Costly Interpretation of Asset Prices

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Abstract

We propose a model in which investors spend effort to interpret the informational content of asset prices. Due to cognitive ability limitations, investors have a cost to process information from prices, but they still choose optimal trading strategies given their beliefs. We show that as long as investors are not fully sophisticated, their interpretation of prices can inject noise into the price system, which serves as a source of endogenous noise trading. Compared to the standard rational expectations equilibrium, our setup features price momentum and yields higher return volatility and excessive trading volume. When investors can study market data to endogenize their sophistication level, research efforts can exhibit strategic complementarity, leading to multiple equilibria.

Key words: Investor sophistication, asset prices, disagreement, trading volume, noise trading, multiplicity

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1 Introduction

Data can be viewed as information only after it has been analyzed. Interpreting data is often costly in terms of time, effort, and other investor resources. This is particular true for market data given the complexity of modern financial markets. In the existing frameworks—such as the traditional rational expectations equilibrium (REE) model (e.g., Grossman, 1976; Radner, 1979), and the more recent REE-disagreement hybrid models (e.g., Banerjee, 2011)—investors perfectly understand the price function and thus can costlessly read into the price to uncover value-relevant information. In this paper, we propose a model to explicitly capture the notion that interpreting price information is costly and the error committed in the inference process can inject noise into the price in a form of endogenous noise trading.

As in the standard REE, the price serves two purposes in our model: clearing the market and conveying information to investors. The informational role of the price in the standard REE is pinned down by assuming that investors perfectly comprehend the price function.[1] We instead relax this assumption and employ a "receiver noise" approach as in Myatt and Wallace (2012) to model how the price affects investors' beliefs. A fully sophisticated investor would extract the best signal possible from the price (which is endogenously determined in equilibrium), while a less sophisticated investor introduces noise in interpreting the price. After investors form their beliefs based on the personalized price signals, they behave as rational Bayesian and make optimal investments in response to their own beliefs. Through market clearing, investors' optimal asset investments in turn endogenously determine the equilibrium price function and hence the best price signal (i.e., the "truth" in investors' personalized signals extracted from the price data).

Our first result is to show that costly price interpretation can inject noise into the price system. This result relates to De Long, Shleifer, Summers, and Waldmann (1990, DSSW) who show that the misperception of irrational traders about asset fundamentals can pose "noise trader risk" to rational arbitrageurs. We extend the idea to an asymmetric information

[1] As discussed by Guesnerie (1992), this comprehension is broadly justified in two ways: the "eductive" justification that relies on the understanding of the logic of the situation faced by economic agents and that is associated with mental activity of agents aiming at "forecasting the forecasts of others;" and the "evolutive" justification that emphasizes the learning possibilities offered by the repetition of the situation and that is associated with the convergence of several versions of learning processes.

setting through imperfect price interpretation. Specifically, in our setting, the equilibrium price is a linear function of the asset fundamental and a noise term. The fundamental element comes from aggregating investors' private value-relevant information, which is the root reason why investors care to learn from the price. The noise term in the price arises from a common error in investors' personalized price signals, which is meant to capture the idea that in processing price data, investors may suffer a common cognitive error (such as "sentiment"/"misperception") or technical error (such as a common bias in data processing algorithms). When investors become more sophisticated, they understand better the true price signal and their trading brings less noise into the price. As investor sophistication approaches to infinity, the asset price approaches the standard REE.

We then examine the behavior of asset prices and trading volume in our economy. Compared to the standard REE (in which investors are infinitely sophisticated), costly interpretation of prices leads to price momentum (future returns depend positively on the current price), excessive return volatility, and excessive trading volume. This result is consistent with the existing empirical evidence (e.g., Jegadeesh and Titman (1993) and Moskowitz, Ooi, and Pedersen (2012) on momentum; Shiller (1981) and LeRoy and Porter (1981) on excess volatility; and Odean (1999) and Barber and Odean (2000) on excess trading). In addition, this result also demonstrates that our setup qualitatively differs from the traditional models with exogenous noise trading such as Grossman and Stiglitz (1980) and Hellwig (1980). For instance, in Hellwig (1980), asset returns exhibit reversals—a high price predicts a future price decline—which is opposite of our prediction.

As investors become gradually more sophisticated, return volatility generally decreases, while both disagreement and trading volume can exhibit a hump shape. This finding echoes Garfinkel (2009) who finds that volume is a better proxy for disagreement than return volatility. It also helps to reconcile the contradictory evidence on the relation between disagreement and return volatility. For instance, Frankel and Foot (1990) and Anderson, Ghysels, and Juergens (2005) document a positive disagreement-volatility relation, while Garfinkel (2009) documents a negative relation.

Return volatility monotonically decreases with sophistication because sophistication improves price informativeness: more sophisticated investors understand the price better and their trading brings more information than noise into the price; as a result, the price is closer to the asset fundamental, thereby lowering the volatility of returns that are the difference between the price and the fundamental.

By contrast, investor sophistication affects disagreement and hence volume through two offsetting channels. On the one hand, investors interpret the price differently. The more sophisticated they are, the higher weight they put on their diverse information extracted from the price in forecasting the asset fundamental, and thus the more likely they may end up with different understandings and trade more. On the other hand, investor sophistication improves price informativeness, which makes the price contain more precise information about the fundamental. So, actively reading information from the price can also cause investors' beliefs to converge, leading to less trading.

When investors start with precise fundamental signals, the second negative effect always dominates so that disagreement and volume monotonically decrease with investor sophistication. This is because when investors are endowed with precise fundamental information, the price signal is particularly accurate after aggregation, and thus the belief-convergence effect is particularly strong. By contrast, when investors are endowed with coarse fundamental information, the positive effect can dominate. For instance, suppose that investors start with extremely coarse fundamental information and extremely low sophistication level, so that their expectations about the asset payoff are close to the prior distribution and thus almost homogeneous. Now if we increase investors' sophistication level, they will start to read different information from the price, and so their expectations will diverge. Nonetheless, when investor sophistication level becomes sufficiently high, disagreement will decrease with sophistication again (i.e., the belief-convergence effect will eventually dominate), because as sophistication approaches to infinity, the asset price perfectly aggregates private information and thus investor disagreement and trading volume will vanish.

We further endogenize investors' sophistication level using a learning technology: investors can spend resources to study market data, and the more resources they spend, the better can they read the price, and so the less noise is incurred in the inference process. We find strategic complementarity in sophistication acquisition and the possibility of multiple equilibria. Specifically, when a representative investor spends more resources to become more sophisticated in reading the price, price informativeness increases and the price conveys more information, which increases the marginal value of attending to price data. This in turn further strengthens investors' ex-ante incentives to study market data. This strategic complementarity implies that multiple sophistication levels can be sustained in equilibrium. Thus, when an exogenous parameter, for instance, the cost of achieving sophistication, changes, there can be jumps in equilibrium sophistication levels. This can correspond to waves of development of algorithmic trading in reality in response to exogenous shocks to the economy, say, some regulation changes.

We also consider an analysis which allows investors to endogenously choose both the sophistication level of studying market data and the precision of fundamental information. Two observations emerge from such an analysis. First, from one investor's perspective, studying market data and acquiring fundamental information are substitutes. This is because studying market data eventually delivers a signal about asset payoff, which lowers an investor's incentive to acquire fundamental information directly. Second, from a cross-investor perspective, studying market data can be a strategic complement to acquiring fundamental information. Intuitively, when all investors acquire more precise fundamental information, the price becomes a more accurate information source in predicting fundamentals, and thus the incentive of a particular investor to study market data can be stronger.

The plan of the paper is as follows. Section 2 reviews related literature. Section 3 presents the model and the equilibrium concept. Section 4 studies the equilibrium in the financial market for given sophistication levels of investors. Section 5 determines the overall equilibrium including the investor's sophistication level and examines potential multiplicity. Section 6 considers an extension with endogenous fundamental information and performs a robustness exercise in a two-type economy. Section 7 concludes the paper. Proofs are gathered in an appendix.

2 Related Literature

As stated, we model investor sophistication by the degree of individual noise added to the best signal possible extracted from the price following a similar approach to Myatt and

Wallace (2002). We extend Myatt and Wallace (2002) by introducing a common term into receiver's noise, which in turn endogenously determines the accuracy of the best price signal.

The common term in receiver's noise can be understood as a form of "sentiment" or "misperception," which therefore connects our paper to the behavioral economics literature (see Shleifer (2000) and Barberis and Thaler (2003) for excellent surveys). In particular, the way we model investors' beliefs shares similarity with DSSW (1990). In DSSW(1990), irrational noise traders misperceive future asset payoffs, and because this misperception is identical across all noise traders, it generates noise trader risk to rational arbitrageurs in financial markets. In our setting, investors suffer misperception when they try to read information from the price and the misperception generates endogenous noise trading that in turn determines the accuracy of price information. In a way, our analysis can be viewed as DSSW cast in an asymmetric information model with endogenous sophistication. Recently, Gârleanu and Pedersen (2016) propose a model to show market efficiency is closely connected to the efficiency of asset management. In our model, market efficiency is determined by how investors (institutions or retail investors) interpret the asset price.

In the accounting literature, some researchers, Indjejikian (1991) and Kim and Verrecchia (1994) for instance, have considered settings in which investors have different interpretations about an exogenous public signal such as earnings announcements. In contrast, in our setting, investors have different interpretations about an endogenous public signal, which is the equilibrium price. In Ganguli and Yang (2009) and Manzano and Vives (2011), investors interpret the price information differently through acquiring information about the noise supply. Our setting differs from these supply-information models in two important ways. First, at the conceptual level, our investors lack the full capacity to interpret prices, while it is not the case in the supply-information models. Second, the supply-information models have focused on uniqueness versus multiplicity of equilibrium, while our analysis has broader implications for prices and volume.

Our approach of modeling investors' understanding of market data shares similarity also to the concept of "rationalizability" (Guesnerie, 1992; Jara-Moroni, 2012) and the "level-k" or "cognitive hierarchy" models (see Crawford, Costa-Gomes, and Iriberri (2013) for a survey). These existing studies make an effort to study whether and how rational expectations can be generated, starting from a more fundamental principle that investors are individually Bayesian rational and best respond to some beliefs. Similarly, under our approach, investors are fully rational at the individual level—more specifically, investors are subjective expected utility (SEU) maximizers (Savage, 1954) and they can perform perfect partial equilibrium analysis—but they cannot read perfectly the information from the price and therefore may not necessarily have the best signal.

Our paper is also related to the recent literature on environment complexity that makes agents fail to account for the informational content of other players' actions in game settings. Eyster and Rabin (2005) develop the concept of "cursed equilibrium," which assumes that each player correctly predicts the distribution of other players' actions, but underestimates the degree to which these actions are correlated with other players' information. Esponda (2008) extends Eyster and Rabin's (2005) concept to "behavioral equilibrium" by endogenizing the beliefs of cursed players. Recently, Esponda and Pouzo (2016) propose the concept of "Berk-Nash equilibrium" to capture that people can have a possibly misspecified view of their environment. In a Berk-Nash equilibrium, each player follows a strategy that is optimal given her belief, and her belief is restricted to be the best fit among the set of beliefs she considers possible, where the notion of best fit is formalized in terms of minimizing the Kullback-Leibler divergence. Although these models are cast in a game theoretical framework, the spirit of our financial market model is similar. In our model, investors' interactions are mediated by an asset price, which is sort of a summary statistic for all the other players' actions.

Eyster, Rabin, and Vayanos (2015) have applied the cursed equilibrium concept to a financial market setting and labeled the resulting equilibrium as the cursed expectations equilibrium (CEE). In a CEE, an investor is a combination of a fully rational REE investor (who correctly reads information from the price) and a naive Walrasian investor (who totally neglects the information in the asset price). The investor in our economy is conceptually related to but different from a partially cursed investor; she can not read from the price function perfectly and has to spend an endogenous cost to infer information from the price. The recent finance literature, such as Banerjee, Kaniel, and Kremer (2009) and Banerjee (2011), have combined REE and disagreement frameworks to allow investors underestimate the precision of other investors' private information (and hence also labeled as "dismissiveness" models). A dismissive investor can be roughly viewed as a combination of a fully sophisticated and a naive agent, and thus conceptually related to the investor in our economy. However, in the dismissiveness model, investors still can read perfectly from the price function and they only disagree about the distribution of other investors' signals.

3 A Model of Costly Interpretation of Asset Prices

3.1 Setup

Environment We consider a one-period economy similar to Hellwig (1980). Two assets are traded in a competitive market: a risk-free asset and a risky asset. The risk-free asset has a constant value of 1 and is in unlimited supply. The risky asset is traded at an endogenous price $\tilde{p}$ and is in zero supply. It pays an uncertain cash flow at the end of the economy, denoted $\tilde{V}$. We assume that $\tilde{V}$ has two elements, a learnable element $\tilde{v}$ and an unlearnable element $\tilde{\xi}$, which are mutually independent and normally distributed. That is, $\tilde{V} = \tilde{v} + \tilde{\xi}$, where $\tilde{v} \sim N\left(0, \tau_v^{-1}\right)$ and $\tilde{\xi} \sim N\left(0, \tau_\xi^{-1}\right)$, with $\tau_v > 0$ and $\tau_\xi > 0$.

There is a continuum $[0, 1]$ of investors who have constant absolute risk aversion (CARA) utility with a risk aversion coefficient of $\gamma > 0$. Investors have fundamental information and trade on it. Specifically, investor i is endowed with the following private signal about the learnable element $\tilde{v}$ in the asset payoff:

$$\tilde{s}_i = \tilde{v} + \tilde{\varepsilon}_i, \text{ with } \tilde{\varepsilon}_i \sim N\left(0, \tau_\varepsilon^{-1}\right),$$

where $\tau_\varepsilon > 0$, and $(\tilde{v}, \tilde{\xi}, \{\tilde{\varepsilon}_i\}_i)$ are mutually independent. In the baseline model, we take the information precision τ_ε as exogenous, and we will endogenize it later on in Section 6. We will refer to both the learnable element $\tilde{v}$ and the total asset payoff $\tilde{V}$ as "fundamentals." The unlearnable element $\tilde{\xi}$ reflects the notion that investors collectively do not know the true payoff from the risky asset.

Belief specification The idea of our paper is to show that the financial market is so complex that investors cannot fully understand its structure so that they cannot perfectly interpret information in asset prices. In traditional REE models, investors look into the asset price to make inference about fundamentals, which is usually modeled as a statistical signal, $\tilde{s}_p$, about the asset fundamental $\tilde{V}$. Economically, investors are sophisticated enough to understand the statistical properties of the price function that links the price $\tilde{p}$ to the fundamental $\tilde{V}$ and thus, they can convert the price $\tilde{p}$ into a signal $\tilde{s}_p$ to extract information about $\tilde{V}$.

In practice, it is questionable that the information in asset prices in modern financial markets can be fully understood by market participants. A better understanding of the market structure needs more effort. Even worse, the very act of extracting information from the price can bring noise into the price, as interpreting prices can involve errors. To capture this idea, we adopt a reduced-form belief specification, which takes the form "truth plus noise." Specifically, after seeing price $\tilde{p}$, investor i interprets it as the following signal:

$$\tilde{s}_{p,i} = \underbrace{\tilde{s}_p}_{\text{truth}} + \underbrace{\tilde{x}_i}_{\text{noise}}, \text{ with } \tilde{x}_i \sim N\left(0, \tau_x^{-1}\right), \tag{1}$$

where $\tilde{s}_p$ is the true signal implied by the price, which is also the best signal that a fully sophisticated investor can obtain in a standard REE setting; and where $\tilde{x}_i$ is the noise in processing the price data, which can come from poor mental reasoning or from technology capacity. We assume that $\tilde{s}_p$ and $\tilde{x}_i$ are mutually independent. We do not model where specification (1) comes from and thus it is a reduced-form belief formation. The standard REE concept corresponds to a situation in which investors fully understand the price function and can convert the price $\tilde{p}$ to the signal $\tilde{s}_p$, so that the noise $\tilde{x}_i$ degenerates to 0 (i.e., $\tau_x = \infty$). In contrast, if $\tau_x = 0$, then investors completely ignore the price information.

