_e\tau_u+\tau_e\tau_{\eta_i}+\tau_u\tau_{\eta_i}}{\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_{\eta_i}}di}\text{ and }a_u=\alpha a_v,$$

with $\alpha>0$ being determined by

$$\alpha\tau_\varepsilon=\int_0^1\frac{\tau_e\left(\tau_u-\alpha\tau_{\eta_i}\right)}{\tau_u+\tau_e\left(\alpha+1\right)^2+\alpha^2\tau_{\eta_i}}di. \tag{18}$$

Proof. See the appendix. ■

In Section 8, we will show that, in the overall equilibrium, all investors will endogenously choose the same sophistication level (i.e., $\tau_{\eta_i}=\tau_\eta$, $\forall i\in[0,1]$), for any smooth, increasing, and weakly convex cost function $C\left(\tau_{\eta_i}\right)$ of achieving sophistication. Under this condition, the financial market equilibrium can be further characterized as follows.

Theorem 2 *When investors have the same sophistication level (i.e., $\tau_{\eta_i}=\tau_\eta$, $\forall i\in[0,1]$), there exists a unique linear equilibrium price function,*

$$\tilde{p}=a_v\tilde{v}+a_u\tilde{u},$$

where

$$a_v = \frac{\tau_\varepsilon + \frac{\tau_e\tau_u + \tau_e\tau_\eta + \tau_u\tau_\eta}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_\eta}}{\tau_v + \tau_\varepsilon + \frac{\tau_e\tau_u + \tau_e\tau_\eta + \tau_u\tau_\eta}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_\eta}} \text{ and } a_u = \alpha a_v,$$

and where $\alpha \in \left(0, \frac{\tau_e\tau_u}{\tau_e\tau_\varepsilon + \tau_e\tau_\eta + \tau_u\tau_\varepsilon}\right)$ *is uniquely determined by the positive real root of the following cubic equation:*

$$\left(\tau_e\tau_\varepsilon + \tau_\varepsilon\tau_\eta\right)\alpha^3 + 2\tau_e\tau_\varepsilon\alpha^2 + \left(\tau_e\tau_\varepsilon + \tau_e\tau_\eta + \tau_u\tau_\varepsilon\right)\alpha - \tau_e\tau_u = 0. \tag{19}$$

Proof. See the appendix. ■

As discussed by Guesnerie (1992, p. 1254), there are broadly two ways to justify the standard REE: the "eductive" justification that relies on the understanding of the logic of the situation faced by economic agents and that is associated with mental activity of agents aiming at "forecasting the forecasts of others;" and the "evolutive" justification that emphasizes the learning possibilities offered by the repetition of the situation and that is associated with the convergence of several versions of learning processes. In our equilibrium investors could be as sophisticated as the usual REE agents (and get the value of α by one of the two methods) but each investor makes a "processing error" in interpreting price data and understands that she makes a mistake and that the other investors also make mistakes. Alternatively, and equivalently, investors have a "sentiment shock" when interpreting prices, and this shock has a common and an idiosyncratic component, and understand that other investors also have sentiment shocks. The following subsection links our equilibrium to the traditional REE more explicitly in a polar case of our economy.

5.2 Polar Cases: REE and Walrasian Economies

When $\tau_\eta = \infty$, investors are fully sophisticated and extract the best signal from the price, so that the economy degenerates to a full REE setup. When $\tau_x = \tau_\eta = 0$, investors completely ignore the price information, and the economy becomes a traditional Walrasian economy. In both settings, we have $\alpha = 0$, although the price functions are different. This result also connects our model to the cursed-expectations equilibrium (CEE) in Eyster, Rabin, and Vayanos (2015). Specifically, the case of $\tau_\eta = \infty$ in our model corresponds to the fully rational case in CEE, while the case of $\tau_x = \tau_\eta = 0$ corresponds to the fully cursed case in CEE. That is, parameter τ_η in our economy conceptually "corresponds" (in the sense of having a parallel) to the degree of cursedness in CEE.

Proposition 1 *When $\tau_e \in (0,\infty), \tau_u \in (0,\infty)$, and $\tau_\eta = \infty$, the price function is*

$$\tilde{p}^{REE} = \tilde{v}.$$

When $\tau_u \in (0,\infty)$ and $\tau_e = \tau_\eta = 0$, or when $\tau_e \in (0,\infty)$ and $\tau_u = \tau_\eta = 0$, the price function is

$$\tilde{p}^{Walrasian} = \frac{\tau_\varepsilon}{\tau_v + \tau_\varepsilon}\tilde{v}.$$

Proof. See the appendix. ■

5.3 Investor Sophistication and Price Informativeness

In the end of this section, we establish a complementarity result. When investors become more sophisticated (i.e., τ_η is higher), the true price signal becomes more precise as well (i.e., α becomes lower so that $\tilde{s}_p$ is a more precise signal in predicting $\tilde{v}$ in (12)). Intuitively, when τ_η is large, investors know well the true price signal $\tilde{s}_p$, and thus their trading brings less noise $\tilde{u}$ into the price. This complementarity result has important implications for the determination of sophistication level in Section 8.

Proposition 2 *When investors become more sophisticated, the price $\tilde{p}$ conveys more precise information about the asset fundamental $\tilde{v}$. That is, $\frac{\partial \alpha}{\partial \tau_\eta} < 0$.*

Proof. See the appendix. ■

6 Trading Volume and Investor Disagreement

We now examine how investor sophistication affects trading volume and disagreement by conducting comparative static analysis for these variables with respect to parameter τ_η. In a full equilibrium setting, an increase in τ_η corresponds to a decrease in some parameter that governs the cost function $C\left(\tau_{\eta_i}\right)$, which will be explored later in Section 8.

6.1 Characterizations of Volume and Disagreement

Suppose that investors have the same sophistication level, i.e., $\tau_{\eta_i} = \tau_\eta$, $\forall i \in [0,1]$. By equation (15), all investors face the same risk level when trading the risky asset, i.e.,

$$Risk \equiv Var\left(\tilde{v}|\tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i\right) = \frac{1}{\tau_v + \tau_\varepsilon + \frac{\tau_e \tau_u + \tau_e \tau_\eta + \tau_u \tau_\eta}{\tau_u + \tau_e (\alpha+1)^2 + \alpha^2 \tau_\eta}}. \tag{20}$$

Then, by the demand function (7) and the market clearing condition (8), the equilibrium price is equal to the average expectation of investors,

$$\tilde{p} = \int_0^1 E\left(\tilde{v}|\, \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i\right) di \equiv \bar{E}\left(\tilde{v}\right), \tag{21}$$

where $\bar{E}$ indicates the average expectation operator.

To focus on the volume generated solely by different costly price interpretations, we assume that investors start with a zero initial position of risky assets. Therefore, the trading volume of investor i is

$$\left|D\left(\tilde{p}; \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i\right)\right| = \left|\frac{E\left(\tilde{v}|\, \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i\right) - \tilde{p}}{\gamma Var\left(\tilde{v}|\, \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i\right)}\right| = \frac{\left|E\left(\tilde{v}|\, \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i\right) - \bar{E}\left(\tilde{v}\right)\right|}{\gamma Var\left(\tilde{v}|\, \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i\right)}. \tag{22}$$

The total trading volume is

$$Volume \equiv \int_0^1 \left|D\left(\tilde{p}; \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i\right)\right| di = \frac{\int_0^1 \left|E\left(\tilde{v}|\, \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i\right) - \bar{E}\left(\tilde{v}\right)\right| di}{\gamma Var\left(\tilde{v}|\, \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i\right)}, \tag{23}$$

where the last equality uses the fact that $Var\left(\tilde{v}|\, \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i\right)$ is independent of i given $\tau_{\eta_i} = \tau_\eta$, $\forall i \in [0,1]$.

By (14), $E\left(\tilde{v}|\, \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i\right) - \bar{E}\left(\tilde{v}\right)$ is normally distributed with mean zero, and thus,

$$\int_0^1 \left|E\left(\tilde{v}|\, \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i\right) - \bar{E}\left(\tilde{v}\right)\right| di = \sqrt{\frac{2}{\pi}} \times Disagreement, \tag{24}$$

where we define

$$Disagreement \equiv \sqrt{Var\left(E\left(\tilde{v}|\, \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i\right) - \bar{E}\left(\tilde{v}\right)\right)}, \tag{25}$$

which is the disagreement across investors' expectations about the fundamental $\tilde{v}$.

Using equations (20), (23), and (24), we have

$$Volume = \sqrt{\frac{2}{\pi}} \frac{Disagreement}{\gamma \times Risk}. \tag{26}$$

The total trading volume is therefore jointly determined by three factors: investors' different expectations about the asset fundamental $\tilde{v}$, investors' risk aversion coefficient γ, and the risk faced by investors in trading the assets. When investors disagree more about the future fundamental $\tilde{v}$, they trade more and so the total trading volume is higher. When investors are less risk averse and when they perceive less risk in trading the assets, they also trade more aggressively, leading to a higher total trading volume.

Now we compute the expressions of $Disagreement$ and $Volume$. By equation (14), we can compute

$$E\left(\tilde{v}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)-\bar{E}\left(\tilde{v}\right)=\frac{\tau_\varepsilon\tilde{\varepsilon}_i+\frac{\tau_e(\tau_u-\alpha\tau_\eta)}{\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_\eta}\tilde{e}_i-\frac{\tau_\eta(\tau_u+\tau_e+\alpha\tau_e)}{\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_\eta}\tilde{\eta}_i,}{\tau_v+\tau_\varepsilon+\frac{\tau_e\tau_u+\tau_e\tau_\eta+\tau_u\tau_\eta}{\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_\eta}}, \tag{27}$$

and thus by (25),

$$Disagreement=\frac{\sqrt{\tau_\varepsilon+\left(\frac{\tau_u-\alpha\tau_\eta}{\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_\eta}\right)^2\tau_e+\left(\frac{\tau_u+\tau_e+\alpha\tau_e}{\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_\eta}\right)^2\tau_\eta}}{\tau_v+\tau_\varepsilon+\frac{\tau_e\tau_u+\tau_e\tau_\eta+\tau_u\tau_\eta}{\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_\eta}}. \tag{28}$$

Thus, investors' disagreement comes from three sources: heterogeneous errors $\tilde{\varepsilon}_i$ in their private fundamental information $\tilde{s}_i$, heterogeneous errors $\tilde{e}_i$ in their prior price interpretation $\tilde{s}_{p,i}$, and heterogeneous errors $\tilde{\eta}_i$ generated from the process of studying market data.