We further specify that noise term $\tilde{x}_i$ in (1) admits a factor structure:

$$\tilde{x}_i = \tilde{u} + \tilde{e}_i, \text{ with } \tilde{u} \sim N\left(0, \tau_u^{-1}\right) \text{ and } \tilde{e}_i \sim N\left(0, \tau_e^{-1}\right), \tag{2}$$

where $(\tilde{u}, \{\tilde{e}_i\}_i)$ is mutually independent and independent of all other random variables. Note that, by equations (1) and (2), we have $\tau_x^{-1} = \tau_u^{-1} + \tau_e^{-1}$. In (2), the idiosyncratic noise $\tilde{e}_i$ is specific to investor i. The common noise $\tilde{u}$ in investors' price signals represents waves of optimism and pessimism, which corresponds to the notion of "sentiment" in the behavioral economics literature (e.g., DSSW, 1990; Baker and Wurgler, 2007; Angeletos and La'o, 2013; Benhabib, Wang, and Wen, 2015). For instance, DSSW (1990) assume that all noise traders misperceive future asset payoff with a common error that generates noise trader risk to rational arbitrageurs. The term $\tilde{u}$ in our setting can also arise from a common bias in data-processing algorithms. As we will show shortly, the random variable $\tilde{u}$ will enter the price endogenously as noise trading emphasized in the noisy REE literature (e.g., Grossman and Stiglitz, 1980; Hellwig, 1980).

Sophistication (attention) Investors can study market data to reduce their noise $\tilde{x}_i$ in (1), thereby bringing the price signal $\tilde{s}_{p,i}$ closer to the best signal $\tilde{s}_p$. We model this noise-reduction process as investors gleaning private information about $\tilde{x}_i$ (e.g., Kim and Verrecchia, 1994). Specifically, investor i can study the market and obtain the following signal about $\tilde{x}_i$:

$$\tilde{z}_i = \tilde{x}_i + \tilde{\eta}_i \text{ with } \tilde{z}_i \sim N(0, \tau_{\eta_i}^{-1}), \tag{3}$$

where $\tilde{\eta}_i$ is independent of all other random variables and independent of each other. Conditional on $\tilde{z}_i$, the noise in investor i's price signal $\tilde{s}_{p,i}$ has a posterior distribution

$$\tilde{x}_i|\tilde{z}_i \sim N\left(\tau_{\eta_i}\left(\tau_x + \tau_{\eta_i}\right)^{-1} \tilde{z}_i, \left(\tau_x + \tau_{\eta_i}\right)^{-1}\right), \tag{4}$$

which indeed has a variance $\left(\tau_x + \tau_{\eta_i}\right)^{-1}$ smaller than the prior variance τ_x^{-1}.

Precision τ_{η_i} captures investor i's ability or "sophistication" level in understanding the asset market. When $\tau_{\eta_i} = \infty$, investors fully understand the market, which reduces our economy to the traditional REE setup. When $\tau_x = \tau_{\eta_i} = 0$, investors cannot understand the price function at all and totally neglect the information in prices, which reduces our economy to the traditional Walrasian economy. Parameter τ_{η_i} is endogenous in the model and it comes from the intensity of studying market data. Being sophisticated is costly, which is reflected by a weakly increasing and convex cost function of precision, $C\left(\tau_{\eta_i}\right)$ (similar to the literature, e.g., Verrecchia, 1982; Vives, 2008; Myatt and Wallace, 2012). Investors choose τ_{η_i} to optimally balance the benefit from being more sophisticated against its cost.

Alternatively, we can interpret sophistication parameter τ_{η_i} as attention: if investors do not pay attention then there is limited learning from the price, but to pay attention is costly. For instance, in the language of Pavan (2014), parameter τ_{η_i} can be thought of as the time investor i devotes to the information source (which is the price in our context) and $C\left(\tau_{\eta_i}\right)$

denotes the attention cost incurred by the investor.[2]

Our belief specification can also be closely connected to the attention structure of Myatt and Wallace (2012). Specifically, the term $\tilde{x}_i$ in (1) corresponds to the notion of "receiver noise" in Myatt and Wallace (2012) and extends it in three important ways. First, in equation (2), we allow both a common noise $\tilde{u}$ and an agent-specific noise $\tilde{e}_i$ in investor i's receiver noise, where Myatt and Wallace (2012) only deal with agent-specific receiver noise. Second, the quality or accuracy of the true underlying signal $\tilde{s}_p$ (i.e., $Var(\alpha\tilde{u})$ in (11)) is endogenous, while it is exogenous in Myatt and Wallace (2012). Third, Myatt and Wallace (2012) assume that paying attention τ_{η_i} can linearly increase the precision of receiver noise. Here, we employ a learning structure to endogenously generate a posterior receiver-noise precision that is linear in τ_{η_i}, as shown by equation (4).

For our analysis, it does not matter which interpretation (sophistication or attention) makes more sense. We use the two words sophistication and attention interchangeably, although the language we use in the rest of the paper follows mostly the first interpretation of sophistication.

Remark 1 (Interpretations) *Our setup admits two interpretations. First, we can interpret investors as trading desks of institutions. Trading desks are responsible for trading assets but they rely on the institutions' research departments to generate information from the prices. Research departments are able to extract the true signal $\tilde{s}_p$ from the price in the form of research reports, but when they pass the signal to trading desks, trading desks add noise in comprehending the reports. Secondly, we can also think of our investors as individuals who have two selves in their minds. The two selves behave in the sense of Kahneman's (2011) two thinking systems. The first self engages in slow and deliberative thinking; she behaves like a research department in our institutional interpretation. The second self is like the trading desk and her thinking is fast but noisy since a decision has to be made.*

[2] Some studies in the rational inattention literature further adopt an entropy-based cost function (e.g., Hellwig, Kohls, and Veldkamp, 2012; Myatt and Wallace, 2012; Matějka and McKay, 2015): the amount of information transmitted is captured by the concept of mutual information; the mutual information uses an agent's attention capacity and an agent can incur a cost to increase the attention capacity. In our context, the mutual information is given by $K \equiv \frac{1}{2}\log\left[\frac{Var(\tilde{s}_p|\tilde{s}_{p,i})}{Var(\tilde{s}_p|\tilde{s}_{p,i},\tilde{z}_i)}\right]$, which captures how much information is transmitted after the investor processes price data. The investor incurs a cost $C(K)$ to process price information more accurately. The recent experimental study by Dean and Neligh (2017) finds supporting evidence for rational inattention but not for the cost function based on mutual information.

3.2 Timeline and Equilibrium Concept

The timeline of our economy is as follows:

1. Investors choose simultaneously a sophistication level τ_{η_i} and a demand function $D(\tilde{p}; \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i)$, which depends on private information $\tilde{s}_i$, individual noisy signal extraction from the price $\tilde{s}_{p,i}$, and individual signal $\tilde{z}_i$ which depends on the sophistication level. In particular, when choosing the sophistication level and the demand function, investor i conditions on the possible realizations of the price $\tilde{p}$.

2. Investors receive their private fundamental information $\tilde{s}_i$ and the signal $\tilde{z}_i$ according to the chosen sophistication level τ_{η_i}, the market clears according to the chosen demand function $D(\tilde{p}; \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i)$, and the price $\tilde{p}$ is realized.

3. Asset payoff $\tilde{V}$ is realized, and investors get paid and consume.

The overall equilibrium in our model is composed of a simultaneous trading and sophistication determination equilibrium. In the financial market subequilibrium, investors maximize their conditional subjective expected utilities and the asset market clears for given sophistication levels τ_{η_i}. This subequilibrium determines the price function and hence the best price signal $\tilde{s}_p$. In the sophistication determination subequilibrium, investors optimally choose their sophistication levels τ_{η_i} to maximize their ex-ante expected utilities taking into account equilibrium demands. In Sections 4, we will consider first a financial market equilibrium taking investors' sophistication level τ_{η_i} as given. In Section 5 we will deal with the overall equilibrium and the determination of sophistication levels.

4 Financial Market Equilibrium

Investors choose investments in assets to maximize their subjective expected utilities. They are price takers but still actively infer information from the price $\tilde{p}$, although adding individual noise in their inference process. Formally, investor i chooses investment D_i in the risky asset to maximize

$$E_i\left[-\exp\left(-\gamma \tilde{W}_i\right)\middle| \tilde{p}, \tilde{s}_i, \tilde{z}_i\right], \tag{5}$$

with her final wealth $\tilde{W}_i$ given by

$$\tilde{W}_i = (\tilde{V} - \tilde{p})D_i - C\left(\tau_{\eta_i}\right), \tag{6}$$

where we have normalized her initial wealth level at 0 and take τ_{η_i} as given.

The operator $E_i\left[\cdot|\,\tilde{p}, \tilde{s}_i, \tilde{z}_i\right]$ in (5) indicates that investor i takes expectation with respect to her own (subjective) belief. Specifically, investor i observes $\{\tilde{p}, \tilde{s}_i, \tilde{z}_i\}$ and needs to forecast her future wealth $\tilde{W}_i$. Since $\tilde{p}$ is in her information set, she takes $\tilde{p}$ as a known constant. Thus, in equation (6), the only random variable she faces is the fundamental $\tilde{V}$. When she predicts $\tilde{V}$, she extracts information from the price by interpreting $\tilde{p}$ as a signal $\tilde{s}_{p,i}$ according to (1). Endowed with signals $\{\tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i\}$, investor i is an SEU investor, and in particular, she is Bayesian. As a consequence, in investor i's mind, the fundamental $\tilde{V}$ follows a normal distribution conditional on $\{\tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i\}$.

The CARA-normal setting implies that investor i's demand for the risky asset is

$$D\left(\tilde{p}; \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i\right) = \frac{E(\tilde{V}|\tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i) - \tilde{p}}{\gamma Var(\tilde{V}|\tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i)}, \tag{7}$$

where $E(\tilde{V}|\tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i)$ and $Var(\tilde{V}|\tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i)$ are the conditional expectation and variance of $\tilde{V}$ given information $\{\tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i\}$. In (7), we have explicitly incorporated $\tilde{s}_{p,i}$ in the demand function to reflect the informational role of the price (i.e., the price helps to predict $\tilde{V}$) and used $\tilde{p}$ per se to capture the substitution role of the price (i.e., a higher price directly leads to a lower demand). Thus, the conditioning on the price in (7) is only used to gauge scarcity as with any other good but the learning on fundamentals is via the private signal $\tilde{s}_{p,i}$ or "price interpretation."

The financial market clears, i.e.,

$$\int_0^1 D\left(\tilde{p}; \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i\right) di = 0 \text{ almost surely.} \tag{8}$$

This market-clearing condition, together with demand function (7), will determine an equilibrium price function

$$\tilde{p} = p(\tilde{v}, \tilde{u}). \tag{9}$$

where $\tilde{v}$ and $\tilde{u}$ come from the aggregation of signals $\tilde{s}_i$, $\tilde{s}_{p,i}$, and $\tilde{z}_i$. In equilibrium, price function (9) will endogenously determine the informational content in the best signal $\tilde{s}_p$.

A financial market equilibrium for given sophistication levels $(\tau_{\eta_i})_{i\in[0,1]}$ is characterized by a price function $p(\tilde{v}, \tilde{u})$ and demand functions $D\left(\tilde{p}; \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i\right)$, such that:

1. $D(\tilde{p}; \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i)$ is given by (7), which maximizes investors' conditional subjective expected utilities given their beliefs;

2. The market clears almost surely, i.e., equation (8) holds;

3. Investors' beliefs are given by (1), (2), and (3), where $\tilde{s}_p$ in (1) is implied by the equilibrium price function $p(\tilde{v}, \tilde{u})$.

4.1 Equilibrium Construction

We consider a linear financial market equilibrium in which the price function takes the following form:

$$\tilde{p} = a_v \tilde{v} + a_u \tilde{u}, \tag{10}$$

where the coefficients a's are endogenous.

By equation (10), provided that $a_v \neq 0$, the price $\tilde{p}$ is equivalent to the following signal in predicting the asset fundamental $\tilde{v}$:

$$\tilde{s}_p = \tilde{v} + \alpha \tilde{u} \text{ with } \alpha \equiv a_u / a_v, \tag{11}$$

which would be the best signal that a fully sophisticated investor can achieve. However, as we mentioned in Section 3.1, investor i cannot fully understand the price function and she can only extract limited information from the price to the extent that she reads a coarser signal as follows:

$$\tilde{s}_{p,i} = \tilde{s}_p + \tilde{x}_i = (\tilde{v} + \alpha \tilde{u}) + (\tilde{u} + \tilde{e}_i) = \tilde{v} + (\alpha + 1)\, \tilde{u} + \tilde{e}_i, \tag{12}$$

where the second equality follows from equations (1) and (2). In other words, our investors add noise to the best signal that a fully sophisticated trader could obtain; that is, it adds noise in the inference process.

Recall that investor i can study market data to further purge the receiver noise $\tilde{x}_i$ in her personalized price signal $\tilde{s}_{p,i}$. This is represented by an access to the private signal $\tilde{z}_i$ in (3). By Bayes' rule, the two signals $\{\tilde{s}_{p,i}, \tilde{z}_i\}$ combine to generate the following signal $\tilde{s}_{pz,i}$

in predicting the fundamental $\tilde{v}$:

$$\begin{aligned}\tilde{s}_{pz,i} &\equiv \tilde{s}_{p,i}-\frac{\tau_\eta\left(\tau_e+\tau_u+\alpha\tau_e\right)}{\tau_e\tau_u+\tau_e\tau_{\eta_i}+\tau_u\tau_{\eta_i}}z_i\\ &= \tilde{v}+\left(\alpha+\frac{\tau_e\left(\tau_u-\alpha\tau_{\eta_i}\right)}{\tau_e\tau_u+\tau_e\tau_{\eta_i}+\tau_u\tau_{\eta_i}}\right)\tilde{u}\\ &\quad+\frac{\tau_e\left(\tau_u-\alpha\tau_{\eta_i}\right)}{\tau_e\tau_u+\tau_e\tau_{\eta_i}+\tau_u\tau_{\eta_i}}\tilde{e}_i-\frac{\tau_\eta\left(\tau_e+\tau_u+\alpha\tau_e\right)}{\tau_e\tau_u+\tau_e\tau_{\eta_i}+\tau_u\tau_{\eta_i}}\tilde{\eta}_i.\end{aligned}$$

The signal $\tilde{s}_{pz,i}$ summarizes the overall information that investor i can extract from the price after spending effort τ_{η_i}. It predicts $\tilde{v}$ with a precision given by

$$\tau_{p,i}=\frac{\tau_e\tau_u+\left(\tau_e+\tau_u\right)\tau_{\eta_i}}{\tau_u+\tau_e\left(\alpha+1\right)^2+\alpha^2\tau_{\eta_i}}. \tag{13}$$

Using Bayes' rule, we can compute

$$E(\tilde{V}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)=E\left(\tilde{v}|\tilde{s}_i,\tilde{s}_{pz,i}\right)=\beta_s\tilde{s}_i+\beta_p\tilde{s}_{p,i}+\beta_z\tilde{z}_i, \tag{14}$$

$$Var(\tilde{V}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)=\left(\tau_v+\tau_\varepsilon+\tau_{p,i}\right)^{-1}+\tau_\xi^{-1}, \tag{15}$$

where the coefficients β's are given in the appendix. Inserting these two expressions into (7), we can compute the expression of $D\left(\tilde{p};\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)$, which is in turn inserted into (8) to compute the equilibrium price as a function of $\tilde{v}$ and $\tilde{u}$. Comparing coefficients with the conjectured price function (10), we can form a system of equations to determine the two unknown price coefficients a_v and a_u.

In Section 5, we will show that, in the overall equilibrium, investors will endogenously choose the same sophistication level (i.e., $\tau_{\eta_i}=\tau_\eta$, $i\in[0,1]$), for any smooth, increasing, and weakly convex cost function $C\left(\tau_{\eta_i}\right)$ of sophistication. This is because investors face a strictly concave program and the optimal sophistication level is uniquely pinned down by the first-order condition. Under the condition $\tau_{\eta_i}=\tau_\eta$, the financial market equilibrium can be characterized as follows.

Proposition 1 (Financial market equilibrium) *Suppose that investors have the same sophistication level (i.e., $\tau_{\eta_i}=\tau_\eta$, $i\in[0,1]$). There exists a unique linear equilibrium price function,*

$$\tilde{p}=a_v\tilde{v}+a_u\tilde{u},$$

where

$$a_v = \frac{\tau_\varepsilon + \tau_p}{\tau_v + \tau_\varepsilon + \tau_p} \text{ and } a_u = \frac{\tau_p}{\tau_v + \tau_\varepsilon + \tau_p} \frac{\tau_u \left(\tau_e + \alpha \tau_e + \alpha \tau_\eta\right)}{\tau_e \tau_u + \tau_e \tau_\eta + \tau_u \tau_\eta},$$

where $\tau_p = \frac{\tau_e \tau_u + (\tau_e + \tau_u)\tau_\eta}{\tau_u + \tau_e (\alpha+1)^2 + \alpha^2 \tau_\eta}$ *and where* $\alpha \equiv \frac{a_u}{a_v} \in \left(0, \frac{\tau_e \tau_u}{\tau_e \tau_\varepsilon + \tau_e \tau_\eta + \tau_u \tau_\varepsilon}\right)$ *is uniquely determined by the positive real root of the following cubic equation:*

$$\left(\tau_e \tau_\varepsilon + \tau_\varepsilon \tau_\eta\right) \alpha^3 + 2\tau_e \tau_\varepsilon \alpha^2 + \left(\tau_e \tau_\varepsilon + \tau_e \tau_\eta + \tau_u \tau_\varepsilon\right) \alpha - \tau_e \tau_u = 0. \tag{16}$$

Note that in Proposition 1, we have $a_u > 0$ for any $\tau_\eta \in (0, \infty)$. That is, costly interpretation of asset prices brings an endogenous noise $\tilde{u}$ into the price system. As $\tau_\eta \to \infty$, investors become fully sophisticated and thus they can extract the best signal from the price. In this limiting case, the noise $\tilde{u}$ will vanish in the price function, which degenerates our economy to the full REE setup. It is worth noting that the full REE with $\tau_\eta = \infty$ is not implementable in demand functions.

Corollary 1 (Full REE) *Given* $(\tau_e, \tau_u, \tau_v, \tau_\xi, \tau_\varepsilon) \in \mathbb{R}^5_{++}$*, as* $\tau_\eta \to \infty$*, the price function converges almost surely to*

$$\tilde{p}^{REE} = \tilde{v}.$$

4.2 Implications of Investor Sophistication

In this subsection, we examine how investor sophistication affects asset prices, investor beliefs, and trading volume. We assume that all investors have a common sophistication level τ_η and conduct comparative static analysis with respect to τ_η. In a full equilibrium setting, an increase in τ_η corresponds to a decrease in some parameter that governs the cost function $C(\tau_\eta)$, which will be explored later in Section 5.

4.2.1 Price Informativeness and Asset Returns

Price informativeness As standard in the literature (e.g., Vives, 2008; Peress, 2010; Ozsoylev and Walden, 2011), we can use the precision $\frac{1}{Var(\tilde{V}|\tilde{p})}$ of stock payoff conditional on its price to measure "price informativeness" (or "market efficiency," "informational efficiency," and "price efficiency"). By equation (10), applying Bayes' rule delivers $\frac{1}{Var(\tilde{V}|\tilde{p})} = \left[\left(\tau_v + \alpha^{-2}\tau_u\right)^{-1} + \tau_\xi^{-1}\right]^{-1}$. Since τ_v, τ_u, and τ_ξ are exogenous constants, we can measure price informativeness inversely by α: a high value of α corresponds to a low value of price informativeness.

We can show that price informativeness increases with investor sophistication (i.e., α decreases with τ_η). Intuitively, when investors pay a lot of attention to study price data, they know well the true price signal $\tilde{s}_p$, and thus their trading brings less noise $\tilde{u}$ into the price. This complementarity result has important implications for determining the sophistication level in Section 5.

In the left two panels of Figure 1, we use solid curves to plot price informativeness α against investor sophistication τ_η. As a comparison, the dashed lines plot the α-value in the standard REE economy (i.e., $\alpha = 0$ for $\tau_\eta = \infty$). We have set the other values at $\tau_\varepsilon \in \{0.02, 2\}$ and $\tau_v = \tau_\xi = \tau_e = \tau_u = \gamma = 1$. In both panels, we observe (1) that costly interpretation of prices injects noise into the price as long as investors are not fully sophisticated (i.e., the solid curves lie above the dashed lines); and (2) that price informativeness increases with sophistication (i.e., the solid curves are downward sloping).

[INSERT FIGURE 1 HERE]

Return volatility Buying the asset at the market costs $\tilde{p}$ per share. Holding it till the end of the economy generates a payoff $\tilde{V}$. Hence, the asset return per share is $\tilde{V} - \tilde{p}$. Return volatility is measured by the standard deviation of asset returns, $\sigma(\tilde{V} - \tilde{p})$.

In the middle two panels of Figure 1, we plot return volatility $\sigma(\tilde{V} - \tilde{p})$ against investor sophistication τ_η with solid curves. Again, the dashed lines correspond to the value in the standard REE economy. We make the following two observations. First, costly interpretation of prices generates higher return volatility than the full REE benchmark (i.e., the solid curves lie above the dashed lines in both panels). This may help to address the volatility puzzle (Shiller, 1981; LeRoy and Porter, 1981), which states that it is difficult to explain the historical volatility of stock returns with any model in which investors are rational and discount rates are constant. Second, return volatility decreases with investor sophistication (i.e., the solid curves are downward sloping). This is because price informativeness increases with τ_η, which implies that sophistication makes the price $\tilde{p}$ closer to the fundamental $\tilde{V}$, driving down the equilibrium return volatility.

Return predictability We now examine whether and how asset returns $\tilde{V} - \tilde{p}$ can be predicted by the price $\tilde{p}$. Empirically, one can run a linear regression from $\tilde{V} - \tilde{p}$ on $\tilde{p}$, i.e., $\tilde{V} - \tilde{p} = intercept + m \times \tilde{p} + error$. The regression coefficient is $m = \frac{Cov(\tilde{V} - \tilde{p}, \tilde{p})}{Var(\tilde{p})}$. In the traditional noisy-REE setting with exogenous noise trading (e.g., Hellwig, 1980), returns exhibit reversals; that is, $m < 0$ (see Banerjee, Kaniel, and Kremer (2009)). This is because exogenous noise demand pushes the price too high and exogenous noisy supply depresses the price too low. In contrast, in our setting with endogenous noise trading due to the common error $\tilde{u}$ in price interpretation, returns exhibit momentum: $m > 0$. This provides an explanation for the price momentum documented in the data (e.g., Jegadeesh and Titman, 1993; Moskowitz, Ooi, and Pedersen, 2012).

The price momentum in our model is an underreaction story. When investors are fully sophisticated ($\tau_\eta = \infty$), the price fully aggregates their private information and there is no return predictability. Formally, by Corollary 1, the price is a martingale ($\tilde{p}^{REE} = E(\tilde{V}|\tilde{p}^{REE})$) and hence the price change is not predictable ($Cov(\tilde{V} - \tilde{p}^{REE}, \tilde{p}^{REE}) = 0$). When investors have limited sophistication, their forecasts do not fully use the information in the price, which in turn causes their trading not to fully incorporate information, thereby making the price underreact to information.

In the right two panels of Figure 1, we plot m against τ_η in solid curves, where the dashed lines still indicate the m-values in a standard REE model. In both panels, we observe that m is indeed positive, indicating that there exists price momentum in our economy. In addition, m can be hump-shaped or decreasing in τ_η, depending on the value of the precision τ_ε of private information. It is intuitive that m decreases with τ_η for large values of τ_η, since m eventually degenerates to 0 as τ_η approaches to infinity.

Figure 1 demonstrates that m can also increase with τ_η for small values of τ_η, which is true when investors have coarse private information (i.e., τ_ε is small). The intuition is as follows. When both τ_ε and τ_η are small, investors have little private information and read little information from the price. In equilibrium, the price is close to being a constant since it does not aggregate much information. This means that the price does not have much predictive power for future returns. Now if we increase τ_η, investors start to pay more attention to the price, and thus their trading starts to inject more information into the price, generating more predictability of asset returns.

The following proposition summarizes the results on price informativeness and asset returns.

Proposition 2 (Price informativeness, return volatility, and price momentum)

(a) *Price informativeness*

As investors become more sophisticated, the price $\tilde{p}$ conveys more precise information about the asset fundamental $\tilde{V}$. That is, $\frac{\partial \alpha}{\partial \tau_\eta} < 0$.

(b) *Return volatility*

(1) *As $\tau_\eta \to \infty$, return volatility approaches $\tau_\xi^{-1/2}$ (i.e., $\lim_{\tau_\eta \to \infty} \sigma(\tilde{V} - \tilde{p}) = \tau_\xi^{-1/2}$).*

(2) *As investor sophistication level τ_η increases, return volatility monotonically decreases if investors' fundamental information is sufficiently coarse or sufficiently precise (i.e., $\frac{\partial \sigma\left(\tilde{V} - \tilde{p}\right)}{\partial \tau_\eta} < 0$ if τ_ε is sufficiently small or sufficiently large).*

(c) *Price momentum*

(1) *When investors are not fully sophisticated, asset returns exhibit price momentum. When $\tau_\eta \to \infty$ there is no return predictability. That is, $m > 0$ for $\tau_\eta \in (0, \infty)$, and $\lim_{\tau_\eta \to \infty} m = 0$.*

(2) *When investors have sufficiently coarse fundamental information, price momentum m increases with investor sophistication τ_η at low values of τ_η, and price momentum m decreases with investor sophistication τ_η at high values of τ_η. When investors have sufficiently precise fundamental information, price momentum m monotonically decreases in investor sophistication τ_η.*

4.2.2 Investor Disagreement and Trading Volume

Disagreement We define investor disagreement as the dispersion across investors' expectations about the fundamental $\tilde{V}$, i.e.,

$$Disagreement \equiv \sqrt{Var\left(E(\tilde{V}|\tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i) - \bar{E}(\tilde{V})\right)}, \tag{17}$$

where

$$\bar{E}(\tilde{V}) \equiv \int_0^1 E(\tilde{V}|\tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i)di = E\left[E(\tilde{V}|\tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i)\,|\tilde{v}, \tilde{u}\right] \tag{18}$$

is the average expectation across investors.

In the two middle panels of Figure 2, we plot *Disagreement* against investor sophistication τ_η in solid curves. The other parameters take the same values as in Figure 1. The dashed lines still plot the values in the standard REE economy with $\tau_\eta = \infty$. By Corollary 1, when $\tau_\eta = \infty$, the price perfectly reveals the aggregate fundamental information, and so investors agree on asset valuation. As a result, $Disagreement = 0$ for $\tau_\eta = \infty$. Comparing the solid curves to dashed lines, we see that costly interpretation of prices adds disagreement among investors.

[INSERT FIGURE 2 HERE]

Disagreement can change with τ_η non-monotonically, depending on the precision τ_ε of investors' private fundamental information. This is due to two opposite forces. First, investors interpret the price in different ways, and so a higher τ_η means that investors' expectations rely more on their diverse information extracted from the price, thereby leading to a larger belief heterogeneity. Second, a higher τ_η implies that the price conveys more precise information about the asset fundamental (see Part (a) of Proposition 2), which tends to make investors' belief converge. In addition, since disagreement vanishes when $\tau_\eta = \infty$, it must be the case that the negative effect dominates for sufficiently large τ_η, so that *Disagreement* decreases with τ_η when τ_η is large. Nonetheless, when τ_η is small, the first positive effect can dominate as well. This possibility will arise when investors' private fundamental information is very coarse (i.e., τ_ε is small). Intuitively, starting from a small τ_ε, before accessing to market data, investors' beliefs are close to the prior and thus do not differ much from each other; after they see the price and interpret it differently, their opinions start to diverge. Taken together, when τ_ε is small, *Disagreement* is hump-shaped in τ_η. When τ_ε is large, *Disagreement* monotonically decreases with τ_η.

Trading volume To focus on the volume generated solely by different price interpretations, we assume that investors start with a zero initial position of risky assets. Therefore, the trading volume of investor i and the total trading volume are, respectively,

$$|D(\tilde{p};\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)| = \left|\frac{E(\tilde{V}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)-\tilde{p}}{\gamma Var(\tilde{V}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)}\right| \text{ and } Volume \equiv \int_0^1 |D(\tilde{p};\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)|\,di. \quad (19)$$

When all investors have the same sophistication level τ_η, they face the same variance in trading the risky asset, i.e.,

$$Risk \equiv Var(\tilde{V}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i) = \frac{1}{\tau_v+\tau_\varepsilon+\tau_p}+\frac{1}{\tau_\xi}, \quad (20)$$

where the second equality follows from equation (15). Hence, by the demand function (7) and the market-clearing condition (8), the equilibrium price is equal to the average expectation of investors,

$$\tilde{p} = \int_0^1 E(\tilde{V}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)di = \int_0^1 E(\tilde{v}|\,\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)\,di \equiv \bar{E}(\tilde{v}). \quad (21)$$

By equations (17)–(21), we can compute

$$Volume = \sqrt{\frac{2}{\pi}}\frac{Disagreement}{\gamma\times Risk}. \quad (22)$$

Thus, the total trading volume is jointly determined by three factors: investors' different expectations about the asset fundamental $\tilde{v}$, investors' risk aversion coefficient γ, and the risk faced by investors in trading the assets. When investors disagree more about the future fundamental $\tilde{V}$, they trade more and so the total trading volume is higher. When investors are less risk averse and when they perceive less risk in trading the assets, they also trade more aggressively, leading to a higher total trading volume.

Remark 2 (Hedging-motivated trade) *The assumption that investors start with no risky assets does not affect our result. Suppose instead that investor i is initially endowed with $\tilde{y}_i$ shares of risky asset, where $\tilde{y}_i \sim N(0,\sigma_y^2)$ is independently and identically distributed across investors. Our baseline model corresponds to a degenerate case of $\sigma_y = 0$. In this extended setting, we can compute that the total trading volume is given by*

$$Volume = \int_0^1 |D(\tilde{p};\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)-\tilde{y}_i|\,di = \sqrt{\frac{2}{\pi}}\frac{Disagreement}{\gamma\times Risk}+\sqrt{\frac{2}{\pi}}\sigma_y.$$

This expression differs from equation (22) only by a constant $\sqrt{\frac{2}{\pi}}\sigma_y$ that captures the volume generated by the endowment heterogeneity.

We continue our numerical example of Figure 2 and plot $Volume$ and $Risk$ against τ_η. We observe that $Risk$ decreases with τ_η. This is because more sophisticated investors glean more information from price data for two reasons. First, a higher sophistication level means that they study market data more intensively and can directly get more information from the price. Second, by Part (a) of Proposition 2, when all investors study the price more intensively, the price itself becomes a more informative signal (i.e., α decreases), and thus each investor can infer more information from the price. As $\tau_\eta \to \infty$, the price aggregates perfectly investors' private information and investors' perceived risk declines to $Var(\tilde{V}|\tilde{v}) = \tau_\xi^{-1}$.

The volume pattern mimics the disagreement pattern. First, comparing the solid curves to the dashed lines, we see that costly price interpretation generates excess trading volume. This result is consistent with the empirical evidence documented in the finance literature (e.g., Odean, 1999; Barber and Odean, 2000). Second, when investors have coarse fundamental information, $Volume$ is hump-shaped in τ_η. When investors have precise fundamental information, $Volume$ monotonically decreases with τ_η.

The literature has long been interested in the tripartite relation among opinion divergence, trading volume, and stock return volatility (e.g., Shalen, 1993). Figures 1 and 2 help to understand some documented empirical findings. First, Garfinkel (2009) constructs an order-based measure for investor opinion divergence and finds that volume is a better proxy for disagreement than return volatility. Garfinkel's disagreement measure is the simple daily standard deviation (across orders) of the distance between each order's requested price and the most recent trade price preceding that order. This measure can be viewed as a close empirical counterpart for our disagreement definition in (17): $E(\tilde{V}|\tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i)$ represents the investor's reservation value in the submitted order, and $\bar{E}(\tilde{V})$ is the prevailing price according to equation (21). Our results indeed show that volume mimics disagreement better than return volatility, in particular, when τ_ε is small: both volume and disagreement are hump-shaped in τ_η, while return volatility is decreasing in τ_η.