Using equations (20), (26), and (28), we have

$$Volume=\frac{1}{\gamma}\sqrt{\frac{2}{\pi}}\sqrt{\tau_\varepsilon+\left(\frac{\tau_u-\alpha\tau_\eta}{\tau_u+\tau_e\left(\alpha+1\right)^2+\alpha^2\tau_\eta}\right)^2\tau_e+\left(\frac{\tau_u+\tau_e+\alpha\tau_e}{\tau_u+\tau_e\left(\alpha+1\right)^2+\alpha^2\tau_\eta}\right)^2\tau_\eta}. \tag{29}$$

Remark 1 *The assumption that investors start with no risky assets does not affect our result. Suppose instead that investor i is initially endowed with $\tilde{y}_i$ shares of risky asset, where $\tilde{y}_i\sim N\left(0,\sigma_y^2\right)$ is independently and identically distributed across investors. Our baseline model corresponds to a degenerate case of $\sigma_y=0$. In the extended setting, we can compute that the total trading volume is given by*

$$Volume=\int_0^1\left|D\left(\tilde{p};\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)-\tilde{y}_i\right|di=\sqrt{\frac{2}{\pi}}\frac{Disagreement}{\gamma\times Risk}+\sqrt{\frac{2}{\pi}}\sigma_y.$$

This expression differs from equation (26) only by a constant $\sqrt{\frac{2}{\pi}}\sigma_y$ that captures the volume generated by the endowment heterogeneity.

6.2 Investor Sophistication, Volume, and Disagreement

We deliver two sets of results. The first set concerns the behavior of $Volume$, $Disagreement$, and $Risk$ as $\tau_\eta\to\infty$. These results are particularly interesting, because as $\tau_\eta\to\infty$, the economy converges to the fully REE setting (see Proposition 1). The second set of results is about how $Volume$, $Disagreement$, and $Risk$ change with τ_η in general.

6.2.1 The Limiting Economy with $\tau_\eta \to \infty$

Suppose $\tau_\eta \to \infty$. Both $Disagreement$ and $Risk$ converge to 0. This is because by Proposition 1, the price approaches to be fully revealing, and thus investors face almost no trading risk and agree on the valuation. However, trading volume diverges to ∞, because the perceived risk shrinks at a higher order than the perceived risk (i.e., τ_η versus $\sqrt{\tau_\eta}$).

In addition, the divergence of $Volume$ comes from investors' price data analysis, that is, the term $\left(\frac{\tau_u+\tau_e+\alpha\tau_e}{\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_\eta}\right)^2\tau_\eta$ in (29). Formally, by equation (29), the trading volume comes from three sources as follows:

$$Volume^2 \propto \underbrace{\tau_\varepsilon}_{\text{diverse fundamental information}} + \underbrace{\left(\frac{\tau_u-\alpha\tau_\eta}{\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_\eta}\right)^2\tau_e}_{\text{diverse prior of price information}} + \underbrace{\left(\frac{\tau_u+\tau_e+\alpha\tau_e}{\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_\eta}\right)^2\tau_\eta}_{\text{diverse noise in studying market data}}. \quad (30)$$

As $\tau_\eta \to \infty$, only the third term in the above expression diverges to ∞, while the first two terms are bounded.

These results seem to describe well the recent high frequency trading in financial markets. As more traders have more sophisticated trading algorithms (i.e., $\tau_\eta \to \infty$), they tend to analyze data more heavily and trade more heavily (i.e., $Volume \to \infty$), although their beliefs may not differ that much (i.e., $Disagreement \to 0$).

Proposition 3 *When investors approach to be fully rational,*
(a) asset prices approach to be fully revealing;
(b) disagreement and perceived risk vanish toward zero; and
(c) trading volume diverges to infinity, which is driven by investors studying market data.
That is, $\lim_{\tau_\eta\to\infty}\tilde{p}=\tilde{v}$ *almost surely,* $\lim_{\tau_\eta\to\infty} Disagreement = \lim_{\tau_\eta\to\infty} Risk = 0$*, and* $\lim_{\tau_\eta\to\infty} Volume = \infty$ *(with only the third term in (30) being divergent).*

Proof. See the appendix. ■

6.2.2 Comparative Statics with Respect to τ_η

Now suppose that τ_η gradually increases from 0 to ∞. As τ_η becomes higher, investors face lower risk in trading assets—i.e., $\frac{\partial Var(\tilde{v}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)}{\partial \tau_\eta} < 0$—because they glean more information from the price data for two reasons. First, a higher sophistication level means that they study market data more intensively and can directly get more information from the price. Second, by Proposition 2, when all investors study data more intensively, the price itself becomes a more informative signal (i.e., α decreases), and thus each investor can infer more information from the price data.

Investor sophistication affects disagreement in two opposite ways. First, in our setting, investors interpret the price differently, and a higher τ_η means that investors' expectations rely more on their diverse information extracted from the price, thereby leading to a larger belief heterogeneity. Second, a higher τ_η implies that the price conveys more precise information about the asset fundamental (see Proposition 2), which tends to make investors' belief converge. By Proposition 3, it must be the case that the second effect dominates for sufficiently large τ_η so that *Disagreement* decreases with τ_η when τ_η is large. Nonetheless, when τ_η is small, the first positive effect on disagreement can dominate too. This possibility will arise when investors' private fundamental information is very coarse (i.e., τ_ε is small). Intuitively, starting from a small τ_ε, before accessing to market data, investors' beliefs are close to the prior and thus do not differ much from each other; after they see the price and interpret it differently, their opinions start to diverge. Taken together, when τ_ε is small, *Disagreement* is hump-shaped in τ_η. When τ_ε is large, *Disagreement* monotonically decreases with τ_η.

The total trading volume increases with investor sophistication. That is, $\frac{\partial Volume}{\partial \tau_\eta} > 0$. Note that by (26), *Volume* increases with *Disagreement* and decreases with *Risk*. Given that *Risk* decreases with τ_η, *Volume* tends to increase with τ_η through the risk channel. When *Disagreement* increases with τ_η—which is true when both τ_ε and τ_η are small—sophistication τ_η increases *Volume* further through the disagreement channel. When *Disagreement* decreases with τ_η, it turns out that the risk channel dominates so that the overall effect of increasing τ_η is to increase *Volume*.

Proposition 4 *When investor sophistication level τ_η increases,*
(a) trading volume increases and perceived risk decreases (i.e., $\frac{\partial Volume}{\partial \tau_\eta} > 0$ and $\frac{\partial Risk}{\partial \tau_\eta} < 0$);
(b) investor disagreement is hump-shaped when investors have coarse private fundamental information, and it monotonically decreases when investors have precise private fundamental information (i.e., for small values of τ_ε, $\frac{\partial Disagreement}{\partial \tau_\eta} < 0$ if and only if τ_η is sufficiently large; for large values of τ_ε, $\frac{\partial Disagreement}{\partial \tau_\eta} < 0$ for all values of τ_η).

Proof. See the appendix. ■

We use Figure 1 to graphically illustrate Proposition 4. In the top three panels, we have set $\tau_\varepsilon = 0.05$, while in the bottom three panels, we have set $\tau_\varepsilon = 1$. All the other parameters are set at 1, i.e., $\tau_v = \tau_e = \tau_u = \gamma = 1$. Consistent with Part (a) of Proposition 4, $Volume$ increases with τ_η and $Risk$ decreases with τ_η, independent of the value of τ_ε. Also consistent with Part (b) of Proposition 4, $Disagreement$ first increases and then decreases with τ_η in the top-middle panel where τ_ε is small, and $Disagreement$ monotonically decreases with τ_η in the bottom-middle panel where τ_ε is relatively large.

[INSERT FIGURE 1 HERE]

7 Investor Sophistication and Welfare

7.1 Welfare Characterization

In this section, we flesh out the normative implications of investor sophistication. We define investors' welfare as follows:

$$Welfare \equiv -\frac{1}{\gamma} \log \left[E \left(e^{-\gamma[(\tilde{v}-\tilde{p})D(\tilde{p};\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)-C(\tau_\eta)]} \right) \right].$$

That is, welfare is measured as the ex-ante equilibrium expected utility (certainty equivalent) of investors, where the expectation is taken with respect to the *objective* distribution of all underlying random variables. This treatment is standard in the behavioral economics literature (e.g., Sandroni and Squintani, 2007; Gennaioli, Shleifer, and Vishny, 2012; Simsek, 2013; and Spinnewijn, 2015). The idea is that in the presence of belief disagreements, investors' perceived welfare is illusory because subjective beliefs misspecify the economy, and thus when conducting normative analysis, one should instead consider actual welfare that is evaluated under the objective distribution.

After some computation, we can express $Welfare$ as follows:

$$Welfare = \frac{1}{2\gamma} \log \left(1 - \gamma^2 \sigma_D^2 \sigma_{\tilde{v}-\tilde{p}}^2\right) - C\left(\tau_\eta\right), \tag{31}$$

where

$$\sigma_D \equiv \sqrt{Var\left(D\left(\tilde{p}; \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i\right)\right)} \text{ and } \sigma_{\tilde{v}-\tilde{p}} \equiv \sqrt{Var\left(\tilde{v} - \tilde{p}\right)} \tag{32}$$

are the volatility of investors' trading positions and the volatility of asset returns $\tilde{v} - \tilde{p}$, respectively. Thus, $Welfare$ decreases with trading volatility σ_D, return volatility $\sigma_{\tilde{v}-\tilde{p}}$, and the cost $C\left(\tau_\eta^*\right)$ of studying market data. It is straightforward that an increase in the exogenous sophistication cost $C\left(\tau_\eta\right)$ directly lowers $Welfare$ in (31). So, our discussion will focus on the welfare effect of sophistication through the two endogenous variables, σ_D and $\sigma_{\tilde{v}-\tilde{p}}$.

Using equations (7), (21), and (27), we can express $D\left(\tilde{p}; \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i\right)$ as follows:

$$D\left(\tilde{p}; \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i\right) = \frac{1}{\gamma}\left[\tau_\varepsilon \tilde{\varepsilon}_i + \frac{\tau_e\left(\tau_u - \alpha\tau_\eta\right)}{\tau_u + \tau_e\left(\alpha+1\right)^2 + \alpha^2\tau_\eta}\tilde{e}_i - \frac{\tau_\eta\left(\tau_u + \tau_e + \alpha\tau_e\right)}{\tau_u + \tau_e\left(\alpha+1\right)^2 + \alpha^2\tau_\eta}\tilde{\eta}_i\right].$$

Thus, the equilibrium holding of each investor is simply a linear combination of the error terms ($\tilde{\varepsilon}_i$, $\tilde{e}_i$, and $\tilde{\eta}_i$) in their signals ($\tilde{s}_i$, $\tilde{s}_{p,i}$, and $\tilde{z}_i$), which is a form of "winner's curse" of trading in financial markets, as explained by Biais, Bossaerts, and Spatt (2010). Intuitively, in our setup, investors do not hedge their background risk and they trade for speculation purposes. Their speculative positions are proportional to the difference between their forecast of the fundamental and the asset price. After aggregation, the price averages out the idiosyncratic errors in investors' private information and as a result, investors end up holding positions related only to the noises in their information.

This winner's curse implies that speculative trading hurts investors' welfare. This observation is intuitively reflected by the expression $\gamma^2\sigma_D^2\sigma_{\tilde{v}-\tilde{p}}^2$ in equation (31), which negatively affects $Welfare$. First, variable σ_D measures the size of speculative trading; the more investors speculate, the more they lose. This result is consistent with the empirical evidence documented in the finance literature (e.g., Odean, 1999; Barber and Odean, 2000). Second, variable $\sigma_{\tilde{v}-\tilde{p}}$ is a measure for the wealth loss per unit trading. That is, a higher return volatility $\sigma_{\tilde{v}-\tilde{p}}$ means that it is more likely for the fundamental $\tilde{v}$ to deviate from the prevailing price $\tilde{p}$, and thus the winner's curse harms investors more. Finally, risk aversion γ translates the wealth loss into welfare loss, since a more risk averse investor is more concerned about wealth fluctuations. Taken together, $\gamma^2\sigma_D^2\sigma_{\tilde{v}-\tilde{p}}^2$ captures the negative welfare implications of the winner's curse.