Second and relatedly, our results also help to reconcile the contradictory empirical findings on the cross-sectional relation between disagreement and return volatility. For instance, Frankel and Foot (1990) and Anderson, Ghysels, and Juergens (2005) document a positive relation, while Garfinkel (2009) documents a negative relation. In our setting, return volatility is always downward sloping in investor sophistication τ_η in Figure 1, but disagreement can exhibit a hump-shape in Figure 2. Thus, to the extent that τ_η is driving the cross-sectional variation, return volatility and disagreement can move in the same or opposite directions.

The following proposition summarizes our results on disagreement and volume.

Proposition 3 (Risk, disagreement, and trading volume)

(a) *Risk*

As investors become more sophisticated, investors perceive lower risk in trading (i.e., $\frac{\partial Risk}{\partial \tau_\eta} < 0$*). As* $\tau_\eta \to \infty$*, risk approaches* τ_ξ^{-1} *(i.e.,* $\lim_{\tau_\eta \to \infty} Risk = \tau_\xi^{-1}$*).*

(b) *Investor disagreement*

(1) *As* $\tau_\eta \to \infty$*, investor disagreement vanishes (i.e.,* $\lim_{\tau_\eta \to \infty} Disagreement = 0$*).*

(2) *When investors have coarse fundamental information, disagreement is hump-shaped in investor sophistication (i.e., for small values of* τ_ε*,* $\frac{\partial Disagreement}{\partial \tau_\eta} < 0$ *if and only if* τ_η *is sufficiently large). When investors have precise fundamental information, disagreement decreases monotonically with sophistication (i.e., for large values of* τ_ε*,* $\frac{\partial Disagreement}{\partial \tau_\eta} < 0$ *for all values of* τ_η*).*

(c) *Trading volume*

(1) *As* $\tau_\eta \to \infty$*, the total trading volume vanishes (i.e.,* $\lim_{\tau_\eta \to \infty} Volume = 0$*).*

(2) *When investors have coarse fundamental information, trading volume is hump-shaped in investor sophistication (i.e., for small values of* τ_ε*,* $\frac{\partial Volume}{\partial \tau_\eta} < 0$ *if and only if* τ_η *is sufficiently large). When investors have precise fundamental information, trading volume decreases monotonically with sophistication (i.e., for large values of* τ_ε*,* $\frac{\partial Volume}{\partial \tau_\eta} < 0$ *for all values of* τ_η*).*

5 Sophistication Level Equilibrium

5.1 Sophistication Determination

Inserting the expression of $D(\tilde{p}; \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i)$ in (7) into the objective function in (5), we can compute the indirect utility function of investor i. In anticipation of this indirect utility, investor i determines the attention level τ_{η_i} to maximize her expected utility before seeing the signals $\tilde{z}_i$ and $\tilde{s}_i$. When computing this conditional expected utility, we assume that investors can condition on the possible realizations of the price $\tilde{p}$, as when choosing the demand function, that is, τ_{η_i} is determined by

$$\max_{\tau_{\eta_i}} E_i \left[E_i \left(-\exp\left\{ -\gamma \left[(\tilde{V} - \tilde{p}) D\left(\tilde{p}; \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i\right) - C\left(\tau_{\eta_i}\right) \right] \right\} \middle| \tilde{p}, \tilde{s}_i, \tilde{z}_i \right) \middle| \tilde{p} \right], \tag{23}$$

where $E_i\left(\cdot|\tilde{p}, \tilde{s}_i, \tilde{z}_i\right)$ and $E_i\left(\cdot|\tilde{p}\right)$ again indicate conditional expectations under investor i's belief which interprets $\tilde{p}$ as a signal $\tilde{s}_{p,i}$ in predicting $\tilde{V}$.

Definition 1 *An overall equilibrium is jointly defined by the following two subequilibria:*

(a) *Financial market subequilibrium, which is characterized by a price function $p(\tilde{v}, \tilde{u})$ and demand functions $D\left(\tilde{p}; \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i\right)$, such that:*

(1) *$D\left(\tilde{p}; \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i\right)$ is given by (7), which maximizes investors' conditional subjective expected utilities given their beliefs;*

(2) *the market clears almost surely, i.e., equation (8) holds;*

(3) *investors' beliefs are given by (1), (2), and (3), where $\tilde{s}_p$ in (1) is implied by the equilibrium price function $p(\tilde{v}, \tilde{u})$ and where the sophistication levels $(\tau_{\eta_i})_{i\in[0,1]}$ are determined by the sophistication level subequilibrium.*

(b) *Sophistication level subequilibrium, which is characterized by sophistication levels $(\tau_{\eta_i})_{i\in[0,1]}$, such that τ_{η_i} solves (23), where investors' beliefs are given by (1)–(3), with $\tilde{s}_p$ in (1) being determined by the price function $p(\tilde{v}, \tilde{u})$ in the financial market subequilibrium.*

Remark 3 (Conditioning on prices) *We assume that when choosing τ_{η_i}, investors can condition on the possible realizations of the price $\tilde{p}$ in equation (23). Alternatively, we can assume that investors choose sophistication levels τ_{η_i} to maximize the unconditional subjective expected utility, i.e.,*

$$\max_{\tau_{\eta_i}} E_i \left[E_i \left(-\exp\left\{ -\gamma \left[(\tilde{V} - \tilde{p}) D\left(\tilde{p}; \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i\right) - C\left(\tau_{\eta_i}\right) \right] \right\} \middle| \tilde{p}, \tilde{s}_i, \tilde{z}_i \right) \right].$$

When computing this unconditional expected utility, we can specify that in investor i's mind, the price is distributed according to $\tilde{p} = f_0 + f_1 \tilde{s}_{p,i}$, where f_0 and f_1 are constants known to investors. The results are identical under this alternative approach.

5.2 Equilibrium Characterization

We now discuss how investors determine their sophistication levels τ_{η_i} in reading price data. By studying market data, investor i obtains a private signal $\tilde{z}_i$ with precision τ_{η_i} about the noise $\tilde{x}_i$ in her personalized price signal $\tilde{s}_{p,i}$ (see equation (3)). Investor i chooses τ_{η_i} to maximize her expected utility before observing $\tilde{z}_i$ and $\tilde{s}_i$ but conditional on the possible realizations of the price $\tilde{p}$.

Specifically, we need to average out $\tilde{z}_i$ and $\tilde{s}_i$ and compute $E_i\left[U\left(\tilde{p};\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)|\tilde{p}\right]$, where

$$U\left(\tilde{p};\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)\equiv E_i\left(-\exp\left\{-\gamma\left[(\tilde{V}-\tilde{p})D\left(\tilde{p};\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)-C\left(\tau_{\eta_i}\right)\right]\right\}\middle|\,\tilde{p},\tilde{s}_i,\tilde{z}_i\right)$$

is investor i's indirect value function. We can insert demand function (7) into the investor's objective function (5) and compute

$$U\left(\tilde{p};\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)=-\exp\left(-\frac{\left[E(\tilde{V}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)-\tilde{p}\right]^2}{2Var(\tilde{V}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)}+\gamma C\left(\tau_{\eta_i}\right)\right).$$

Using the above expression of $U\left(\tilde{p};\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)$, we have

$$E_i\left[U\left(\tilde{p};\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)|\tilde{p}\right]=E_i\left[-\exp\left(-\frac{\left[E(\tilde{V}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)-\tilde{p}\right]^2}{2Var(\tilde{V}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)}+\gamma C\left(\tau_{\eta_i}\right)\right)\middle|\,\tilde{p}\right]. \tag{24}$$

In computing the right-hand-side of (24), investor i will treat $\tilde{p}$ as a constant since her computation is conditional on $\tilde{p}$. In her mind, $E(\tilde{V}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)-\tilde{p}$ is normally distributed with mean and variance given respectively by

$$E_i\left[E(\tilde{V}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)-\tilde{p}|\tilde{p}\right] = E(\tilde{V}|\tilde{s}_{p,i})-\tilde{p}, \tag{25}$$

$$Var_i\left[E(\tilde{V}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)-\tilde{p}|\tilde{p}\right] = Var(\tilde{V}|\tilde{s}_{p,i})-Var(\tilde{V}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i), \tag{26}$$

where the equalities follow from the fact that investor i's beliefs satisfy the Bayes' law given that she is an SEU maximizer after extracting information from the price. In other words, $\tilde{p}$ in the above moment computations only serves its substitution role (i.e., as in demand for any other good, a higher price means a higher cost and so the agent will buy fewer goods), while when we think about the investor inferring information from the price, we always "translate" $\tilde{p}$ into the signal $\tilde{s}_{p,i}$ in terms of predicting $\tilde{V}$ to model this inference process. Taken together, in investor i's mind, she believes that $\left[E(\tilde{V}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)-\tilde{p}\right]^2$ follows a noncentral chi-square distribution in (24).

Using equations (25) and (26) and applying the moment generating function for a non-central chi-square distribution, we can compute

$$E_i\left[U\left(\tilde{p};\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)|\tilde{p}\right]=-\sqrt{\frac{Var(\tilde{V}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)}{Var(\tilde{V}|\tilde{s}_{p,i})}}\exp\left\{-\frac{\left[E(\tilde{V}|\tilde{s}_{p,i})-\tilde{p}\right]^2}{2Var(\tilde{V}|\tilde{s}_{p,i})}+\gamma C\left(\tau_{\eta_i}\right)\right\}.$$

We use B to denote the certainty equivalent of the above conditional expected utility, which refers to the net benefit of studying market data. That is,

$$\begin{aligned}B &\equiv -\frac{1}{\gamma}\log\left(-E_i\left[U\left(\tilde{p};\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)|\tilde{p}\right]\right)\\ &= -\frac{\log\left[Var(\tilde{V}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)\right]}{2\gamma}-C\left(\tau_{\eta_i}\right)+\frac{\log Var(\tilde{V}|\tilde{s}_{p,i})+\frac{\left[E(\tilde{V}|\tilde{s}_{p,i})-\tilde{p}\right]^2}{Var(\tilde{V}|\tilde{s}_{p,i})}}{2\gamma}. \end{aligned}\tag{27}$$

Note that the third term in (27) is independent of the choice variable τ_{η_i}, and so we ignore it and only retain the first two terms to represent B. Using equation (15) to express out $Var(\tilde{V}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)$ in (27) yields the following expression for the benefit B of studying price data:

$$B\left(\tau_{\eta_i};\alpha\right)\propto-\frac{1}{2\gamma}\log\left[\frac{1}{\tau_v+\tau_\varepsilon+\frac{\tau_e\tau_u+\tau_e\tau_{\eta_i}+\tau_u\tau_{\eta_i}}{\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_{\eta_i}}}+\frac{1}{\tau_\xi}\right]-C\left(\tau_{\eta_i}\right),\tag{28}$$

where we have explicitly expressed B as a function of τ_{η_i} and α, the unknowns in the overall equilibrium. Thus, each investor i's optimal sophistication level $\tau^*_{\eta_i}$ is determined by

$$\tau^*_{\eta_i}=\arg\max_{\tau_{\eta_i}}B\left(\tau_{\eta_i};\alpha\right)\tag{29}$$

where α is determined by equation (16) in Proposition 1. The overall equilibrium is jointly characterized by (29) (for $i\in[0,1]$) and (16), in terms of variables $\left(\tau^*_{\eta_i}\right)_{i\in[0,1]}$ and α^*.

For a given α, since $\lim_{\tau_{\eta_i}\to\infty}B\left(\tau_{\eta_i};\alpha\right)=-\infty$ for an increasing and weakly convex $C\left(\tau_{\eta_i}\right)$, we know that $B\left(\tau_{\eta_i};\alpha\right)$ has a maximum over the range of $\tau_{\eta_i}\geq 0$. It is also easy to check that $B\left(\tau_{\eta_i};\alpha\right)$ is strictly concave in τ_{η_i}, and thus the maximum is characterized by the first-order condition (FOC) as follows:

$$\begin{cases}\left.\frac{\partial B\left(\tau_{\eta_i};\alpha\right)}{\partial\tau_{\eta_i}}\right|_{\tau_{\eta_i}=0}\leq 0, & \text{if } \tau_{\eta_i}=0,\\ \frac{\partial B\left(\tau_{\eta_i};\alpha\right)}{\partial\tau_{\eta_i}}=0, & \text{if } \tau_{\eta_i}>0.\end{cases}\tag{30}$$

Note that each investor i faces the same problem in determining τ_{η_i}, and thus as we mentioned in Section 4, investors end up with the same choice of τ_{η_i}. That is, in equilibrium, we have $\tau^*_{\eta_i}=\tau^*_\eta$ for any $i\in[0,1]$. As a result, the overall equilibrium is characterized by the following two conditions in terms of two unknowns τ_η^* and α^*: (a) condition (30), the FOC in investors' sophistication decision problem which determines τ_η for a given α; and (b) equation (16) in Proposition 1, the financial market equilibrium condition which uniquely determines α for a given τ_η. Formally, we have the following proposition characterizing the overall equilibrium.

Proposition 4 (Overall equilibrium) *Suppose that $C(\tau_\eta)$ is smooth, increasing, and weakly convex. Then:*

(a) (Existence) *There exists an overall equilibrium and the equilibrium is symmetric. That is, in equilibrium, all investors choose the same sophistication level (i.e., $\tau_{\eta_i}^* = \tau_\eta^*$ for $i \in [0,1]$). The financial market equilibrium is given by Proposition 1 accordingly at the equilibrium sophistication level τ_η^*.*

(b) (Characterization) *Let*

$$\phi(\tau_\eta) \equiv \frac{\partial B(\tau_\eta; \alpha(\tau_\eta))}{\partial \tau_\eta},$$

where the function $\alpha(\tau_\eta)$ is implicitly determined by equation (16). The equilibrium sophistication level τ_η is determined by the following conditions:

(1) *If $\phi(0) \leq 0$, then $\tau_\eta^* = 0$ is an equilibrium sophistication level;*

(2) *If $\phi(\tau_\eta^*) = 0$ for some $\tau_\eta^* > 0$, then this value of τ_η^* is an equilibrium sophistication level.*

5.3 Complementarity, Multiplicity, and Market Fragility

Proposition 4 establishes the existence of an overall symmetric equilibrium. Whether the equilibrium is unique is determined by the shape of $\phi(\tau_\eta)$. Specifically, if $\phi(\tau_\eta)$ is downward sloping, then the equilibrium must be unique. In contrast, when $\phi(\tau_\eta)$ has an upward sloping segment, multiplicity can arise. The complementarity result in Part (a) of Proposition 2 has implications for this possibility of multiplicity, because it determines the shape of $\phi(\tau_\eta)$.

Formally, by the Chain rule, we have

$$\phi'(\tau_\eta) = \underbrace{\frac{\partial^2 B(\tau_\eta, \alpha(\tau_\eta))}{\partial \alpha \partial \tau_\eta}}_{(-,\text{ for large } \tau_\xi)} \times \underbrace{\frac{\partial \alpha}{\partial \tau_\eta}}_{\text{complementarity } (-)} + \underbrace{\frac{\partial^2 B(\tau_\eta; \alpha(\tau_\eta))}{\partial \tau_\eta^2}}_{\text{SOC of (29) } (-)}. \tag{31}$$

The second term in equation (31) is simply the second-order condition (SOC) of investors' sophistication determination problem (29), which is always negative given that the objective function $B(\tau_\eta; \alpha)$ is globally concave in τ_η. The term $\frac{\partial^2 B(\tau_\eta, \alpha(\tau_\eta))}{\partial \alpha \partial \tau_\eta}$ tends to be negative: when α increases, the price signal is not very useful, and thus its marginal value of being more attentive to price data is low. Then, combining with the complementarity result in Proposition 2 (i.e., $\frac{\partial \alpha}{\partial \tau_\eta} < 0$), we can have $\frac{\partial^2 B(\tau_\eta, \alpha(\tau_\eta))}{\partial \alpha \partial \tau_\eta} \frac{\partial \alpha}{\partial \tau_\eta} > 0$, which counter balances the second negative term in (31). This complementarity result can be so strong that it dominates so that $\phi'(\tau_\eta)$ can be upward sloping at some region, which admits multiple equilibria.

Proposition 5 (Multiplicity) *The complementarity effect can dominate so that there can be multiple overall equilibria.*

We prove Proposition 5 using a constructive example. In Figure 3, we choose a linear cost function, $C(\tau_\eta) = k\tau_\eta$ with $k > 0$. We then plot the function $\phi(\tau_\eta)$ in the top two panels. Similar to Figure 1, in Panel a1 of Figure 3, we set $\tau_\varepsilon = 0.02$, while in Panel a2 of Figure 3, we set $\tau_\varepsilon = 2$. All the other parameters in both panels are as follows: $\tau_v = \tau_\xi = \tau_e = \tau_u = \gamma = 1$ and $k = 0.02$. We find that in Panel a1, there exist three equilibrium levels of τ_η^*: $\{0, 0.08, 3.14\}$. Among these three equilibria, the middle one is unstable (i.e., $\phi(\tau_\eta)$ crosses zero from below), while the other two equilibria are stable. In Panel a2, there exists a unique equilibrium level of $\tau_\eta^* = 0.83$, which is stable.

[INSERT FIGURE 3 HERE]

This multiplicity result provides another source of market fragility in the sense that a small change in the market environment can cause a significant change in equilibrium outcomes. To illustrate this point, we use Panel b of Figure 3 to examine the implications of changing the sophistication cost in interpreting market data. Specifically, we now allow the cost parameter k to continuously change and plot the equilibrium values of τ_η^* against k. For instance, a decrease in k can be interpreted as an advance in computation technology that allows for easier implementation of complex algorithms. When there are multiple equilibria, we use dashed segments to indicate the unstable equilibrium. We see that as k decreases, τ_η^* increases as long as investors coordinate on a particular stable equilibrium (say, the one with a larger value of τ_η^*). This is intuitive: as the cost k of studying market data becomes lower, investors will devote more effort to study the price and become more sophisticated.

The multiplicity suggests that a slight change in k can lead to jumps in τ_η^*. For instance, suppose that investors coordinate on a stable equilibrium with a higher value of τ_η^*. Then, when k is close to 0.038, and when it drops slightly, the equilibrium value of τ_η^* can jump from 0 to 0.984. In reality, this outcome can correspond to a wave of development of algorithmic trading in financial markets, which is caused by technology progress. A natural experiment in this context is the introduction of automated quote dissemination on the New York Stock Exchange in 2003, which is studied by Hendershott, Jones, and Menkveld (2011). This change corresponds to an exogenous decrease in the cost k of processing market data.

6 Extension and Variation

6.1 Endogenous Fundamental Information Acquisition

In this subsection, we extend our baseline model by allowing investors to acquire both fundamental information and price information, so that we can study how the two types of information interact with each other.

Setup At the beginning of the economy, investor i now makes three simultaneous decisions on a precision level τ_{ε_i} of private fundamental information $\tilde{s}_i$, a sophistication level τ_{η_i} of studying market data, and a demand function $D(\tilde{p}; \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i)$. Acquiring precision τ_{ε_i} and τ_{η_i} incurs a cost according to a weakly increasing and convex function, $C\left(\tau_{\eta_i}, \tau_{\varepsilon_i}\right)$. The other features of the model remain unchanged. In particular, the financial market equilibrium is still characterized by Proposition 1.

With slight abuse of notation, we can follow the steps in deriving equation (28) and compute the benefit of studying market data and acquiring fundamental information as follows:

$$\begin{aligned} B\left(\tau_{\eta_i}, \tau_{\varepsilon_i}; \alpha\right) &\equiv -\frac{1}{\gamma} \log\left(-E_i\left[U\left(\tilde{p}; \tilde{s}_i, \tilde{s}_{p,i}, \tilde{z}_i\right)|\tilde{p}\right]\right) \\ &\propto -\frac{1}{2\gamma} \log\left[\frac{1}{\tau_v + \tau_{\varepsilon_i} + \frac{\tau_e\tau_u + \tau_e\tau_{\eta_i} + \tau_u\tau_{\eta_i}}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_{\eta_i}}} + \frac{1}{\tau_\xi}\right] - C\left(\tau_{\eta_i}, \tau_{\varepsilon_i}\right). \end{aligned}$$

Investor i's decisions on $\left(\tau_{\eta_i}, \tau_{\varepsilon_i}\right)$ are determined by $\left(\tau^*_{\eta_i}, \tau^*_{\varepsilon_i}\right) = \arg\max_{\left(\tau_{\eta_i}, \tau_{\varepsilon_i}\right)} B\left(\tau_{\eta_i}, \tau_{\varepsilon_i}; \alpha\right)$. The maximum is characterized by the two FOCs, i.e., the FOC with respect to τ_{η_i} (equation (30)) and the FOC with respect to τ_{ε_i}, given by

$$\begin{cases} \left. \frac{\partial B\left(\tau_{\eta_i}, \tau_{\varepsilon_i}; \alpha\right)}{\partial \tau_{\varepsilon_i}} \right|_{\tau_{\varepsilon_i}=0} \leq 0, & \text{if } \tau_{\varepsilon_i} = 0, \\ \frac{\partial B\left(\tau_{\eta_i}, \tau_{\varepsilon_i}; \alpha\right)}{\partial \tau_{\varepsilon_i}} = 0, & \text{if } \tau_{\varepsilon_i} > 0. \end{cases} \tag{32}$$

For a smooth cost function $C\left(\tau_{\eta_i}, \tau_{\varepsilon_i}\right)$, the objective function $B\left(\tau_{\eta_i}, \tau_{\varepsilon_i}; \alpha\right)$ is strictly concave in $\left(\tau_{\eta_i}, \tau_{\varepsilon_i}\right)$, which implies that $\tau^*_{\eta_i} = \tau^*_{\eta}$ and $\tau^*_{\varepsilon_i} = \tau^*_{\varepsilon}$ for $i \in [0, 1]$.

Thus, the overall equilibrium is characterized by the following three conditions in terms of three unknowns $\left(\tau^*_{\eta}, \tau^*_{\varepsilon}, \alpha^*\right)$: (a) condition (30), which determines τ_η for given $(\tau_\varepsilon, \alpha)$; (b) condition (32), which determines τ_ε for given (τ_η, α); and (c) equation (16), which uniquely determines α for given $(\tau_\eta, \tau_\varepsilon)$.

Complements and substitutes We examine two kinds of complements/substitutes, depending on whether the notions are defined on one investor's own actions (e.g., Admati and Pfleiderer, 1987) or on actions of other investors (e.g., Grossman and Stiglitz, 1980; Goldstein and Yang, 2015). The former focuses on the perspective of one investor. Specifically, the two types of information are complements (substitutes) if the marginal benefit of one investor to acquire one type of information increases (decreases) with the precision of the other type of information acquired by the same investor. In our setting, we can show that $\frac{\partial^2 B\left(\tau_{\eta_i}, \tau_{\varepsilon_i}; \alpha\right)}{\partial \tau_{\varepsilon_i} \partial \tau_{\eta_i}} < 0$, which implies that the two signals $\tilde{s}_i$ and $\tilde{z}_i$ are substitutes in terms of one investor's own actions. Intuitively, the reason that an investor studies market data is that she tries to extract payoff relevant information from asset prices; thus, if the investor already knows a lot of fundamental information, her incentive to study market data will weaken.

The second kind of complement/substitute notions focus more on the *strategic* interactions among investors, by studying how one investor's marginal benefit of acquiring a particular type of information is affected by other investors acquiring information. In our setting, acquiring fundamental information is a strategic substitute of acquiring any type of information. That is, $\frac{\partial^2 B\left(\tau_{\eta_i}, \tau_{\varepsilon_i}; \alpha\right)}{\partial \tau_{\eta} \partial \tau_{\varepsilon_i}} < 0$ and $\frac{\partial^2 B\left(\tau_{\eta_i}, \tau_{\varepsilon_i}; \alpha\right)}{\partial \tau_{\varepsilon} \partial \tau_{\varepsilon_i}} < 0$. This is because when all investors acquire more information of any type, the market price will convey more accurate information on asset payoff, which therefore lowers a particular investor's incentive to acquire fundamental information on her own, since she can read more information from the price.

However, acquiring price information can be a strategic complement of acquiring any type of information. This is true when residual uncertainty is low. That is, $\frac{\partial^2 B\left(\tau_{\eta_i},\tau_{\varepsilon_i};\alpha\right)}{\partial\tau_\eta\partial\tau_{\eta_i}} > 0$ and $\frac{\partial^2 B\left(\tau_{\eta_i},\tau_{\varepsilon_i};\alpha\right)}{\partial\tau_\varepsilon\partial\tau_{\eta_i}} > 0$ for sufficiently large values of τ_ξ. Intuitively, when all investors acquire more information of any type, the price will become a more accurate information source in predicting asset payoff, which makes it more attractive for any investor to study market data more intensively. Note that this strategic complementarity is identical to the one discussed in Section 5 for the baseline model (i.e., the first term in equation (31) is positive for large values of τ_ξ).

Proposition 6 (Complements/substitutes)

(a) (Own actions) *The two types of information are a substitute from one investor's perspective. That is,* $\frac{\partial^2 B\left(\tau_{\eta_i},\tau_{\varepsilon_i};\alpha\right)}{\partial\tau_{\eta_i}\partial\tau_{\varepsilon_i}} < 0$*.*

(b) (Cross investors)

(1) *Acquiring fundamental information is a strategic substitute of acquiring any type of information. That is,* $\frac{\partial^2 B\left(\tau_{\eta_i},\tau_{\varepsilon_i};\alpha\right)}{\partial\tau_\eta\partial\tau_{\varepsilon_i}} < 0$ *and* $\frac{\partial^2 B\left(\tau_{\eta_i},\tau_{\varepsilon_i};\alpha\right)}{\partial\tau_\varepsilon\partial\tau_{\varepsilon_i}} < 0$*.*

(2) *When the asset's residual uncertainty is sufficiently low, studying market data is a strategic complement of acquiring any type of information (i.e., when* τ_ξ *sufficiently large,* $\frac{\partial^2 B\left(\tau_{\eta_i},\tau_{\varepsilon_i};\alpha\right)}{\partial\tau_\eta\partial\tau_{\eta_i}} > 0$ *and* $\frac{\partial^2 B\left(\tau_{\eta_i},\tau_{\varepsilon_i};\alpha\right)}{\partial\tau_\varepsilon\partial\tau_{\eta_i}} > 0$*). When residual uncertainty is high, studying market data can be a strategic complement or a strategic substitute of acquiring any type of information (i.e., when* τ_ξ *is sufficiently small, the signs of* $\frac{\partial^2 B\left(\tau_{\eta_i},\tau_{\varepsilon_i};\alpha\right)}{\partial\tau_\eta\partial\tau_{\eta_i}}$ *and* $\frac{\partial^2 B\left(\tau_{\eta_i},\tau_{\varepsilon_i};\alpha\right)}{\partial\tau_\varepsilon\partial\tau_{\eta_i}}$ *are ambiguous).*

Effects of information acquisition costs We now use Figure 4 to conduct a comparative static analysis with respect to information acquisition costs. Unlike Figure 3, we here assume the cost function takes a quadratic form: $C\left(\tau_{\eta_i},\tau_{\varepsilon_i}\right) = \frac{\kappa_\eta\tau_{\eta_i}^2+\kappa_\varepsilon\tau_{\varepsilon_i}^2}{2}$, where $\kappa_\eta > 0$ and $\kappa_\varepsilon > 0$. This assumption is made for two reasons. First, it rules out corner solutions, so that the equilibrium can be characterized by a system of three equations. Second, the strict convexity of $C\left(\tau_{\eta_i},\tau_{\varepsilon_i}\right)$ tends to make the equilibrium unique. Intuitively, relative to a linear cost function, a quadratic cost function makes the second term in equation (31) larger in absolute values.

[INSERT FIGURE 4 HERE]

In the top panels of Figure 4, we plot the equilibrium values of $\left(\tau_{\eta}^{*}, \tau_{\varepsilon}^{*}, \alpha^{*}\right)$ against the cost parameter κ_{η} of acquiring sophistication, while fix the cost parameter κ_{ε} of acquiring fundamental information at 0.02. In the bottom panels of Figure 4, we plot $\left(\tau_{\eta}^{*}, \tau_{\varepsilon}^{*}, \alpha^{*}\right)$ against κ_{ε} while fixing $\kappa_{\eta} = 0.02$. All the other parameter values are set at 1. We observe that as we increase the cost of learning one type of information, investors acquire less of that type but more of the other type of information. Overall, price informativeness decreases.

Proposition 6 helps to understand these results. Take the top panels as an example. When κ_{η} increases, intuitively, all investors, including investor i, acquire less price information. That is, $\tau_{\eta_i}^{*}$ and τ_{η}^{*} decrease in Panel a1. This will cause investor i to acquire more fundamental information in Panel a2, because acquiring fundamental information is a substitute of studying market data, both from one investor's perspective and from the cross-investor perspective. The pattern of price informativeness in Panel a3 is driven by the decreased τ_{η}^{*}.

6.2 Two Types of Investors

In this subsection, we consider a variation of the baseline model and show that our main results are robust.

Setup Investors are still endowed with a private fundamental signal $\tilde{s}_i$, and their initial interpretation about the price is still represented by signal $\tilde{s}_{p,i}$. Now we divide investors into two groups: sophisticated (with a measure μ) and unsophisticated (with a measure $1-\mu$). Sophisticated investors can completely purge out the receiver noise $\tilde{x}_i$ and thus have access to the best price signal $\tilde{s}_p$. Unsophisticated investors still keep interpreting price information as $\tilde{s}_{p,i}$. Becoming sophisticated incurs a fixed cost $c > 0$. This economy corresponds to our general setting with a step cost function, $C\left(\tau_{\eta_i}\right) = 0$ for $\tau_{\eta_i} = 0$, and $C\left(\tau_{\eta_i}\right) = c$ for $\tau_{\eta_i} \in (0, \infty]$. Sophisticated investors choose an infinite sophistication level, while unsophisticated investors choose a zero sophistication level (i.e., $\tau_{\eta_i} = \infty$ for

$i \in [0, \mu]$, and $\tau_{\eta_i} = 0$ for $i \in (\mu, 1]$). All the other features of the model remain unchanged. This two-type setting matches well the original DSSW (1990) setup with noise traders and arbitrageurs.