The welfare loss $\gamma^2\sigma_D^2\sigma_{\tilde{v}-\tilde{p}}^2$ is also related to the idea of "speculative variance" studied by Simsek (2013). In Simsek's (2013) setting, investors trade for two purposes, risk-sharing and speculation. Speculative variance refers to the part of portfolio risk that is driven by speculation based on heterogeneous beliefs. Speculative variance tends to harm welfare and it is greater when the assets feature greater belief disagreement, both features consistent with our model. Specifically, our investors have no background risks and trade only for speculation. As a result, trading in our setting has no risk-sharing benefits, which is therefore always "excessive" from a welfare perspective. In addition, similar to Simsek (2013), the welfare loss $\gamma^2\sigma_D^2\sigma_{\tilde{v}-\tilde{p}}^2$ in our setting is also greater when investors exhibit greater disagreement about the asset fundamental (see equation (A13) in the appendix).

Remark 2 *As in Remark 1, we can consider an extension in which investors are initially endowed with $\tilde{y}_i$ shares of risky asset, where $\tilde{y}_i \sim N\left(0,\sigma_y^2\right)$ is independently and identically distributed across investors. In this extended setting, investors trade both for speculation and for hedging. We can compute*

$$Welfare = \frac{1}{2\gamma}\log\left[1 - \underbrace{\gamma^2\sigma_D^2\sigma_{\tilde{v}-\tilde{p}}^2}_{winner's\ curse} + \underbrace{\gamma^4\sigma_y^2\sigma_D^2\left(\sigma_p^2\sigma_{\tilde{v}-\tilde{p}}^2 - \sigma_{\tilde{v}-\tilde{p},\tilde{p}}^2\right)}_{risk\text{-}sharing\ benefit} - \underbrace{\gamma^2\sigma_y^2\sigma_p^2}_{wealth\ fluctuation}\right] - C\left(\tau_\eta\right),$$

where $\sigma_{\tilde{v}-\tilde{p},\tilde{p}} \equiv Cov\left(\tilde{v}-\tilde{p},\tilde{p}\right), \sigma_p^2 = Var\left(\tilde{p}\right), \sigma_{\tilde{v}-\tilde{p}}^2 \equiv Var\left(\tilde{v}-\tilde{p}\right)$ and $\sigma_D^2 \equiv Var\left(D\left(\tilde{p};\tilde{y}_i,\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)\right)$. Comparing the above equation with equation (31), we find that the welfare in the extended economy has two additional terms in addition to the "winner's curse" caused by excessive speculation: (1) $\gamma^4\sigma_y^2\sigma_D^2\left(\sigma_p^2\sigma_{\tilde{v}-\tilde{p}}^2 - \sigma_{\tilde{v}-\tilde{p},\tilde{p}}^2\right)$, which captures the welfare gain from risk sharing; and (2) $\gamma^2\sigma_y^2\sigma_p^2$, which captures the welfare loss from wealth fluctuations (i.e., with an endowment of $\tilde{y}_i$ shares of risky asset, investor i's initial wealth is $\tilde{p}\tilde{y}_i$, which has a variance of $\sigma_y^2\sigma_p^2$ that hurts the risk-averse investor). We can show that our welfare results continue to hold when γ or σ_y are small.

7.2 Welfare Implications of Investor Sophistication

Suppose that investor sophistication τ_η increases and we now examine how trading volatility σ_D, return volatility $\sigma_{\tilde{v}-\tilde{p}}$, and $Welfare$ respond. For $Welfare$, we will assume $C(\tau_\eta) = 0$ in equation (31), since this term is exogenous and we want to focus on how investor sophistication affects welfare through its effect on the endogenous terms σ_D and $\sigma_{\tilde{v}-\tilde{p}}$.

Note that by the definitions of $Volume$ and σ_D in (23) and (32), we have

$$Volume = \sqrt{\frac{2}{\pi}}\sigma_D. \tag{33}$$

Thus, trading volatility σ_D increases with τ_η, because $Volume$ increases with τ_η by Proposition 4. We can also show that return volatility $\sigma_{\tilde{v}-\tilde{p}}$ tends to decrease with τ_η. Intuitively, by Proposition 2, price informativeness increases with τ_η, which implies that sophistication makes the price $\tilde{p}$ closer to the fundamental $\tilde{v}$, driving down the equilibrium return volatility.

Given that σ_D and $\sigma_{\tilde{v}-\tilde{p}}$ respond differently to τ_η, the overall welfare effect of τ_η can be ambiguous. Note that the return-volatility channel is rooted in the informativeness effect of sophistication. When investors have very precise private fundamental information (i.e., when τ_ε is high), we expect that the price is very informative and thus the positive return-volatility channel is particularly strong. Indeed, we can show that when τ_ε is high, this is the case and so $Welfare$ monotonically increases with τ_η. In contrast, when τ_ε is low, the negative excessive-trading channel can dominate, so that $Welfare$ exhibits a U-shape with respect to τ_η.

Proposition 5 *Suppose $C(\tau_\eta) = 0$. When investor sophistication level τ_η increases,*
(a) trading volatility increases (i.e., $\frac{\partial \sigma_D}{\partial \tau_\eta} > 0$);
(b) return volatility decreases if investors' fundamental information is sufficiently coarse or sufficiently precise (i.e., $\frac{\partial \sigma_{\tilde{v}-\tilde{p}}}{\partial \tau_\eta} < 0$ if τ_ε is sufficiently small or sufficiently large);
(c) welfare is U-shaped when investors have coarse private fundamental information, and it monotonically increases when investors have precise private fundamental information (i.e., for small values of τ_ε, $\frac{\partial Welfare}{\partial \tau_\eta} > 0$ if and only if τ_η is sufficiently large; for large values of τ_ε, $\frac{\partial Welfare}{\partial \tau_\eta} > 0$ for all values of τ_η).

Proof. See the appendix. ∎

Figure 2 graphically illustrates Proposition 5. As in Figure 1, in the top three panels, we have set $\tau_\varepsilon = 0.05$, while in the bottom three panels, we have set $\tau_\varepsilon = 1$. All the other parameters are set at 1—i.e., $\tau_v = \tau_e = \tau_u = \gamma = 1$. We have also assumed $C(\tau_\eta) = 0$. Consistent with Proposition 5, σ_D increases with τ_η and $\sigma_{\tilde{v}-\tilde{p}}$ decreases with τ_η, independent of the value of τ_ε. Also consistent with Part (c) of Proposition 5, $Welfare$ first decreases and then increases with τ_η in the top-left panel where τ_ε is small, while $Welfare$ monotonically increases with τ_η in the bottom-left panel where τ_ε is relatively large.

[INSERT FIGURE 2 HERE]

Similar to Proposition 3, we can also establish a proposition for the limiting economy of large τ_η as follows.

Proposition 6 *Suppose $C(\tau_\eta) = 0$. When investors approach to be fully rational, trading volatility goes to infinity, and return volatility and welfare converge to zero. That is,* $\lim_{\tau_\eta \to \infty} \sigma_D = \infty$ *and* $\lim_{\tau_\eta \to \infty} \sigma_{\tilde{v}-\tilde{p}} = \lim_{\tau_\eta \to \infty} Welfare = 0$.

Proof. See the appendix. ■

8 Sophistication Level Equilibrium

8.1 Equilibrium Characterization

We now discuss how investors determine their sophistication levels τ_{η_i} in reading the price data. By studying market data, investor i obtains a private signal $\tilde{z}_i$ with precision τ_{η_i} about the noise $\tilde{x}_i$ in her personalized price signal $\tilde{s}_{p,i}$ (see equation (3)). As we discussed in Section 4, investor i chooses τ_{η_i} to maximize her expected utility before observing $\tilde{z}_i$ but conditional on the possible realizations of the price $\tilde{p}$ and the private fundamental information $\tilde{s}_i$.

Specifically, we need to average out $\tilde{z}_i$ and compute $E_i\left[V(\tilde{p};\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)\,|\tilde{p},\tilde{s}_i\right]$, where

$$V(\tilde{p};\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i) \equiv E_i\left(-e^{-\gamma\left[(\tilde{v}-\tilde{p})D(\tilde{p};\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)-C\left(\tau_{\eta_i}\right)\right]}\middle|\,\tilde{p},\tilde{s}_i,\tilde{z}_i\right)$$

is investor i's indirect value function. We can insert the demand function (7) into the investor's objective function (6) and compute

$$V\left(\tilde{p};\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)=-\exp\left(-\frac{\left[E\left(\tilde{v}\middle|\,\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)-\tilde{p}\right]^2}{2Var\left(\tilde{v}\middle|\,\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)}+\gamma C\left(\tau_{\eta_i}\right)\right). \tag{34}$$

Using the above expression of $V\left(\tilde{p};\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)$, we have

$$E_i\left[V\left(\tilde{p};\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)|\tilde{p},\tilde{s}_i\right]=E_i\left[-\exp\left(-\frac{\left[E\left(\tilde{v}\middle|\,\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)-\tilde{p}\right]^2}{2Var\left(\tilde{v}\middle|\,\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)}+\gamma C\left(\tau_{\eta_i}\right)\right)\middle|\,\tilde{p},\tilde{s}_i\right]. \tag{35}$$

In computing the right-hand-side of (35), investor i will treat $\tilde{p}$ as a constant since her computation is conditional on $\tilde{p}$. In her mind, $E\left(\tilde{v}|\,\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)-\tilde{p}$ is normally distributed with mean and variance given respectively by

$$E_i\left[E\left(\tilde{v}|\,\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)-\tilde{p}|\tilde{p},\tilde{s}_i\right] = E\left(\tilde{v}|\,\tilde{s}_{p,i},\tilde{s}_i\right)-\tilde{p}, \tag{36}$$

$$Var_i\left[E\left(\tilde{v}|\,\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)-\tilde{p}|\tilde{p},\tilde{s}_i\right] = Var\left(\tilde{v}|\tilde{s}_{p,i},\tilde{s}_i\right)-Var\left(\tilde{v}|\,\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right), \tag{37}$$

where the equalities follow from the fact that investor i's beliefs satisfy the Bayes' law given that she is a SEU maximizer. In other words, $\tilde{p}$ in the above moment computations only serves its substitution role (i.e., as in any other commodity demand, a higher price means a higher cost and so the agent will buy fewer commodities), while when we think about the investor inferring information from the price, we always "translate" $\tilde{p}$ into the signal $\tilde{s}_{p,i}$ in terms of predicting $\tilde{v}$ to model this inference process. Taken together, in investor i's mind, she believes that $\left[E\left(\tilde{v}|\,\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)-\tilde{p}\right]^2$ follows a noncentral chi-square distribution in (35).