Financial market equilibrium The price function is still given by equation (10). The CARA-normal setting implies that sophisticated investor i's demand for the risky asset is

$$D_S(\tilde{p}; \tilde{s}_i, \tilde{s}_p) = \frac{E(\tilde{V}|\tilde{s}_i, \tilde{s}_p) - \tilde{p}}{\gamma Var(\tilde{V}|\tilde{s}_i, \tilde{s}_p)},$$

where by Bayes' rule, we have

$$\begin{aligned} E(\tilde{V}|\tilde{s}_i, \tilde{s}_p) &= \beta_s^S \tilde{s}_i + \beta_p^S \tilde{s}_p, \\ Var(\tilde{V}|\tilde{s}_i, \tilde{s}_p) &= \frac{1}{\tau_v + \tau_\varepsilon + \alpha^{-2}\tau_u} + \frac{1}{\tau_\xi}, \end{aligned}$$

with

$$\beta_s^S = \frac{\tau_\varepsilon}{\tau_v + \tau_\varepsilon + \alpha^{-2}\tau_u} \text{ and } \beta_p^S = \frac{\alpha^{-2}\tau_u}{\tau_v + \tau_\varepsilon + \alpha^{-2}\tau_u}.$$

Similarly, we can compute the demand for the risky asset of an unsophisticated investor i as follows:

$$D_U(\tilde{p}; \tilde{s}_i, \tilde{s}_{p,i}) = \frac{E(\tilde{V}|\tilde{s}_i, \tilde{s}_{p,i}) - \tilde{p}}{\gamma Var(\tilde{V}|\tilde{s}_i, \tilde{s}_{p,i})},$$

where

$$\begin{aligned} E(\tilde{V}|\tilde{s}_i, \tilde{s}_{p,i}) &= \beta_s^U \tilde{s}_i + \beta_p^U \tilde{s}_{p,i}, \\ Var(\tilde{V}|\tilde{s}_i, \tilde{s}_{p,i}) &= \frac{1}{\tau_v + \tau_\varepsilon + \frac{\tau_e\tau_u}{\tau_e + \tau_u + 2\alpha\tau_e + \alpha^2\tau_e}} + \frac{1}{\tau_\xi}, \end{aligned}$$

with

$$\beta_s^U = \frac{\tau_\varepsilon}{\tau_v + \tau_\varepsilon + \frac{\tau_e\tau_u}{\tau_e + \tau_u + 2\alpha\tau_e + \alpha^2\tau_e}} \text{ and } \beta_p^U = \frac{\frac{\tau_e\tau_u}{\tau_e + \tau_u + 2\alpha\tau_e + \alpha^2\tau_e}}{\tau_v + \tau_\varepsilon + \frac{\tau_e\tau_u}{\tau_e + \tau_u + 2\alpha\tau_e + \alpha^2\tau_e}}.$$

Inserting the above demand functions into the following market clearing condition,

$$\int_0^\mu D_S(\tilde{p}; \tilde{s}_i, \tilde{s}_p)\, di + \int_0^{1-\mu} D_U(\tilde{p}; \tilde{s}_i, \tilde{s}_{p,i})\, di = 0,$$

we can compute the implied price function, which is in turn compared with the conjectured price function, yielding the following fifth-order polynomial of α that determines the financial market equilibrium:

$$F(\alpha; \mu) = A_5\alpha^5 + A_4\alpha^4 + A_3\alpha^3 + A_2\alpha^2 + A_1\alpha + A_0 = 0,$$

where

$$A_5 = -\tau_e \tau_\varepsilon (\tau_v + \tau_\xi + \tau_\varepsilon), A_4 = -2\tau_e \tau_\varepsilon (\tau_v + \tau_\xi + \tau_\varepsilon),$$

$$A_3 = -\tau_\varepsilon (\tau_e \tau_u + \tau_e \tau_v + \tau_u \tau_v + \tau_e \tau_\xi + \tau_e \tau_\varepsilon + \tau_u \tau_\xi + \tau_u \tau_\varepsilon),$$

$$A_2 = (1-\mu) \tau_e \tau_u (\tau_v + \tau_\xi - \tau_\varepsilon),$$

$$A_1 = -(1-\mu) \tau_u \tau_\varepsilon (\tau_e + \tau_u), \text{ and } A_0 = (1-\mu) \tau_e \tau_u^2.$$

It is clear that there always exists a solution to $F(\alpha; \mu) = 0$, and hence existence is established. After we compute α, the price coefficients are as follows:

$$a_v = \frac{\mu \frac{\beta_s^S + \beta_p^S}{Var(\tilde{V}|\tilde{s}_i, \tilde{s}_p)} + (1-\mu) \frac{\beta_s^U + \beta_p^U}{Var(\tilde{V}|\tilde{s}_i, \tilde{s}_{p,i})}}{\frac{\mu}{Var(\tilde{V}|\tilde{s}_i, \tilde{s}_p)} + \frac{1-\mu}{Var(\tilde{V}|\tilde{s}_i, \tilde{s}_{p,i})}},$$

$$a_u = \frac{\mu \frac{\beta_p^S \alpha}{Var(\tilde{V}|\tilde{s}_i, \tilde{s}_p)} + (1-\mu) \frac{\beta_p^U (\alpha+1)}{Var(\tilde{V}|\tilde{s}_i, \tilde{s}_{p,i})}}{\frac{\mu}{Var(\tilde{V}|\tilde{s}_i, \tilde{s}_p)} + \frac{1-\mu}{Var(\tilde{V}|\tilde{s}_i, \tilde{s}_{p,i})}}.$$

Sophistication equilibrium We can follow the computation as in the main text and find the net benefit of becoming sophisticated as follows:

$$\Pi(\mu; \alpha) = \frac{1}{2\gamma} \log \left[\frac{Var(\tilde{V}|\tilde{s}_i, \tilde{s}_{p,i})}{Var(\tilde{V}|\tilde{s}_i, \tilde{s}_p} \right] - c.$$

In this setup, the fraction μ of sophisticated investors serves the same role as the sophistication level τ_η in the baseline model. Function $\Pi(\mu; \alpha)$ corresponds to the marginal benefit function $\frac{\partial B(\tau_{\eta_i}; \alpha)}{\partial \tau_{\eta_i}}$ in the main text.

Note that in equilibrium, α is implicitly determined by μ through $F(\alpha; \mu) = 0$, the condition characterizing the financial market equilibrium. That is, $\alpha = \alpha(\mu)$. Inserting this implicit function into $\Pi(\mu; \alpha)$ yields the following function,

$$\Phi(\mu) \equiv \Pi(\mu; \alpha(\mu)),$$

which is the counterpart of $\phi(\tau_\eta)$ in the main text. The equilibrium fraction μ^* is defined by the following three conditions:

(a) If $\Phi(0) \leq 0$, then $\mu^* = 0$ is an equilibrium fraction of sophisticated investors;

(b) If $\Phi(1) \geq 0$, then $\mu^* = 1$ is an equilibrium fraction of sophisticated investors; and

(c) If $\Phi(\mu^*) = 0$ for some $\mu^* \in (0,1)$, then μ^* constitutes an interior equilibrium fraction of sophisticated investors.

Results We now show that our main qualitative results continue to hold in this variant model. We use Figure 5 to plot the following equilibrium outcomes against the fraction μ of sophisticated investors: price informativeness measure α, return volatility $\sigma(\tilde{v}-\tilde{p})$, price momentum $m = \frac{Cov(\tilde{v}-\tilde{p},\tilde{p})}{Var(\tilde{p})}$, average aggregate trading volume $E(Volume)$, and the function $\Phi(\mu)$ of becoming sophisticated. The other parameters are fixed at $\tau_v = \tau_\xi = \tau_e = \tau_u = \tau_\varepsilon = \gamma = 1$ and $c = 0$. The standard REE corresponds to the economy with $\mu = 1$. The values in standard REE are ploted in dashed lines.

[INSERT FIGURE 5 HERE]

We make the following observations that confirm the main results in our baseline model. First, compared to the standard REE (with $\mu = 1$), costly price interpretation injects noise into the price (i.e., $\alpha > 0$ for $\mu < 1$), generates excess return volatility and trading volume (i.e., $\sigma(\tilde{v}-\tilde{p}) > \sqrt{\frac{1}{\tau_\xi}}$ and $E(Volume) > 0$ for $\mu < 1$), and leads to price momentum (i.e., $m > 0$ for $\mu < 1$). Second, studying market data can exhibit strategic complementarity (i.e., $\Phi(\mu)$ can be upward sloping for a certain range of μ). This can lead to multiple equilibrium values of μ^*, depending on the choice of c. Third, the presence of more sophisticated investors increase price informativeness and lower return volatility (i.e., α and $\sigma(\tilde{v}-\tilde{p})$ are decreasing in μ).

7 Conclusion

We construct a model to capture the notion that investors have to spend effort to interpret price data in financial markets. In our model, investors actively infer information from the price but their information processing is noisy. Investors can reduce the processing noise by spending more resources to study market data and become more sophisticated. We still maintain the assumption that investors are individually Bayesian rational—i.e., after reading price data and form their beliefs, investors hold optimal trading positions according to their own beliefs (and so they are only boundedly rational in extracting information from the price). We find that imperfect price interpretation can inject noise into the price system. Compared to the standard REE, our model generates price momentum, excessive return volatility, and excessive trading volume. As investor sophistication increases, return volatility decreases, while disagreement and volume can exhibit a hump shape. Finally, we endogenize investors' sophistication level using a learning technology and find that studying market data exhibits strategic complementarity that can lead to multiple equilibria.

Appendix: Proofs

Proof of Proposition 1

Using Bayes' rule, we can compute

$$E\left(\tilde{v}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i\right)=E(\tilde{v}|\tilde{s}_i,\tilde{s}_{pz,i})=\beta_s\tilde{s}_i+\beta_p\tilde{s}_{p,i}+\beta_z\tilde{z}_i,$$

where

$$\begin{aligned}\beta_s &= \frac{\tau_\varepsilon}{\tau_v+\tau_\varepsilon+\tau_p},\\ \beta_p &= \frac{\tau_p}{\tau_v+\tau_\varepsilon+\tau_p},\\ \beta_z &= -\frac{\tau_\eta\left(\tau_e+\tau_u+\alpha\tau_e\right)}{\tau_e\tau_u+\tau_e\tau_\eta+\tau_u\tau_\eta}\frac{\tau_p}{\tau_v+\tau_\varepsilon+\tau_p}.\end{aligned}$$

Note that when $\tau_{\eta_i}=\tau_\eta$, all investors have the same conditional variance $Var(\tilde{V}|\tilde{s}_i,\tilde{s}_{p,i},\tilde{z}_i)$. Thus, using the demand function and the market clearing condition, we can show that

$$\tilde{p}=\int_0^1 E(\tilde{v}|\tilde{s}_i,\tilde{s}_{pz,i})di=E\left[E(\tilde{v}|\tilde{s}_i,\tilde{s}_{pz,i})\,|\tilde{v},\tilde{u}\right].$$

Inserting the expression of $E(\tilde{v}|\tilde{s}_i,\tilde{s}_{pz,i})$ and comparing the coefficients of the conjectured price function (10), we have:

$$a_v=\beta_s+\beta_p \text{ and } a_u=\beta_p\left(\alpha+1\right)+\beta_z.$$

Plugging in the expressions of β's into the above two conditions leads to the expressions of a's in Proposition 1.

Inserting the expressions of a's into $\alpha=\frac{a_u}{a_v}$ and simplifying yield to the cubic (16) that determines the value of α. Denote the LHS of (16) by $f\left(\alpha\right)$. That is,

$$f\left(\alpha\right)\equiv\left(\tau_e\tau_\varepsilon+\tau_\varepsilon\tau_\eta\right)\alpha^3+2\tau_e\tau_\varepsilon\alpha^2+\left(\tau_e\tau_\varepsilon+\tau_e\tau_\eta+\tau_u\tau_\varepsilon\right)\alpha-\tau_e\tau_u.$$

We can compute $f\left(0\right)=-\tau_e\tau_u<0$ and $f\left(\frac{\tau_e\tau_u}{\tau_e\tau_\varepsilon+\tau_e\tau_\eta+\tau_u\tau_\varepsilon}\right)>0$, and thus by the intermediate value theorem, there exists a solution $\alpha\in\left(0,\frac{\tau_e\tau_u}{\tau_e\tau_\varepsilon+\tau_e\tau_\eta+\tau_u\tau_\varepsilon}\right)$ such that $f\left(\alpha\right)=0$. This result establishes the existence of a financial market equilibrium.

We can compute the discriminant of the cubic $f\left(\alpha\right)$ as follows:

$$\Delta = -\tau_\varepsilon \left(\begin{array}{c} 4\tau_e^3\tau_\eta^4 + 4\tau_e^4\tau_\eta^3 + 4\tau_e\tau_u^3\tau_\varepsilon^3 + 4\tau_e^3\tau_u\tau_\varepsilon^3 + 4\tau_e^4\tau_u\tau_\varepsilon^2 + 27\tau_e^4\tau_u^2\tau_\varepsilon \\ +12\tau_e^3\tau_\varepsilon\tau_\eta^3 + 4\tau_e^3\tau_\varepsilon^3\tau_\eta + 8\tau_e^4\tau_\varepsilon\tau_\eta^2 + 4\tau_e^4\tau_\varepsilon^2\tau_\eta + 4\tau_u^3\tau_\varepsilon^3\tau_\eta + 8\tau_e^2\tau_u^2\tau_\varepsilon^3 \\ +36\tau_e^3\tau_u^2\tau_\varepsilon^2 + 12\tau_e^3\tau_\varepsilon^2\tau_\eta^2 + 12\tau_e\tau_u^2\tau_\varepsilon^2\tau_\eta^2 + 24\tau_e^2\tau_u\tau_\varepsilon^2\tau_\eta^2 + 27\tau_e^2\tau_u^2\tau_\varepsilon\tau_\eta^2 \\ +48\tau_e^2\tau_u^2\tau_\varepsilon^2\tau_\eta + 36\tau_e^4\tau_u\tau_\varepsilon\tau_\eta + 12\tau_e\tau_u^2\tau_\varepsilon^3\tau_\eta + 12\tau_e^2\tau_u\tau_\varepsilon\tau_\eta^3 + 12\tau_e^2\tau_u\tau_\varepsilon^3\tau_\eta \\ +48\tau_e^3\tau_u\tau_\varepsilon\tau_\eta^2 + 52\tau_e^3\tau_u\tau_\varepsilon^2\tau_\eta + 54\tau_e^3\tau_u^2\tau_\varepsilon\tau_\eta \end{array} \right),$$

which is negative. Thus, there exists a unique real root, which establishes the uniqueness of a financial market equilibrium. QED.

Proof of Corollary 1

Lemma 1 *Given $(\tau_e, \tau_u, \tau_\varepsilon) \in \mathbb{R}^3_{++}$ and let $\tau_\eta \to \infty$. We have:*
(a) $\alpha\tau_\eta^{-1}$ is bounded; (b) $\alpha = O\left(\tau_\eta^{-1}\right) \to 0$; and (c) $\alpha^2\tau_\eta = O\left(\tau_\eta^{-1}\right) \to 0$.

Proof. By the bounds of α in Proposition 1, we have

$$\begin{aligned} 0 &< \alpha < \frac{\tau_e\tau_u}{\tau_e\tau_\varepsilon + \tau_e\tau_\eta + \tau_u\tau_\varepsilon} \\ &\Rightarrow 0 < \alpha\tau_\eta < \frac{\tau_e\tau_u}{\tau_e\tau_\varepsilon + \tau_e\tau_\eta + \tau_u\tau_\varepsilon}\tau_\eta = \tau_u - \frac{\tau_u\tau_\varepsilon\left(\tau_e + \tau_u\right)}{\tau_e\tau_\varepsilon + \tau_e\tau_\eta + \tau_u\tau_\varepsilon} < \tau_u \\ &\Rightarrow \alpha\tau_\eta = O\left(1\right) \text{ (i.e., } \alpha\tau_\eta \text{ is finite).} \end{aligned}$$

Parts (b) and (c) follows directly from Part (a). ■

By Lemma 1, we have

$$\tau_p = \frac{\tau_e\tau_u + \tau_e\tau_\eta + \tau_u\tau_\eta}{\tau_u + \tau_e\left(\alpha + 1\right)^2 + \alpha^2\tau_\eta} \propto \tau_\eta.$$

By the expressions of a_v and a_u in Proposition 1, we have

$$a_v \propto \frac{\tau_\varepsilon + \tau_\eta}{\tau_v + \tau_\varepsilon + \tau_\eta} \to 1 \text{ and } a_u = a_v\alpha \to 0.$$

QED.

Proof of Proposition 2

Part (a) Price informativeness By the proof for Proposition 1, we know that α is determined by $f\left(\alpha\right) = 0$. Using the implicit function theorem, we can compute:

$$\frac{\partial\alpha}{\partial\tau_\eta} = -\frac{\tau_\varepsilon\alpha^3 + \tau_e\alpha}{3\left(\tau_e\tau_\varepsilon + \tau_\varepsilon\tau_\eta\right)\alpha^2 + 4\tau_e\tau_\varepsilon\alpha + \left(\tau_e\tau_\varepsilon + \tau_e\tau_\eta + \tau_u\tau_\varepsilon\right)} < 0. \tag{A1}$$

Part (b) Return volatility Using Proposition 1, we can compute

$$\sigma(\tilde{V}-\tilde{p})=\sqrt{Var\left(\tilde{v}-\tilde{p}\right)+\tau_{\xi}^{-1}}$$

where

$$Var\left(\tilde{v}-\tilde{p}\right)=\frac{\tau_v+\left[\frac{(1+\alpha)\tau_e+\alpha\tau_\eta}{\tau_u+(1+\alpha)^2\tau_e+\alpha^2\tau_\eta}\right]^2\tau_u}{\left(\tau_v+\tau_\varepsilon+\frac{\tau_e\tau_u+\tau_e\tau_\eta+\tau_u\tau_\eta}{\tau_u+(1+\alpha)^2\tau_e+\alpha^2\tau_\eta}\right)^2}. \tag{A2}$$

Part (b1) simply follows combining Lemma 1 and the above expression of $Var\left(\tilde{v}-\tilde{p}\right)$. In order to prove Part (b2), we establish the following lemma.