Using equations (36)–(37) and applying the moment generating function for a noncentral chi-square distribution, we can compute

$$E_i\left[V\left(\tilde{p};\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)|\tilde{p},\tilde{s}_i\right]=-\sqrt{\frac{Var\left(\tilde{v}|\,\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)}{Var\left(\tilde{v}|\tilde{s}_{p,i},\tilde{s}_i\right)}}\exp\left\{-\frac{\left[E\left(\tilde{v}|\,\tilde{s}_{p,i},\tilde{s}_i\right)-\tilde{p}\right]^2}{2Var\left(\tilde{v}|\tilde{s}_{p,i},\tilde{s}_i\right)}+\gamma C\left(\tau_{\eta_i}\right)\right\}.$$

We use B to denote the certainty equivalent of the above conditional expected utility, which refers to the net benefit of studying market data. That is,

$$\begin{aligned} B &\equiv -\frac{1}{\gamma}\log\left(-E_i\left[V\left(\tilde{p};\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)|\tilde{p},\tilde{s}_i\right]\right) \\ &= \frac{1}{2\gamma}\log\frac{1}{Var\left(\tilde{v}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)}-C\left(\tau_{\eta_i}\right)+\frac{\log Var\left(\tilde{v}|\tilde{s}_{p,i},\tilde{s}_i\right)+\frac{\left[E\left(\tilde{v}|\tilde{s}_{p,i},\tilde{s}_i\right)-\tilde{p}\right]^2}{Var\left(\tilde{v}|\tilde{s}_{p,i},\tilde{s}_i\right)}}{2\gamma}. \end{aligned} \tag{38}$$

In (38), the third term $\frac{\log Var\left(\tilde{v}|\tilde{s}_{p,i},\tilde{s}_i\right)+\frac{\left[E\left(\tilde{v}|\tilde{s}_{p,i},\tilde{s}_i\right)-\tilde{p}\right]^2}{Var\left(\tilde{v}|\tilde{s}_{p,i},\tilde{s}_i\right)}}{2\gamma}$ is independent of the choice variable τ_{η_i}, and so we ignore it and only retain the first two terms to represent B. Using equation

(15) to express out $Var(\tilde{v}|\tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i)$ in (38) yields the following expression for the benefit B of studying the price data:

$$B\left(\tau_{\eta_i};\alpha\right) \propto \frac{1}{2\gamma}\log\left[\tau_v + \tau_\varepsilon + \frac{\tau_e\tau_u + \tau_e\tau_{\eta_i} + \tau_u\tau_{\eta_i}}{\tau_u + \tau_e\left(\alpha+1\right)^2 + \alpha^2\tau_{\eta_i}}\right] - C\left(\tau_{\eta_i}\right), \tag{39}$$

where we have explicitly expressed B as a function of τ_{η_i} and α, the two unknowns in the full equilibrium. Thus, each investor i's sophistication level τ_{η_i} is determined by

$$\max_{\tau_{\eta_i}} B\left(\tau_{\eta_i};\alpha\right) \tag{40}$$

where α is determined by equation (18) in Theorem 1. The overall equilibrium is jointly characterized by (40) (for $i \in [0,1]$) and (18), in terms of variables $\left(\tau_{\eta_i}\right)_{i\in[0,1]}$ and α.

Since $\lim_{\tau_{\eta_i}\to\infty} B\left(\tau_{\eta_i};\alpha\right) = -\infty$ for an increasing and weakly convex $C\left(\tau_{\eta_i}\right)$, we know that $B\left(\tau_{\eta_i};\alpha\right)$ has a maximum over the range of $\tau_{\eta_i} \geq 0$. It is also easy to check that $B\left(\tau_{\eta_i};\alpha\right)$ is concave in τ_{η_i}, and thus the maximum is characterized by the first-order condition (FOC) as follows:

$$\begin{cases} \left.\frac{\partial B\left(\tau_{\eta_i};\alpha\right)}{\partial \tau_{\eta_i}}\right|_{\tau_{\eta_i}=0} \leq 0, & \text{if } \tau_{\eta_i} = 0, \\ \frac{\partial B\left(\tau_{\eta_i};\alpha\right)}{\partial \tau_{\eta_i}} = 0, & \text{if } \tau_{\eta_i} > 0. \end{cases} \tag{41}$$

Note that each investor i faces the same problem in determining τ_{η_i}, and thus as we mentioned in Section 5, investors end up with the same choice of τ_{η_i}. That is, in equilibrium, we have $\tau_{\eta_i} = \tau_\eta$ for any $i \in [0,1]$. As a result, the overall equilibrium is characterized by the following two conditions in terms of two unknowns τ_η and α: (a) condition (41), the FOC in investors' sophistication decision problem which determines τ_η for a given α, and (b) equation (19) in Theorem 2, the financial market equilibrium condition which uniquely determines α for a given τ_η. Formally, we have the following theorem characterizing the overall equilibrium.

Theorem 3 *Suppose that $C(\tau_\eta)$ is smooth, increasing, and weakly convex. Then, there exists an overall equilibrium. Let*

$$\phi\left(\tau_\eta\right) \equiv \frac{\partial B\left(\tau_\eta;\alpha\left(\tau_\eta\right)\right)}{\partial \tau_\eta}, \tag{42}$$

where the function $\alpha(\tau_\eta)$ is implicitly determined by equation (19). The equilibrium sophistication level τ_η is determined by the following conditions:

(a) If $\phi(0) \leq 0$, then $\tau_\eta^* = 0$ is an equilibrium sophistication level;

(b) If $\phi\left(\tau_{\eta}^{*}\right) = 0$ for some $\tau_{\eta}^{*} > 0$, then this value of τ_{η}^{*} is an equilibrium sophistication level. The financial market equilibrium is given by Theorem 2 accordingly at the equilibrium sophistication level τ_{η}^{*}.

Proof. See the appendix. ■

8.2 Complementarity and Multiplicity

Theorem 3 establishes the existence of an overall equilibrium. Whether the equilibrium is unique is determined by the shape of $\phi(\tau_{\eta})$. Specifically, if $\phi(\tau_{\eta})$ is downward sloping, then the equilibrium must be unique. In contrast, when $\phi(\tau_{\eta})$ is upward sloping, multiplicity can arise. The complementarity result in Proposition 2 has implications for this possibility of multiplicity, because it determines the shape of $\phi(\tau_{\eta})$.

Formally, by the Chain rule, we have

$$\phi'(\tau_{\eta}) = \underbrace{\frac{\partial^2 B(\tau_{\eta}, \alpha(\tau_{\eta}))}{\partial \alpha \partial \tau_{\eta}}}_{(-)} \times \underbrace{\frac{\partial \alpha}{\partial \tau_{\eta}}}_{\text{complementarity } (-)} + \underbrace{\frac{\partial^2 B(\tau_{\eta}; \alpha(\tau_{\eta}))}{\partial \tau_{\eta}^2}}_{\text{SOC of (40) } (-)}. \tag{43}$$

The second term in equation (43) is simply the second-order condition (SOC) of investors' sophistication determination problem (40), which is always negative given that the objective function $B(\tau_{\eta}; \alpha)$ is globally concave in τ_{η}. We can also show $\frac{\partial^2 B(\tau_{\eta}, \alpha(\tau_{\eta}))}{\partial \alpha \partial \tau_{\eta}} < 0$, i.e., when α increases, the price signal is not very useful and so its marginal value of being more attentive to the price data is low. Then, combining with the complementarity result in Proposition 2 (i.e., $\frac{\partial \alpha}{\partial \tau_{\eta}} < 0$), we have $\frac{\partial^2 B(\tau_{\eta}, \alpha(\tau_{\eta}))}{\partial \alpha \partial \tau_{\eta}} \frac{\partial \alpha}{\partial \tau_{\eta}} > 0$, which counter balances the second negative term in (43). This complementarity result can be so strong that it dominates so that $\phi'(\tau_{\eta})$ can be upward sloping at some region, which admits multiple equilibria.

Proposition 7 *The complementarity effect can dominate so that there can be multiple overall equilibria.*

We prove Proposition 7 using a constructive example. In Figure 3, we choose a linear cost function, $C(\tau_{\eta}) = k\tau_{\eta}$ with $k > 0$. We then plot the function $\phi(\tau_{\eta})$. Similar to Figure 1, in the top panel, we set $\tau_{\varepsilon} = 0.05$, while in the bottom panel, we set $\tau_{\varepsilon} = 1$. All the other parameters in both panels are as follows: $\tau_v = \tau_e = \tau_u = \gamma = 1$ and $k = 0.08$. We find that in the top panel, there exist three equilibrium levels of τ_η^*: $\{0, 0.29, 3.61\}$. Among these three equilibria, the middle one is unstable (i.e., $\phi(\tau_\eta)$ crosses zero from below), while the other two equilibria are stable. In the bottom panel, there exists a unique equilibrium level of $\tau_\eta^* = 2.96$, which is stable.

[INSERT FIGURE 3 HERE]

8.3 Implications of the Cost of Studying Market Data

In this section, we examine the implications of changing the cost of becoming more sophisticated in interpreting market data. Specifically, we continue to use the parametric example in Figure 3 with a linear cost function $C(\tau_\eta) = k\tau_\eta$, but now we allow the cost parameter k to continuously change and plot the equilibrium values of τ_η^* and α^* in Figure 4. For instance, a decrease in k can be interpreted as an advance in computation technology that allows for easier implementation of complex algorithms.

[INSERT FIGURE 4 HERE]

As we explained in Figure 3, for the chosen parameter values, there can be multiple equilibria. When multiplicity happens, we use dashed segments to indicate the unstable equilibrium in Figure 4. We see that as k decreases, τ_η^* increases and α^* decreases as long as investors coordinate on a particular stable equilibrium (say, the one with a larger value of τ_η^*).

It is intuitive that as the cost k of studying market data becomes lower, investors will devote more effort to study the price and become more sophisticated. The multiplicity also suggests that a slight change in k can lead to jumps in τ_η^*. For instance, suppose that investors coordinate on a stable equilibrium with a higher value of τ_η^*. Then, when k is close to 0.1, and when it drops slightly, the equilibrium value of τ_η^* can jump from 0 to 2. This outcome can correspond to a wave of development of algorithmic trading in financial markets, which is caused by technology progress. A natural experiment in this context is the introduction of automated quote dissemination on the New York Stock Exchange in 2003, which is studied by Hendershott, Jones, and Menkveld (2011). This change corresponds to an exogenous decrease in the cost k of processing market data.

Variable α^* negatively measures price informativeness. By Proposition 2, α^* decreases with τ_η^*. Thus, as k decreases, α^* decreases, since τ_η^* increases. Economically, when the cost of studying market data is lower, investors study market data more intensively and glean more precise information from the price, which makes the price more responsive to the fundamental. Again, there can be jumps in the equilibrium value of α^* in response to small changes in the cost k of studying market data.

9 Conclusion

We construct a model to capture the notion that investors have to spend effort to interpret the price data in financial markets. In our model, investors actively infer information from the price but they do not fully understand the price function. Investors can understand better the price function by spending more resources to study market data and become more sophisticated. We still maintain the assumption that investors are individually Bayesian rational—i.e., after reading the price data and form their beliefs, investors hold optimal trading positions according to their own beliefs. In equilibrium, as investors' sophistication level increases, aggregate trading volume increases, and disagreement across investors' expectations about the fundamental can exhibit a hump-shape. When investors approach to be fully sophisticated, the equilibrium approaches to a standard rational-expectations equilibrium, while trading volume diverges to infinity. This divergence is driven by investors actively studying market data. Investor sophistication affects welfare through two offsetting channels: it harms welfare through a winner's curse but improves welfare by reducing equilibrium return volatility. Finally, we endogenize investors' sophistication level using a learning technology and find that studying market data exhibits strategic complementarity that can lead to multiple equilibria.

Appendix: Proofs

Proof of Theorem 1

The expressions of a's follow directly from equation (16). Rearranging equation (17) yields equation (18) in the theorem. When $\alpha = 0$, the left-hand-side (LHS) of (18) is 0, while its right-hand-side (RHS) is positive. When $\alpha \to \infty$, the LHS of (18) approaches ∞, while its RHS is negative. By the intermediate value theorem, there exists a solution of $\alpha > 0$ to equation (18). QED.

Proof of Theorem 2

The expressions of a_v and a_u follow from Theorem 1 and the fact $\tau_{\eta_i} = \tau_\eta$ for $i \in [0, 1]$. Using $\tau_{\eta_i} = \tau_\eta$ and equation (18), we can obtain equation (19) in Theorem 2.

Denote the LHS of (19) by $f(\alpha)$. That is,

$$f(\alpha) \equiv (\tau_e\tau_\varepsilon + \tau_\varepsilon\tau_\eta)\,\alpha^3 + 2\tau_e\tau_\varepsilon\alpha^2 + (\tau_e\tau_\varepsilon + \tau_e\tau_\eta + \tau_u\tau_\varepsilon)\,\alpha - \tau_e\tau_u.$$

We can compute $f(0) = -\tau_e\tau_u < 0$ and $f\left(\frac{\tau_e\tau_u}{\tau_e\tau_\varepsilon+\tau_e\tau_\eta+\tau_u\tau_\varepsilon}\right) > 0$, and thus by the intermediate value theorem, there exists a solution $\alpha \in \left(0, \frac{\tau_e\tau_u}{\tau_e\tau_\varepsilon+\tau_e\tau_\eta+\tau_u\tau_\varepsilon}\right)$ such that $f(\alpha) = 0$. This result establishes the existence of a financial market equilibrium.

We can compute the discriminant of the cubic $f(\alpha)$ as follows:

$$\Delta = -\tau_\varepsilon \left(\begin{array}{c} 4\tau_e^3\tau_\eta^4 + 4\tau_e^4\tau_\eta^3 + 4\tau_e\tau_u^3\tau_\varepsilon^3 + 4\tau_e^3\tau_u\tau_\varepsilon^3 + 4\tau_e^4\tau_u\tau_\varepsilon^2 + 27\tau_e^4\tau_u^2\tau_\varepsilon \\ +12\tau_e^3\tau_\varepsilon\tau_\eta^3 + 4\tau_e^3\tau_\varepsilon^3\tau_\eta + 8\tau_e^4\tau_\varepsilon\tau_\eta^2 + 4\tau_e^4\tau_\varepsilon^2\tau_\eta + 4\tau_u^3\tau_\varepsilon^3\tau_\eta + 8\tau_e^2\tau_u^2\tau_\varepsilon^3 \\ +36\tau_e^3\tau_u^2\tau_\varepsilon^2 + 12\tau_e^3\tau_\varepsilon^2\tau_\eta^2 + 12\tau_e\tau_u^2\tau_\varepsilon^2\tau_\eta^2 + 24\tau_e^2\tau_u\tau_\varepsilon^2\tau_\eta^2 + 27\tau_e^2\tau_u^2\tau_\varepsilon\tau_\eta^2 \\ +48\tau_e^2\tau_u^2\tau_\varepsilon^2\tau_\eta + 36\tau_e^4\tau_u\tau_\varepsilon\tau_\eta + 12\tau_e\tau_u^2\tau_\varepsilon^3\tau_\eta + 12\tau_e^2\tau_u\tau_\varepsilon\tau_\eta^3 + 12\tau_e^2\tau_u\tau_\varepsilon^3\tau_\eta \\ +48\tau_e^3\tau_u\tau_\varepsilon\tau_\eta^2 + 52\tau_e^3\tau_u\tau_\varepsilon^2\tau_\eta + 54\tau_e^3\tau_u^2\tau_\varepsilon\tau_\eta \end{array} \right),$$

which is negative. Thus, there exists a unique real root, which establishes the uniqueness of a financial market equilibrium. QED.

Proof of Proposition 1

Suppose $\tau_e, \tau_u \in (0, \infty)$ and $\tau_\eta \to \infty$. The upper bound $\frac{\tau_e\tau_u}{\tau_e\tau_\varepsilon+\tau_e\tau_\eta+\tau_u\tau_\varepsilon}$ of α in Theorem 2 goes to 0. So, we have $\alpha \to 0$. In addition, $\alpha < \frac{\tau_e\tau_u}{\tau_e\tau_\varepsilon+\tau_e\tau_\eta+\tau_u\tau_\varepsilon} \Rightarrow \alpha^2\tau_\eta < \left(\frac{\tau_e\tau_u}{\tau_e\tau_\varepsilon+\tau_e\tau_\eta+\tau_u\tau_\varepsilon}\right)^2 \tau_\eta \to 0$

as $\tau_\eta \to \infty$, and thus $\alpha^2\tau_\eta \to 0$ as $\tau_\eta \to \infty$. Thus, by the expressions of a_v and a_u in Theorem 2, we have

$$a_v = \frac{\tau_\varepsilon + \frac{\tau_e\tau_u+\tau_e\tau_\eta+\tau_u\tau_\eta}{\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_\eta}}{\tau_v + \tau_\varepsilon + \frac{\tau_e\tau_u+\tau_e\tau_\eta+\tau_u\tau_\eta}{\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_\eta}} \to 1 \text{ and } a_u = a_v\alpha \to 0.$$

Suppose $\tau_u \in (0,\infty)$, $\tau_e \to 0$, and $\tau_\eta \to 0$. Again, $\alpha < \frac{\tau_e\tau_u}{\tau_e\tau_\varepsilon+\tau_e\tau_\eta+\tau_u\tau_\varepsilon} \to 0$. By the expressions of a_v and a_u in Theorem 2, we have

$$a_v = \frac{\tau_\varepsilon + \frac{\tau_e\tau_u+\tau_e\tau_\eta+\tau_u\tau_\eta}{\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_\eta}}{\tau_v + \tau_\varepsilon + \frac{\tau_e\tau_u+\tau_e\tau_\eta+\tau_u\tau_\eta}{\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_\eta}} \to \frac{\tau_\varepsilon}{\tau_v+\tau_\varepsilon} \text{ and } a_u = a_v\alpha \to 0.$$

Similarly, if $\tau_e \in (0,\infty)$, $\tau_u \to 0$, and $\tau_\eta \to 0$, we have the same result. QED.

Proof of Proposition 2

From the proof for Theorem 2, we know that α is determined by $f(\alpha) = 0$, where $f(\alpha)$ crosses 0 from below. Using the implicit function theorem, we can compute:

$$\frac{\partial\alpha}{\partial\tau_\eta} = -\frac{\tau_\varepsilon\alpha^3 + \tau_e\alpha}{3(\tau_e\tau_\varepsilon + \tau_\varepsilon\tau_\eta)\alpha^2 + 4\tau_e\tau_\varepsilon\alpha + (\tau_e\tau_\varepsilon + \tau_e\tau_\eta + \tau_u\tau_\varepsilon)} < 0. \tag{A1}$$

QED.

Proof of Proposition 3

Suppose that $\tau_\eta \to \infty$.

Part (a): By Proposition 1, we have $\lim_{\tau_\eta\to\infty} \tilde{p} = \tilde{v}$ almost surely.

Part (b): In the proof of Proposition 1, we have shown $\lim_{\tau_\eta\to\infty} \alpha = 0$ and $\lim_{\tau_\eta\to\infty} \alpha^2\tau_\eta = 0$. Thus,

$$\tau_v + \tau_\varepsilon + \frac{\tau_e\tau_u + \tau_e\tau_\eta + \tau_u\tau_\eta}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_\eta} = O(\tau_\eta). \tag{A2}$$

Therefore,

$$Risk = \frac{1}{\tau_v + \tau_\varepsilon + \frac{\tau_e\tau_u+\tau_e\tau_\eta+\tau_u\tau_\eta}{\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_\eta}} = O\left(\frac{1}{\tau_\eta}\right) \to 0.$$

In addition, $\alpha < \frac{\tau_e\tau_u}{\tau_e\tau_\varepsilon+\tau_e\tau_\eta+\tau_u\tau_\varepsilon} \Rightarrow \alpha\tau_\eta < \frac{\tau_e\tau_u}{\tau_e\tau_\varepsilon+\tau_e\tau_\eta+\tau_u\tau_\varepsilon}\tau_\eta = \tau_u - \tau_u\tau_\varepsilon\frac{\tau_e+\tau_u}{\tau_e\tau_\varepsilon+\tau_e\tau_\eta+\tau_u\tau_\varepsilon} < \tau_u$, and thus, $\alpha\tau_\eta = O(1)$. Thus,

$$\frac{\tau_u - \alpha\tau_\eta}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_\eta} = O(1). \tag{A3}$$

Also, by $\lim_{\tau_\eta\to\infty} \alpha = 0$ and $\lim_{\tau_\eta\to\infty} \alpha^2\tau_\eta = 0$, we have

$$\lim_{\tau_\eta\to\infty} \frac{\tau_u + \tau_e + \alpha\tau_e}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_\eta} = 1. \tag{A4}$$

As a result, the numerator in (28) is $O\left(\sqrt{\tau_\eta}\right)$. Combined with (A2), we have $Disagreement = O\left(\frac{1}{\sqrt{\tau_\eta}}\right) \to 0$.

Part (c): Since $Risk = O\left(\frac{1}{\tau_\eta}\right)$ and $Disagreement = O\left(\frac{1}{\sqrt{\tau_\eta}}\right)$, we have $Volume = O\left(\sqrt{\tau_\eta}\right) \to \infty$ by equation (26). The proof of Part (b) also shows that

$$\tau_\varepsilon + \left(\frac{\tau_u - \alpha\tau_\eta}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_\eta}\right)^2 \tau_e = O(1).$$

Thus, the divergence of volume is driven by $\left(\frac{\tau_u+\tau_e+\alpha\tau_e}{\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_\eta}\right)^2 \tau_\eta$ in equation (30). QED.

Proof of Proposition 4

Part (a): Using the expression of $\frac{\partial\alpha}{\partial\tau_\eta}$ in (A1), we can directly compute the expression of

$$\frac{\partial}{\partial\tau_\eta}\left[\left(\frac{\tau_u - \alpha\tau_\eta}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_\eta}\right)^2 \tau_e + \left(\frac{\tau_u + \tau_e + \alpha\tau_e}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_\eta}\right)^2 \tau_\eta\right],$$

which is positive. Thus, by (29), we have $\frac{\partial Volume}{\partial\tau_\eta} > 0$.