Lemma 2 *Given $(\tau_e,\tau_u,\tau_\eta)\in\mathbb{R}^3_{++}$. (a) $\lim_{\tau_\varepsilon\to 0}\alpha=\frac{\tau_u}{\tau_\eta}$ and $\lim_{\tau_\varepsilon\to 0}\frac{\partial\alpha}{\partial\tau_\eta}=-\frac{\tau_u}{\tau_\eta^2}$. (b) As $\tau_\varepsilon\to\infty$, we have $\alpha=O\left(\tau_\varepsilon^{-1}\right)\to 0$ and $\frac{\partial\alpha}{\partial\tau_\eta}\propto-\frac{\tau_e\alpha}{(\tau_e+\tau_u)\tau_\varepsilon}\to 0$.*

Proof. (a) By the proof for Proposition 1, we know that α is determined by $f\left(\alpha\right)=0$, where $f\left(\alpha\right)$ crosses 0 from below. As τ_ε increases, $f\left(\alpha\right)$ shifts upward. Since f crosses zero from below, we know that α decreases with τ_ε. So, α is bounded as τ_ε goes to 0. By (16), we know that as $\tau_\varepsilon\to 0$,

$$\left(\tau_e\tau_\varepsilon+\tau_e\tau_\eta+\tau_u\tau_\varepsilon\right)\alpha\propto\tau_e\tau_u\Rightarrow\alpha\propto\frac{\tau_u}{\tau_\eta}.$$

Inserting $\alpha\propto\frac{\tau_u}{\tau_\eta}$ into the expression of $\frac{\partial\alpha}{\partial\tau_\eta}$ in equation (A1), we can show $\frac{\partial\alpha}{\partial\tau_\eta}\propto-\frac{\tau_e\alpha}{(\tau_e+\tau_u)\tau_\varepsilon}$.

(b) Let $\tau_\varepsilon\to\infty$. By Proposition 1,

$$\begin{aligned}\alpha &\in \left(0,\frac{\tau_e\tau_u}{\tau_e\tau_\varepsilon+\tau_e\tau_\eta+\tau_u\tau_\varepsilon}\right)\Rightarrow\\ 0 &< \lim_{\tau_\varepsilon\to\infty}\alpha\tau_\varepsilon<\lim_{\tau_\varepsilon\to\infty}\frac{\tau_e\tau_u}{\tau_e\tau_\varepsilon+\tau_e\tau_\eta+\tau_u\tau_\varepsilon}\tau_\varepsilon=\frac{\tau_e\tau_u}{\tau_e+\tau_u}\\ &\Longrightarrow \lim_{\tau_\varepsilon\to\infty}\alpha\tau_\varepsilon\text{ is finite}\Rightarrow\alpha=O\left(\tau_\varepsilon^{-1}\right).\end{aligned}$$

Inserting $\alpha=O\left(\tau_\varepsilon^{-1}\right)$ into the expression of $\frac{\partial\alpha}{\partial\tau_\eta}$ in equation (A1), we can show $\frac{\partial\alpha}{\partial\tau_\eta}\propto-\frac{\tau_e\alpha}{(\tau_e+\tau_u)\tau_\varepsilon}=-O(\tau_\varepsilon^{-2})$. ■

To show Part (b2) of Proposition 2, we first use equation (A2) to directly compute the derivative of $\frac{\partial\log Var(\tilde{v}-\tilde{p})}{\partial\tau_\eta}$, and then combine with Lemma 2 to show that both $\lim_{\tau_\varepsilon\to 0}\frac{\partial\log Var(\tilde{v}-\tilde{p})}{\partial\tau_\eta}<0$ and $\lim_{\tau_\varepsilon\to\infty}\frac{\partial\log Var(\tilde{v}-\tilde{p})}{\partial\tau_\eta}<0$. For instance, using Part (a) of Lemma 2, we can compute: $\lim_{\tau_\varepsilon\to 0}\frac{\partial\log Var(\tilde{v}-\tilde{p})}{\partial\tau_\eta}=-\frac{2\tau_\eta\left(\tau_u^2\tau_v+\tau_u\tau_\eta^2+\tau_v\tau_\eta^2+3\tau_u\tau_v\tau_\eta\right)}{\left(\tau_u^2\tau_v+\tau_u\tau_\eta^2+\tau_v\tau_\eta^2+2\tau_u\tau_v\tau_\eta\right)\left(\tau_\eta^2+\tau_u\tau_v+\tau_v\tau_\eta\right)}<0$. Similarly, we can show $\lim_{\tau_\varepsilon\to\infty}\frac{\partial\log Var(\tilde{v}-\tilde{p})}{\partial\tau_\eta}<0$.

Part (c) Price momentum Direct computation shows $m = \frac{Cov(\tilde{v},\tilde{p})}{Var(\tilde{p})} - 1$. We can use Proposition 1 to compute:

$$\frac{Cov(\tilde{v},\tilde{p})}{Var(\tilde{p})} = \frac{\left(\tau_\varepsilon + \frac{\tau_e\tau_u + (\tau_e+\tau_u)\tau_\eta}{\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_\eta}\right)\left(\tau_v + \tau_\varepsilon + \frac{\tau_e\tau_u + (\tau_e+\tau_u)\tau_\eta}{\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_\eta}\right)}{\left(\tau_\varepsilon + \frac{\tau_e\tau_u + (\tau_e+\tau_u)\tau_\eta}{\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_\eta}\right)^2 + \left(\frac{\tau_e+\alpha\tau_e+\alpha\tau_\eta}{\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_\eta}\right)^2\tau_v\tau_u}. \tag{A3}$$

Part (c1): Using the above expression, we can show $\frac{Cov(\tilde{v},\tilde{p})}{Var(\tilde{p})} > 1$, and so $m = \frac{Cov(\tilde{v},\tilde{p})}{Var(\tilde{p})} - 1 > 0$. Combining expression (A3) and Lemma 1 yields $\lim_{\tau_\eta\to\infty} m = 0$.

Part (c2): Note that $\frac{\partial m}{\partial \tau_\eta}$ and $\frac{\partial}{\partial \tau_\eta}\log\left[\frac{Cov(\tilde{v},\tilde{p})}{Var(\tilde{p})}\right]$ have the same sign. So, let us examine $\frac{\partial}{\partial \tau_\eta}\log\left[\frac{Cov(\tilde{v},\tilde{p})}{Var(\tilde{p})}\right]$.

Let $\tau_\varepsilon \to 0$. Using Part (a) of Lemma 2 and expression (A3), we can show:

$\frac{\partial}{\partial \tau_\eta}\log\left[\frac{Cov(\tilde{v},\tilde{p})}{Var(\tilde{p})}\right] \propto -(\tau_e+\tau_u)\tau_\eta^4 - \tau_u(\tau_e+2\tau_u)\tau_\eta^3 + \tau_u(2\tau_u^2+\tau_e\tau_v-\tau_u\tau_v)\tau_\eta^2 + \tau_e\tau_u^2\tau_v\tau_\eta + 2\tau_u^4\tau_v$.

Thus, $\frac{\partial}{\partial \tau_\eta}\log\left[\frac{Cov(\tilde{v},\tilde{p})}{Var(\tilde{p})}\right] < 0$ for large values of τ_η, and $\frac{\partial}{\partial \tau_\eta}\log\left[\frac{Cov(\tilde{v},\tilde{p})}{Var(\tilde{p})}\right] > 0$ for small values of τ_η.

Now let $\tau_\varepsilon \to \infty$. Using Part (b) of Lemma 2 and expression (A3), we can show $\frac{\partial}{\partial \tau_\eta}\log\left[\frac{Cov(\tilde{v},\tilde{p})}{Var(\tilde{p})}\right] \propto -\frac{\tau_v}{\tau_\varepsilon^2} < 0$. QED.

Proof of Proposition 3

Part (a) Risk By equation (20), $\frac{\partial Risk}{\partial \tau_\eta}$ and $\frac{\partial \tau_p}{\partial \tau_\eta}$ have the opposite sign. Direct computation shows

$$\frac{\partial \tau_p}{\partial \tau_\eta} = \frac{(\tau_e+\tau_u+\alpha\tau_e)^2 - 2(\tau_e\tau_u+\tau_e\tau_\eta+\tau_u\tau_\eta)(\tau_e(\alpha+1)+\alpha\tau_\eta)\frac{\partial\alpha}{\partial\tau_\eta}}{\left(\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_\eta\right)^2} > 0,$$

since $\frac{\partial \alpha}{\partial \tau_\eta} < 0$ by Part (a) of Proposition 2.

Using Lemma 1 and the expression of τ_p, we can show $\lim_{\tau_\eta\to\infty}\tau_p = \infty$, and thus, $\lim_{\tau_\eta\to\infty} Risk = \frac{1}{\tau_\xi}$.

Part (b) Disagreement Direct computation shows:

$$Disagreement = \frac{\sqrt{\tau_\varepsilon + \left(\frac{\tau_u-\alpha\tau_\eta}{\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_\eta}\right)^2\tau_e + \left(\frac{\tau_e+\tau_u+\alpha\tau_e}{\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_\eta}\right)^2\tau_\eta}}{\tau_v+\tau_\varepsilon+\frac{\tau_e\tau_u+(\tau_e+\tau_u)\tau_\eta}{\tau_u+\tau_e(\alpha+1)^2+\alpha^2\tau_\eta}}. \tag{A4}$$

Part (b1) simply follows combining Lemma 1 and the above expression of *Disagreement*.

To prove part (b2), we first use (A4) to compute $\frac{\partial \log Disagreement}{\partial \tau_\varepsilon}$ and then combine with Lemma 2. Specifically:

As $\tau_\varepsilon \to 0$, we have

$$\begin{aligned}\frac{\partial \log Disagreement}{\partial \tau_\eta} &\propto \frac{-\tau_\eta^2 + 3\tau_u\tau_v + \tau_v\tau_\eta}{2\tau_\eta\left(\tau_\eta^2 + \tau_u\tau_v + \tau_v\tau_\eta\right)} \Rightarrow \\ \frac{\partial \log Disagreement}{\partial \tau_\eta} &> 0 \Longleftrightarrow \tau_\eta^2 - \tau_v\tau_\eta - 3\tau_u\tau_v < 0 \\ &\Longleftrightarrow \tau_\eta < \frac{\tau_v + \sqrt{\tau_v^2 + 12\tau_u\tau_v}}{2}.\end{aligned}$$

As $\tau_\varepsilon \to \infty$, we have

$$\frac{\partial \log Disagreement}{\partial \tau_\eta} \propto -\frac{1}{2\tau_\varepsilon} < 0.$$

Part (c) Trading volume Direct computation shows

$$Volume = \frac{1}{\gamma}\sqrt{\frac{2}{\pi}}\frac{\sqrt{\tau_\varepsilon + \left(\frac{\tau_u - \alpha\tau_\eta}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_\eta}\right)^2\tau_e + \left(\frac{\tau_e + \tau_u + \alpha\tau_e}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_\eta}\right)^2\tau_\eta}}{1 + \frac{1}{\tau_\xi}\left(\tau_v + \tau_\varepsilon + \frac{\tau_e\tau_u + (\tau_e + \tau_u)\tau_\eta}{\tau_u + \tau_e(\alpha+1)^2 + \alpha^2\tau_\eta}\right)}. \tag{A5}$$

Part (c1) simply follows combining Lemma 1 and the above expression of $Volume$.

To prove part (c2), we first use (A5) to compute $\frac{\partial \log Volume}{\partial \tau_\varepsilon}$ and then combine with Lemma 2. Specifically:

As $\tau_\varepsilon \to 0$, we have

$$\begin{aligned}\frac{\partial \log Volume}{\partial \tau_\eta} &\propto \frac{-\tau_\eta^2 + (\tau_v + \tau_\xi)\tau_\eta + 3\tau_u(\tau_v + \tau_\xi)}{2\tau_\eta\left(\tau_\eta^2 + \tau_u\tau_v + \tau_u\tau_\xi + \tau_v\tau_\eta + \tau_\xi\tau_\eta\right)} \Rightarrow \\ \frac{\partial \log Volume}{\partial \tau_\eta} &> 0 \Longleftrightarrow \tau_\eta^2 - (\tau_v + \tau_\xi)\tau_\eta - 3\tau_u(\tau_v + \tau_\xi) > 0 \\ &\Longleftrightarrow \tau_\eta < \frac{(\tau_v + \tau_\xi) + \sqrt{(\tau_v + \tau_\xi)^2 + 12\tau_u(\tau_v + \tau_\xi)}}{2}.\end{aligned}$$

As $\tau_\varepsilon \to \infty$, we have

$$\frac{\partial \log Volume}{\partial \tau_\eta} \propto -\frac{1}{2\tau_\varepsilon} < 0.$$

QED.

Proof of Proposition 4

We prove Parts (a) and (b) simultaneously. Direct computation shows

$$\frac{\partial B\left(\tau_{\eta_i};\alpha\right)}{\partial \tau_{\eta_i}} = \frac{\tau_\xi\left(\tau_u + \tau_e + \alpha\tau_e\right)^2}{2\gamma G_1 G_2} - C'(\tau_{\eta_i}),$$

where

$$\begin{aligned}G_1 &= \left(\tau_e+\tau_u+\alpha^2\tau_v+\alpha^2\tau_\varepsilon\right)\tau_{\eta_i}\\&\quad+\tau_e\tau_u+\tau_e\tau_v+\tau_u\tau_v+\tau_e\tau_\varepsilon+\tau_u\tau_\varepsilon\\&\quad+\alpha^2\tau_e\tau_v+\alpha^2\tau_e\tau_\varepsilon+2\alpha\tau_e\tau_v+2\alpha\tau_e\tau_\varepsilon,\\G_2 &= \left(\tau_e+\tau_u+\alpha^2\tau_v+\alpha^2\tau_\xi+\alpha^2\tau_\varepsilon\right)\tau_{\eta_i}\\&\quad+\tau_e\tau_u+\tau_e\tau_v+\tau_u\tau_v+\tau_e\tau_\xi+\tau_e\tau_\varepsilon+\tau_u\tau_\xi+\tau_u\tau_\varepsilon\\&\quad+\alpha^2\tau_e\tau_v+\alpha^2\tau_e\tau_\xi+\alpha^2\tau_e\tau_\varepsilon+2\alpha\tau_e\tau_v+2\alpha\tau_e\tau_\xi+2\alpha\tau_e\tau_\varepsilon.\end{aligned}$$

Since both G_1 and G_2 are monotonically increasing in τ_{η_i}, $\frac{\partial B\left(\tau_{\eta_i};\alpha\right)}{\partial\tau_{\eta_i}}$ is decreasing in τ_{η_i} and thus $B\left(\tau_{\eta_i};\alpha\right)$ is strictly concave in τ_{η_i}. Thus, for any α, there exists a unique solution to problem (29), which implies that each investor i's optimal choice of τ_{η_i} is identical in equilibrium (i.e., $\tau_{\eta_i}^*=\tau_\eta^*$). This establishes symmetry in Part (a).

Function $\phi\left(\tau_\eta\right)$ is defined by plugging $\alpha\left(\tau_\eta\right)$ into $\frac{\partial B\left(\tau_{\eta_i};\alpha\right)}{\partial\tau_{\eta_i}}$. Then, combining with the fact $\tau_{\eta_i}=\tau_\eta$ and condition (30), we can get the two conditions (1) and (2) in Part (b), which characterize the equilibrium value of τ_η. The only remaining task is to show that $\phi\left(\tau_\eta\right)$ must satisfy either (1) or (2) for $\tau_\eta\geq 0$, so that there exists an overall equilibrium as stated in Part (a).

Apparently, if $\phi(0)\leq 0$, then condition (1) is trivially satisfied, so that $\tau_\eta=0$ constitutes an equilibrium.