Direct computation shows

$$\begin{aligned}
&\frac{\partial}{\partial\tau_\eta}\frac{\tau_e\tau_u + \tau_e\tau_\eta + \tau_u\tau_\eta}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_\eta} \\
&= \frac{(\tau_e + \tau_u + \alpha\tau_e)^2 - 2(\tau_e\tau_u + \tau_e\tau_\eta + \tau_u\tau_\eta)(\tau_e(\alpha+1) + \alpha\tau_\eta)\frac{\partial\alpha}{\partial\tau_\eta}}{\left(\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_\eta\right)^2} > 0,
\end{aligned}$$

since $\frac{\partial\alpha}{\partial\tau_\eta} < 0$. Thus, by equation (15), we have

$$\frac{\partial}{\partial\tau_\eta}\frac{\tau_e\tau_u + \tau_e\tau_\eta + \tau_u\tau_\eta}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_\eta} > 0 \Rightarrow \frac{\partial Var(\tilde{v}|\tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i)}{\partial\tau_\eta} < 0.$$

Part (b): We consider two cases: $\tau_\varepsilon = 0$ and $\tau_\varepsilon = \infty$.

Case 1: $\tau_\varepsilon = 0$

For any given $\tau_\eta \in (0, \infty)$, using (19), we can compute $\alpha = \frac{\tau_u}{\tau_\eta}$. Inserting $\tau_\varepsilon = 0$ and $\alpha = \frac{\tau_u}{\tau_\eta}$ into the expression of *Disagreement* in (28) delivers

$$Disagreement^2 = \frac{\tau_\eta^3}{\left(\tau_\eta^2 + \tau_u\tau_v + \tau_v\tau_\eta\right)^2}. \tag{A5}$$

Direct computation shows

$$\frac{\partial Disagreement^2}{\partial\tau_\eta} = \frac{\tau_\eta^2\left(-\tau_\eta^2 + \tau_v\tau_\eta + 3\tau_u\tau_v\right)}{\left(\tau_\eta^2 + \tau_u\tau_v + \tau_v\tau_\eta\right)^3} \Rightarrow$$

$$\frac{\partial Disagreement^2}{\partial\tau_\eta} < 0 \Longleftrightarrow \tau_\eta > \frac{\tau_v + \sqrt{\tau_v^2 + 12\tau_u\tau_v}}{2}.$$

Case 2: $\tau_\varepsilon = \infty$

Note that for a given $\tau_\eta > 0$, we have $\alpha \in \left(0, \frac{\tau_e \tau_u}{\tau_e\tau_\varepsilon + \tau_e\tau_\eta + \tau_u\tau_\varepsilon}\right)$. Thus,

$$\lim_{\tau_\varepsilon \to \infty} \frac{\tau_e \tau_u}{\tau_e\tau_\varepsilon + \tau_e\tau_\eta + \tau_u\tau_\varepsilon} = 0 \Rightarrow \lim_{\tau_\varepsilon \to \infty} \alpha = 0, \tag{A6}$$

$$\lim_{\tau_\varepsilon \to \infty} \left(\frac{\tau_e \tau_u}{\tau_e\tau_\varepsilon + \tau_e\tau_\eta + \tau_u\tau_\varepsilon}\right)^3 \tau_\varepsilon = 0 \Rightarrow \lim_{\tau_\varepsilon \to \infty} \alpha^3 \tau_\varepsilon = 0. \tag{A7}$$

So, by the expression of $\frac{\partial \alpha}{\partial \tau_\eta}$ in (A1), we have

$$\lim_{\tau_\varepsilon \to \infty} \frac{\partial \alpha}{\partial \tau_\eta} = 0. \tag{A8}$$

By (28), we can show

$$\frac{\partial \log Disagreement^2}{\partial \tau_\eta} = \frac{\frac{\partial}{\partial \tau_\eta}\left[\left(\frac{\tau_u - \alpha\tau_\eta}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_\eta}\right)^2 \tau_e + \left(\frac{\tau_u + \tau_e + \alpha\tau_e}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_\eta}\right)^2 \tau_\eta\right]}{\tau_\varepsilon + \left(\frac{\tau_u - \alpha\tau_\eta}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_\eta}\right)^2 \tau_e + \left(\frac{\tau_u + \tau_e + \alpha\tau_e}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_\eta}\right)^2 \tau_\eta} - 2\frac{\frac{\partial}{\partial \tau_\eta}\frac{\tau_e\tau_u + \tau_e\tau_\eta + \tau_u\tau_\eta}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_\eta}}{\left(\tau_v + \tau_\varepsilon + \frac{\tau_e\tau_u + \tau_e\tau_\eta + \tau_u\tau_\eta}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_\eta}\right)}.$$

By (A6)–(A8), direct computation shows

$$\lim_{\tau_\varepsilon \to \infty} \frac{\partial \frac{\tau_e\tau_u + \tau_e\tau_\eta + \tau_u\tau_\eta}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_\eta}}{\partial \tau_\eta} = 1,$$

$$\lim_{\tau_\varepsilon \to \infty} \frac{\partial\left[\left(\frac{\tau_u - \alpha\tau_\eta}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_\eta}\right)^2 \tau_e + \left(\frac{\tau_u + \tau_e + \alpha\tau_e}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_\eta}\right)^2 \tau_\eta\right]}{\partial \tau_\eta} = 1.$$

Thus, using the above expressions and (A6), we have

$$\frac{\partial \log Disagreement^2}{\partial \tau_\eta} = -\frac{1}{\tau_\varepsilon} + o\left(\frac{1}{\tau_\varepsilon}\right). \tag{A9}$$

As a result, when τ_ε is large, $\frac{\partial \log Disagreement^2}{\partial \tau_\eta} < 0$. QED.

Proof of Proposition 5

Part (a): This part follows from equation (33) and Proposition 4.

Part (b): Using Theorem 2, we can compute

$$\sigma^2_{\tilde{v}-\tilde{p}} = \frac{\tau_v + \left[\frac{(1+\alpha)\tau_e + \alpha\tau_\eta}{\tau_u + (1+\alpha)^2\tau_e + \alpha^2\tau_\eta}\right]^2 \tau_u}{\left(\tau_v + \tau_\varepsilon + \frac{\tau_e\tau_u + \tau_e\tau_\eta + \tau_u\tau_\eta}{\tau_u + (1+\alpha)^2\tau_e + \alpha^2\tau_\eta}\right)^2}. \tag{A10}$$

We again consider two cases: $\tau_\varepsilon = 0$ and $\tau_\varepsilon = \infty$.

Case 1: $\tau_\varepsilon = 0$

For any given $\tau_\eta \in (0, \infty)$, using (19), we can compute $\lim_{\tau_\varepsilon \to 0} \alpha = \frac{\tau_u}{\tau_\eta}$. Inserting $\alpha = \frac{\tau_u}{\tau_\eta}$

and $\tau_\varepsilon = 0$ into (A10), we can compute

$$\lim_{\tau_\varepsilon \to 0} \sigma^2_{\tilde{v}-\tilde{p}} = \frac{\tau_u^2\tau_v + \tau_u\tau_\eta^2 + \tau_v\tau_\eta^2 + 2\tau_u\tau_v\tau_\eta}{\left(\tau_\eta^2 + \tau_u\tau_v + \tau_v\tau_\eta\right)^2}.$$

Direct computation shows

$$\lim_{\tau_\varepsilon \to 0} \frac{\partial\sigma^2_{\tilde{v}-\tilde{p}}}{\partial\tau_\eta} = -\frac{2\tau_\eta((\tau_v + \tau_u)\tau_\eta^2 + 3\tau_u\tau_v\tau_\eta + \tau_u^2\tau_v)}{(\tau_\eta^2 + (\tau_u + \tau_\eta)\tau_v)^3} < 0.$$

Thus, $\sigma_{\tilde{v}-\tilde{p}}$ decreases with τ_η for sufficiently small τ_ε.

Case 2: $\tau_\varepsilon = \infty$

Using equations (A6) and (A8), we can directly compute the derivative of $\sigma^2_{\tilde{v}-\tilde{p}}$ in equation (A10), yielding

$$\frac{\partial\sigma^2_{\tilde{v}-\tilde{p}}}{\partial\tau_\eta} = -\frac{2\left(\tau_e^2\tau_u + \tau_e^2\tau_v + \tau_u^2\tau_v + 2\tau_e\tau_u\tau_v\right)}{\left(\tau_e + \tau_u\right)^2}\frac{1}{\tau_\varepsilon^3} + o\left(\frac{1}{\tau_\varepsilon^3}\right).$$

Thus, for sufficiently large τ_ε, we also have $\frac{\partial\sigma^2_{\tilde{v}-\tilde{p}}}{\partial\tau_\eta} < 0$.

Part (c): When $C\left(\tau_\eta\right) = 0$ in equation (31), we have

$$\frac{\partial Welfare}{\partial\tau_\eta} > 0 \Longleftrightarrow \frac{\partial\gamma^2\sigma_D^2\sigma^2_{\tilde{v}-\tilde{p}}}{\partial\tau_\eta} < 0. \tag{A11}$$

Direct computation shows

$$\sigma_D^2 = \frac{1}{\gamma^2}\left[\tau_\varepsilon + \left(\frac{\tau_u - \alpha\tau_\eta}{\tau_u + (1+\alpha)^2\tau_e + \alpha^2\tau_\eta}\right)^2\tau_e + \left(\frac{\tau_u + (1+\alpha)\tau_e}{\tau_u + (1+\alpha)^2\tau_e + \alpha^2\tau_\eta}\right)^2\tau_\eta\right]. \tag{A12}$$

Using (28), (A10), and (A12), we have

$$\gamma^2\sigma_D^2\sigma^2_{\tilde{v}-\tilde{p}} = Disagreement^2 \times \left[\tau_v + \left(\frac{(1+\alpha)\tau_e + \alpha\tau_\eta}{\tau_u + (1+\alpha)^2\tau_e + \alpha^2\tau_\eta}\right)^2\tau_u\right]. \tag{A13}$$

Case 1: $\tau_\varepsilon = 0$

Using $\lim_{\tau_\varepsilon\to 0}\alpha = \frac{\tau_u}{\tau_\eta}$ and equation (A5), we can compute,

$$\lim_{\tau_\varepsilon \to 0}\gamma^2\sigma_D^2\sigma^2_{\tilde{v}-\tilde{p}} = \frac{\tau_\eta^3\left(\tau_u^2\tau_v + \tau_u\tau_\eta^2 + \tau_v\tau_\eta^2 + 2\tau_u\tau_v\tau_\eta\right)}{\left(\tau_u + \tau_\eta\right)^2\left(\tau_\eta^2 + \tau_u\tau_v + \tau_v\tau_\eta\right)^2}.$$

Then, direct computation shows that as $\tau_\varepsilon \to 0$,

$$\frac{\partial\gamma^2\sigma_D^2\sigma^2_{\tilde{v}-\tilde{p}}}{\partial\tau_\eta} = \frac{\tau_\eta^2\left(3\tau_u^2\tau_v + \tau_u\tau_\eta^2 + \tau_v\tau_\eta^2 + 4\tau_u\tau_v\tau_\eta\right)}{\left(\tau_u + \tau_\eta\right)^3\left(\tau_\eta^2 + \tau_u\tau_v + \tau_v\tau_\eta\right)^3}Q\left(\tau_\eta\right),$$

where

$$Q\left(\tau_\eta\right) = -\tau_\eta^3 + \left(\tau_u + \tau_v\right)\tau_\eta^2 + 2\tau_u\tau_v\tau_\eta + \tau_u^2\tau_v.$$

Note that $Q\left(\tau_\eta\right) = 0$ has a unique positive root, and thus there exists $\bar{\tau}_\eta \in (0,\infty)$ such that

$$Q\left(\tau_\eta\right) < 0 \Longleftrightarrow \tau_\eta > \bar{\tau}_\eta.$$

As a result, by condition (A11), we have

$$\frac{\partial Welfare}{\partial \tau_\eta} > 0 \Longleftrightarrow \frac{\partial \gamma^2 \sigma_D^2 \sigma_{\tilde{v}-\tilde{p}}^2}{\partial \tau_\eta} < 0 \Longleftrightarrow Q(\tau_\eta) < 0 \Longleftrightarrow \tau_\eta > \bar{\tau}_\eta,$$

that is, $Welfare$ is U-shaped in τ_η when τ_ε is sufficiently small.

Case 2: $\tau_\varepsilon = \infty$

Using equations (A6) and (A8), we can show

$$\lim_{\tau_\varepsilon \to \infty} \frac{\partial}{\partial \tau_\eta} \log \left[\tau_v + \left(\frac{(1+\alpha)\tau_e + \alpha \tau_\eta}{\tau_u + (1+\alpha)^2 \tau_e + \alpha^2 \tau_\eta} \right)^2 \tau_u \right] = 0.$$

Then, by equations (A9) and (A13),

$$\lim_{\tau_\varepsilon \to \infty} \frac{\partial \log \gamma^2 \sigma_D^2 \sigma_{\tilde{v}-\tilde{p}}^2}{\partial \tau_\eta} = \lim_{\tau_\varepsilon \to \infty} \frac{\partial \log Disagreement^2}{\partial \tau_\eta} < 0.$$

So, when τ_ε is sufficiently large, $Welfare$ monotonically increases in τ_η, by condition (A11).

QED.

Proof of Proposition 6

Note that in the proof of Proposition 3, we have shown

$$\lim_{\tau_\eta \to \infty} Volume = O(\sqrt{\tau_\eta}).$$

Hence, by (33),

$$\lim_{\tau_\eta \to \infty} \sigma_D^2 = O(\tau_\eta) \to \infty.$$

Given $\lim_{\tau_\eta \to \infty} \alpha = 0$, $\lim_{\tau_\eta \to \infty} \alpha^2 \tau_\eta = 0$, and $\alpha \tau_\eta = O(1)$ (see the proof of Proposition 3), by equation (A10), we have

$$\lim_{\tau_\eta \to \infty} \sigma_{\tilde{v}-\tilde{p}}^2 = \frac{O(1)}{O(\tau_\eta^2)} = O(\tau_\eta^{-2}) \to 0.$$

Thus, by (31), when $C(\tau_\eta) = 0$, we have

$$\gamma^2 \sigma_D^2 \sigma_{\tilde{v}-\tilde{p}}^2 = O\left(\frac{1}{\tau_\eta}\right) \to 0 \Rightarrow Welfare = \frac{1}{2\gamma} \log(1 - \gamma^2 \sigma_D^2 \sigma_{\tilde{v}-\tilde{p}}^2) \to 0.$$

QED.

Proof of Theorem 3

Function $\phi(\tau_\eta)$ in equation (42) is defined by plugging $\alpha(\tau_\eta)$ into $\frac{\partial B(\tau_{\eta_i};\alpha)}{\partial \tau_{\eta_i}}$. Then, combining with the fact $\tau_{\eta_i} = \tau_\eta$ and condition (41), we can get the two conditions (a) and (b) characterizing the equilibrium value of τ_η. The only remaining task is to show that $\phi(\tau_\eta)$ must satisfy either (a) or (b) for $\tau_\eta \geq 0$, so that there exists an overall equilibrium.

Apparently, if $\phi(0) \leq 0$, then condition (a) is trivially satisfied, so that $\tau_\eta = 0$ constitutes an equilibrium.

Now suppose $\phi(0) > 0$. We will show $\lim_{\tau_\eta \to \infty} \phi(\tau_\eta) < 0$, so that by the intermediate value theorem, condition (b) must be satisfied. Direct computation shows

$$\frac{\partial B(\tau_\eta, \alpha(\tau_\eta))}{\partial \tau_\eta} = \frac{1}{2\gamma} \frac{\frac{(\tau_e + \tau_u + \alpha\tau_e)^2}{(\tau_e + \tau_u + 2\alpha\tau_e + \alpha^2\tau_e + \alpha^2\tau_\eta)^2}}{\tau_v + \tau_\varepsilon + \frac{\tau_e\tau_u + \tau_e\tau_\eta + \tau_u\tau_\eta}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_\eta}} - C'(\tau_\eta).$$

In the proof of Proposition 1, we have shown that $\alpha \to 0$ and $\alpha^2\tau_\eta \to 0$, as $\tau_\eta \to \infty$. Thus, the term $\frac{1}{2\gamma} \frac{\frac{(\tau_e + \tau_u + \alpha\tau_e)^2}{(\tau_e + \tau_u + 2\alpha\tau_e + \alpha^2\tau_e + \alpha^2\tau_\eta)^2}}{\tau_v + \tau_\varepsilon + \frac{\tau_e\tau_u + \tau_e\tau_\eta + \tau_u\tau_\eta}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_\eta}} \propto \frac{1}{2\gamma} \frac{1}{\tau_v + \tau_\varepsilon + \tau_\eta} \to 0$, as $\tau_\eta \to \infty$. When $C(\tau_\eta)$ is weakly convex, we know $\lim_{\tau_\eta \to \infty} C'(\tau_\eta)$ is bounded below. Thus, we have $\lim_{\tau_\eta \to \infty} \phi(\tau_\eta) < 0$. QED.

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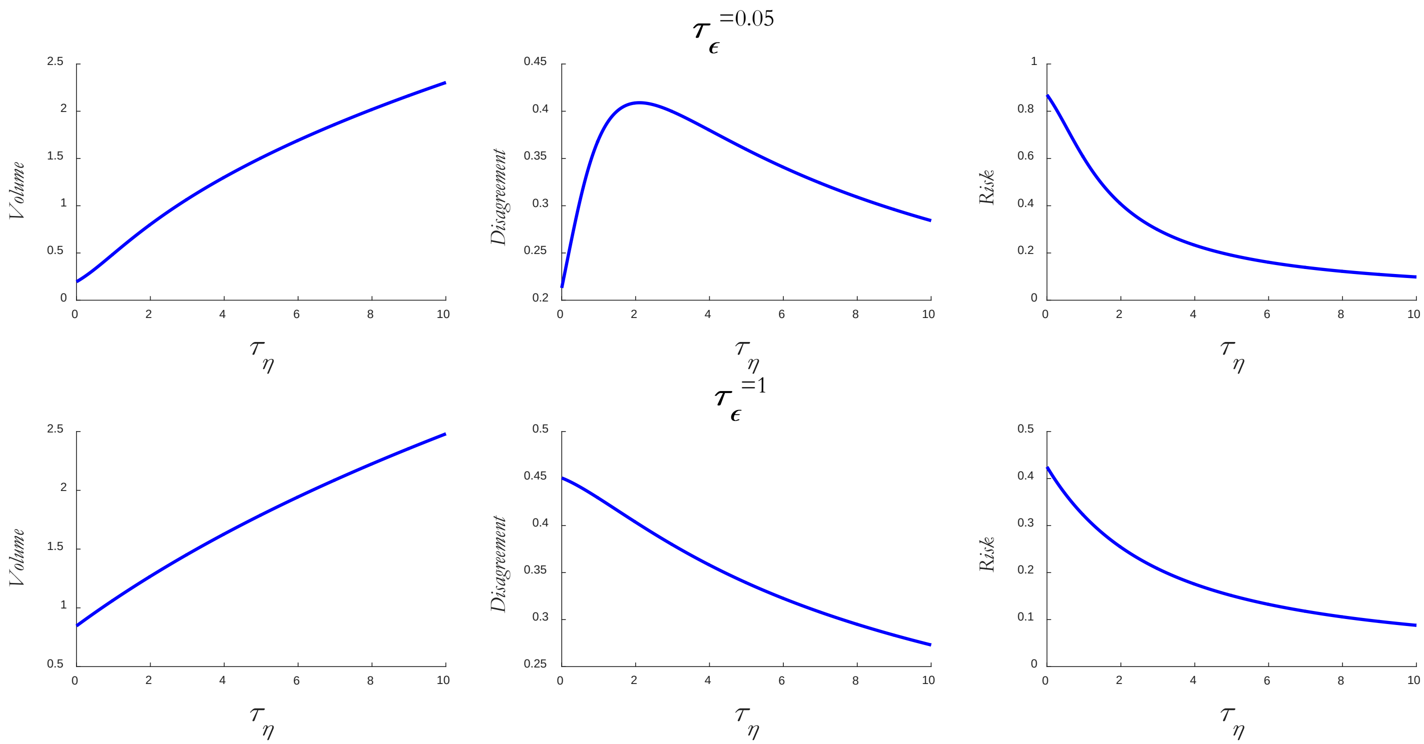
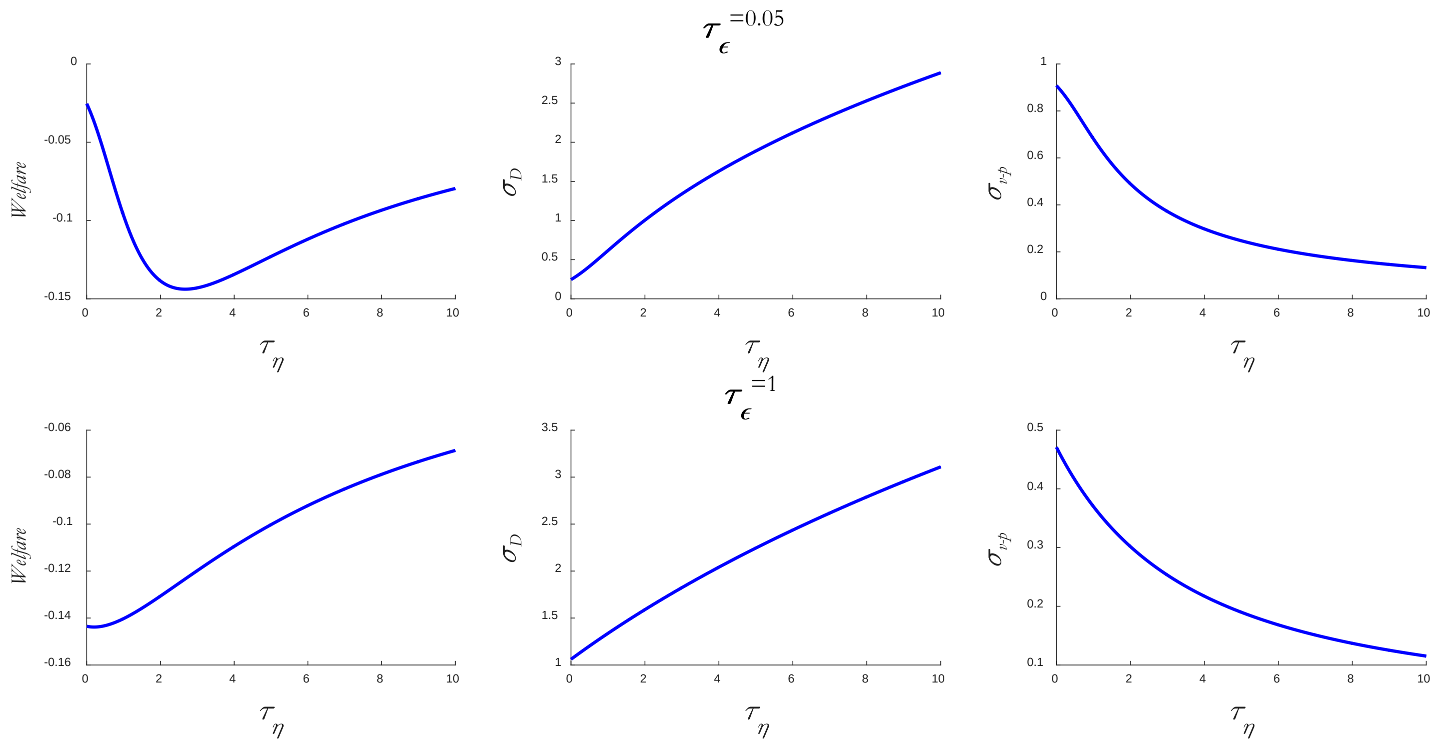
Figure 1 The Effect of Sophistication on Volume, Disagreement, and Trading Risk



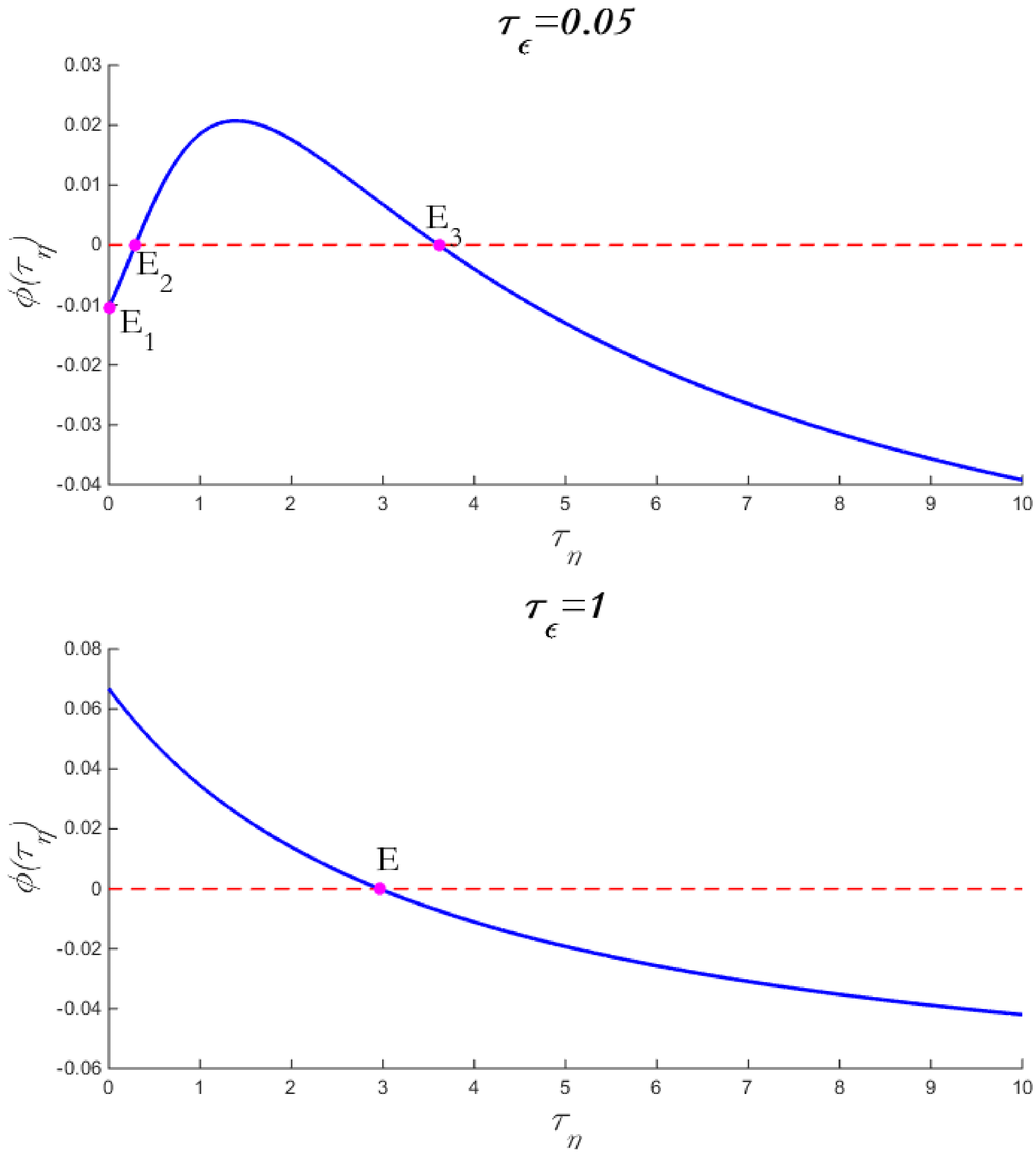
All investors have the same sophistication level, i.e., $\tau_{\eta_i} = \tau_\eta$. This figure plots the implications of sophistication level τ_η for trading volume, disagreement, and the perceived trading risk. In the top three panels, we have set $\tau_\varepsilon = 0.05$, while in the bottom panel, we set $\tau_\varepsilon = 1$. The other parameters are set as follows: $\tau_v = \tau_e = \tau_u = \gamma = 1$.

Figure 2 Welfare Implications of Investor Sophistication



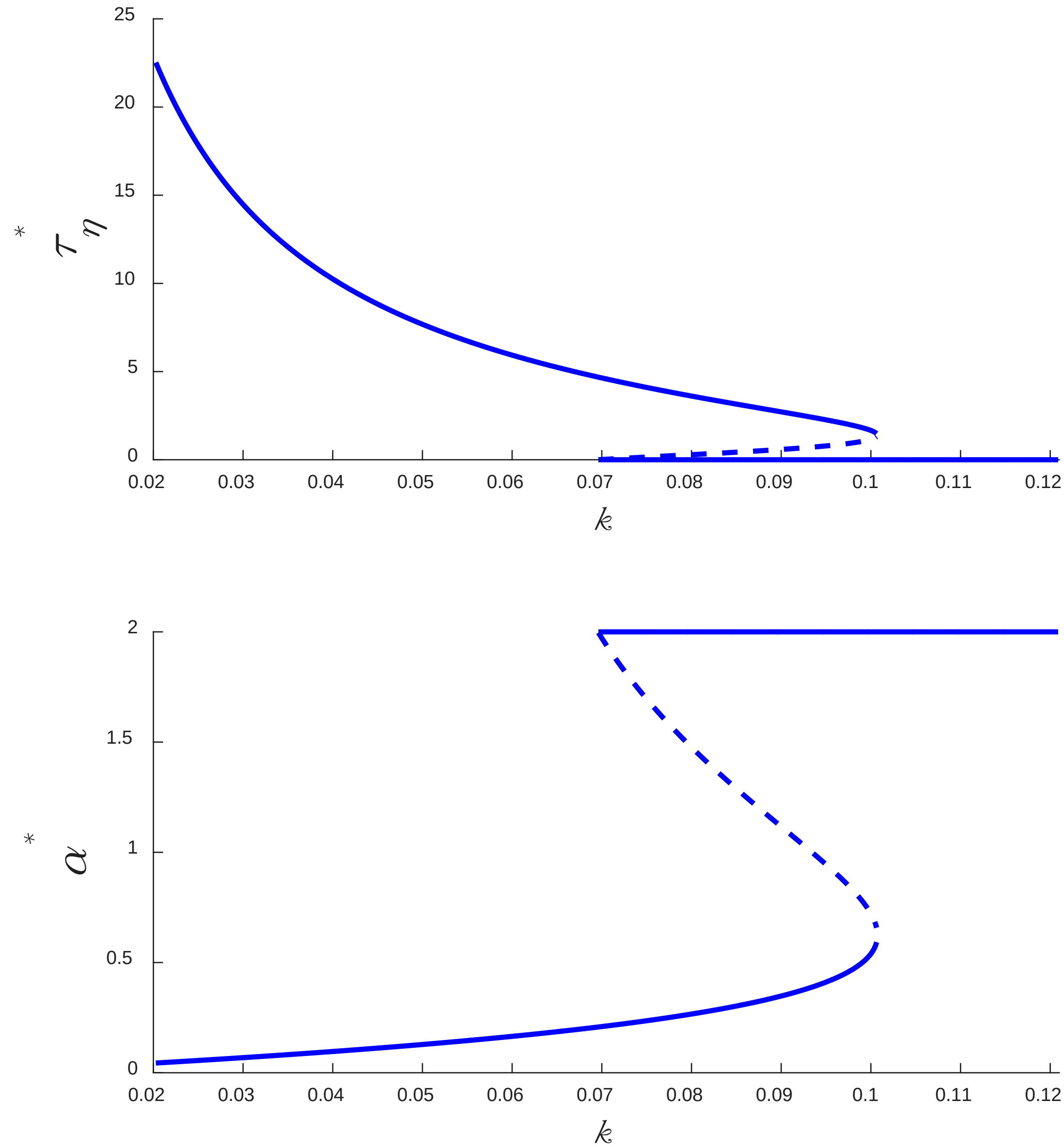
All investors have the same sophistication level, i.e., $\tau_{\eta_i} = \tau_\eta$. This figure plots the implications of sophistication level τ_η for investor welfare, trading volatility σ_D, and return volatility σ_{v-p}. In the top three panels, we have set $\tau_\varepsilon = 0.05$, while in the bottom panel, we set $\tau_\varepsilon = 1$. The other parameters are set as follows: $\tau_v = \tau_e = \tau_u = \gamma = 1$. We also assume $C(\tau_{\eta_i}) = 0$.

Figure 3 Multiplicity vs. Uniqueness of Equilibrium



This figure plots the function of $\phi(\tau_\eta) = \frac{\partial B(\tau_\eta;\alpha(\tau_\eta))}{\partial \tau_\eta}$, which determines the equilibrium level of investor sophistication. Investors' cost function of acquiring sophistication is $C(\tau_{\eta_i}) = k\tau_{\eta_i}$. In the top panel, we set $\tau_\varepsilon = 0.05$, while in the bottom panel, we set $\tau_\varepsilon = 1$. In both panels, the other parameters are set as follows: $\tau_v = \tau_e = \tau_u = \gamma = 1$ and $k = 0.08$.

Figure 4 The Effect of the Cost k of Studying Market Data



This figure plots the effect of the cost of becoming more sophisticated in interpreting market data on the equilibrium values of τ_η^* and α^*, when the cost function takes the form $C(\tau_{\eta_i}) = k\tau_{\eta_i}$. The other parameter values are: $\tau_v = \tau_e = \tau_u = \gamma = 1$ and $\tau_\varepsilon = 0.05$. The dashed segments indicate unstable equilibria.