Now suppose $\phi(0)>0$. We will show $\lim_{\tau_\eta\to\infty}\phi\left(\tau_\eta\right)<0$, so that by the intermediate value theorem, condition (2) must be satisfied. Using the expression of $\frac{\partial B\left(\tau_{\eta_i};\alpha\right)}{\partial\tau_{\eta_i}}$, we have

$$\frac{\partial B\left(\tau_\eta,\alpha\left(\tau_\eta\right)\right)}{\partial\tau_\eta}=\frac{\tau_\xi\left(\tau_u+\tau_e+\alpha\tau_e\right)^2}{2\gamma G_1G_2}-C'(\tau_\eta).$$

By Lemma 1, we know that $\alpha\to 0$ and $\alpha^2\tau_\eta\to 0$, as $\tau_\eta\to\infty$. Thus, $G_1\propto\left(\tau_e+\tau_u\right)\tau_\eta$ and $G_2\propto\left(\tau_e+\tau_u\right)\tau_\eta$, as $\tau_\eta\to\infty$. When $C\left(\tau_\eta\right)$ is weakly convex, $\lim_{\tau_\eta\to\infty}C'\left(\tau_\eta\right)$ is bounded below. Therefore, we have $\lim_{\tau_\eta\to\infty}\phi\left(\tau_\eta\right)<0$. QED.

Proof of Proposition 6

Part (a) Own actions Using the expression of $B\left(\tau_{\eta_i},\tau_{\varepsilon_i};\alpha\right)$, we can compute:

$$\frac{\partial^2 B\left(\tau_{\eta_i},\tau_{\varepsilon_i};\alpha\right)}{\partial\tau_{\eta_i}\partial\tau_{\varepsilon_i}}\propto-\frac{\left(\tau_e+\tau_u+\alpha\tau_e\right)^2}{\left(\tau_u+\tau_e\left(\alpha+1\right)^2+\alpha^2\tau_{\eta_i}\right)^2}<0.$$

Part (b) Cross investors Note that $(\tau_\varepsilon, \tau_\eta)$ affects $B\left(\tau_{\eta_i}, \tau_{\varepsilon_i}; \alpha\right)$ only through variable α. We already know that $\frac{\partial \alpha}{\partial \tau_\eta} < 0$ by Part (a) of Proposition 2. Applying the implicit function theorem to equation (16), we can compute

$$\frac{\partial \alpha}{\partial \tau_\varepsilon} = -\frac{(\tau_e + \tau_\varepsilon)\,\alpha^3 + 2\tau_e \alpha^2 + (\tau_e + \tau_u)\,\alpha}{3\,(\tau_e \tau_\varepsilon + \tau_\varepsilon \tau_\eta)\,\alpha^2 + 4\tau_e \tau_\varepsilon \alpha + (\tau_e \tau_\varepsilon + \tau_e \tau_\eta + \tau_u \tau_\varepsilon)} < 0.$$

Thus, the signs of $\frac{\partial^2 B\left(\tau_{\eta_i}, \tau_{\varepsilon_i}; \alpha\right)}{\partial \tau_\eta \partial \tau_{\eta_i}}$, $\frac{\partial^2 B\left(\tau_{\eta_i}, \tau_{\varepsilon_i}; \alpha\right)}{\partial \tau_\varepsilon \partial \tau_{\eta_i}}$, $\frac{\partial^2 B\left(\tau_{\eta_i}, \tau_{\varepsilon_i}; \alpha\right)}{\partial \tau_\eta \partial \tau_{\varepsilon_i}}$, and $\frac{\partial^2 B\left(\tau_{\eta_i}, \tau_{\varepsilon_i}; \alpha\right)}{\partial \tau_\varepsilon \partial \tau_{\varepsilon_i}}$ are negative of the signs of $\frac{\partial^2 B\left(\tau_{\eta_i}, \tau_{\varepsilon_i}; \alpha\right)}{\partial \alpha \partial \tau_{\eta_i}}$ and $\frac{\partial^2 B\left(\tau_{\eta_i}, \tau_{\varepsilon_i}; \alpha\right)}{\partial \alpha \partial \tau_{\varepsilon_i}}$.

By the FOC (32) of acquiring fundamental information, we can compute

$$\begin{aligned}
\frac{\partial^2 B\left(\tau_{\eta_i}, \tau_{\varepsilon_i}; \alpha\right)}{\partial \alpha \partial \tau_{\varepsilon_i}} &> 0 \Rightarrow \\
\frac{\partial^2 B\left(\tau_{\eta_i}, \tau_{\varepsilon_i}; \alpha\right)}{\partial \tau_\varepsilon \partial \tau_{\varepsilon_i}} &= \frac{\partial^2 B\left(\tau_{\eta_i}, \tau_{\varepsilon_i}; \alpha\right)}{\partial \alpha \partial \tau_{\varepsilon_i}} \frac{\partial \alpha}{\partial \tau_\varepsilon} < 0, \\
\frac{\partial^2 B\left(\tau_{\eta_i}, \tau_{\varepsilon_i}; \alpha\right)}{\partial \tau_\eta \partial \tau_{\varepsilon_i}} &= \frac{\partial^2 B\left(\tau_{\eta_i}, \tau_{\varepsilon_i}; \alpha\right)}{\partial \alpha \partial \tau_{\varepsilon_i}} \frac{\partial \alpha}{\partial \tau_\eta} < 0,
\end{aligned}$$

which establishes Part (b1).

Finally, we prove Part (b2). Using the FOC (30) of acquiring price information, we can show

$$\begin{aligned}
\frac{\partial}{\partial \alpha} \frac{\partial B\left(\tau_{\eta_i}, \tau_{\varepsilon_i}; \alpha\right)}{\partial \tau_{\eta_i}} \propto\ & -H_1\left(\tau_v, \tau_u, \tau_e, \alpha, \tau_{\varepsilon_i}, \tau_{\eta_i}\right) \tau_\xi \\
& + H_2\left(\tau_v, \tau_u, \tau_e, \alpha, \tau_{\varepsilon_i}, \tau_{\eta_i}\right) \times H_3\left(\tau_v, \tau_u, \tau_e, \alpha, \tau_{\varepsilon_i}, \tau_{\eta_i}\right),
\end{aligned}$$

where $H_1\left(\tau_v, \tau_u, \tau_e, \alpha, \tau_{\varepsilon_i}, \tau_{\eta_i}\right) > 0$, $H_2\left(\tau_v, \tau_u, \tau_e, \alpha, \tau_{\varepsilon_i}, \tau_{\eta_i}\right) > 0$, and

$$\begin{aligned}
& H_3\left(\tau_v, \tau_u, \tau_e, \alpha, \tau_{\varepsilon_i}, \tau_{\eta_i}\right) \\
=\ & \tau_e\left(\tau_e \tau_u + \tau_e \tau_{\eta_i} + \tau_u \tau_{\eta_i}\right) \\
& - (\tau_v + \tau_{\varepsilon_i})\left[\tau_e\left(\tau_e + \tau_{\eta_i}\right)\alpha^2 + 2\left(\tau_e + \tau_{\eta_i}\right)(\tau_e + \tau_u)\,\alpha + \tau_e\,(\tau_e + \tau_u)\right].
\end{aligned}$$

Note that α is not affected by τ_ξ, because equation (16) that determines α is involved only with $(\tau_u, \tau_e, \tau_\varepsilon, \tau_\eta)$. Thus, when τ_ξ is sufficiently large, we must have $\frac{\partial}{\partial \alpha} \frac{\partial B\left(\tau_{\eta_i}, \tau_{\varepsilon_i}; \alpha\right)}{\partial \tau_{\eta_i}} < 0$.

Now suppose τ_ξ is sufficiently small. Then, the sign of $\frac{\partial}{\partial \alpha} \frac{\partial B\left(\tau_{\eta_i}, \tau_{\varepsilon_i}; \alpha\right)}{\partial \tau_{\eta_i}}$ is determined by the sign of H_3. The term H_3 can be either positive or negative. For instance, suppose $\tau_{\varepsilon_i} = \tau_\varepsilon \to 0$. Then, $\alpha \to \frac{\tau_u}{\tau_\eta}$. At equilibrium, we also have $\tau_{\eta_i} = \tau_\eta$. Thus, $H_3 \to \frac{(\tau_e \tau_u + \tau_e \tau_\eta + \tau_u \tau_\eta)\left(\tau_e \tau_\eta^2 - \tau_v (\tau_e + 2\tau_u) \tau_\eta - \tau_e \tau_u \tau_v\right)}{\tau_\eta^2}$, which implies that the sign of H_3 is determined by $\tau_e \tau_\eta^2 - \tau_v\,(\tau_e + 2\tau_u)\,\tau_\eta - \tau_e \tau_u \tau_v$, which can be either positive or negative. QED.

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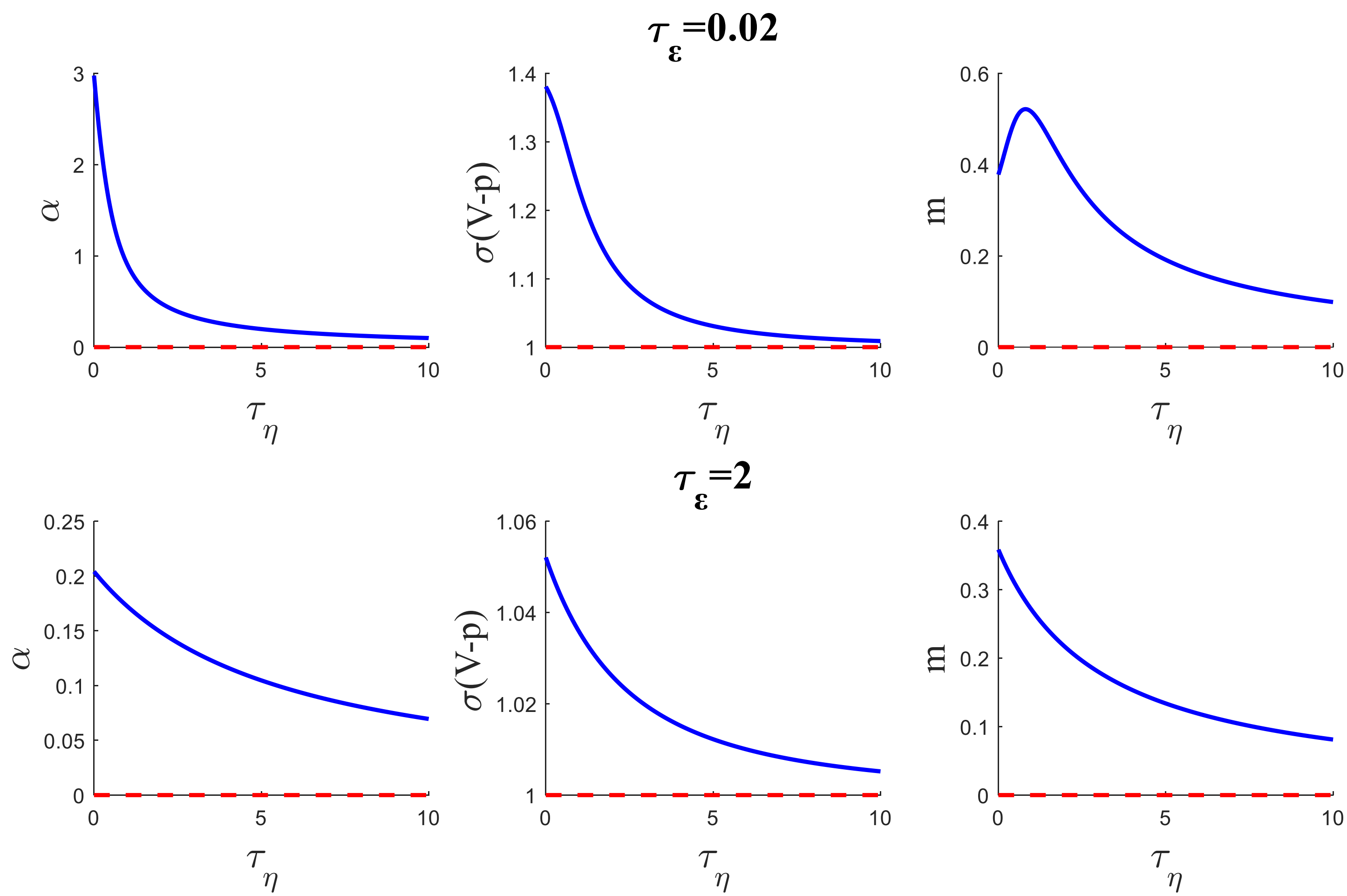
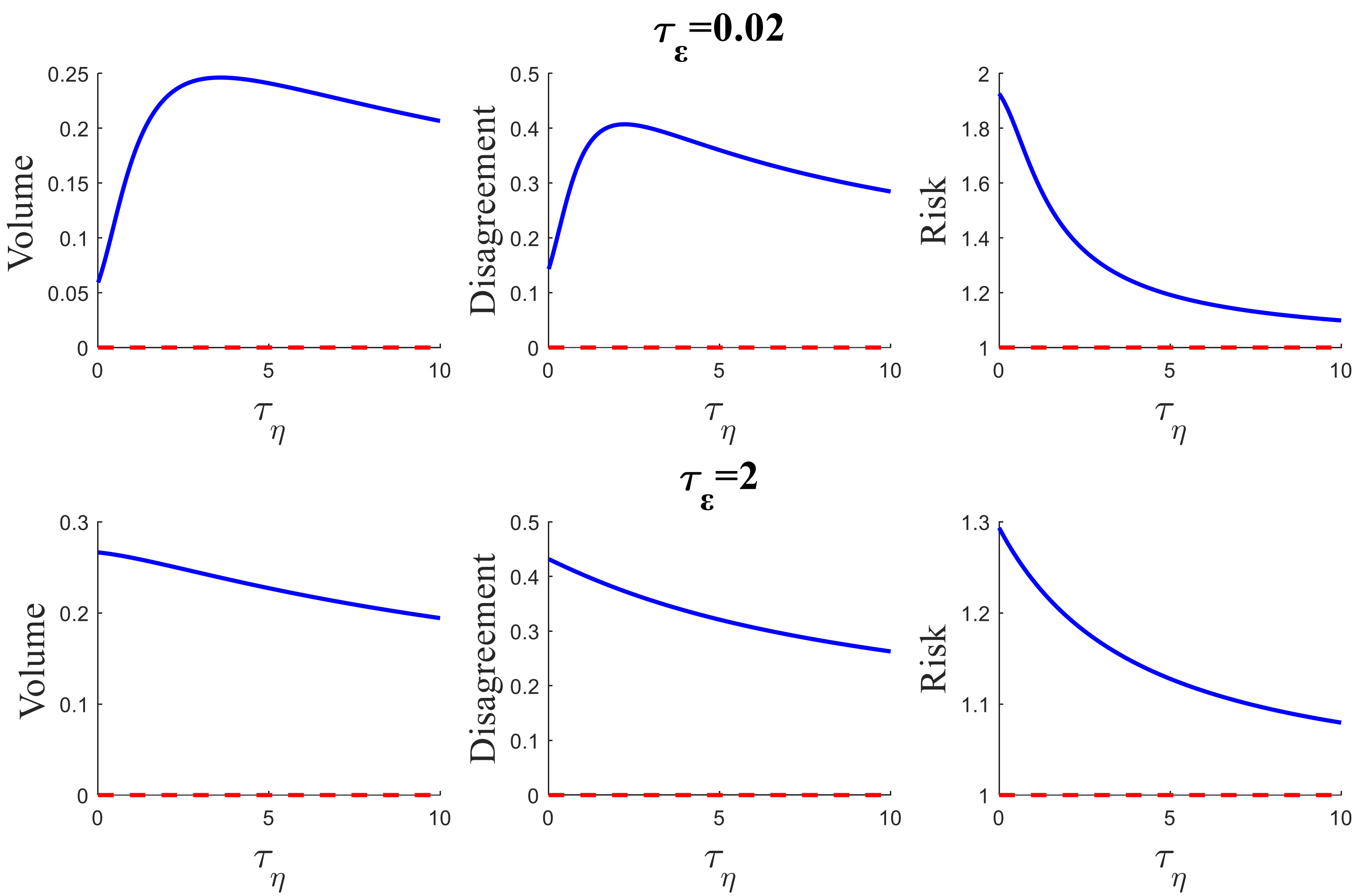
Figure 1: Price Informativeness, Return Volatilty, and Price Momentum



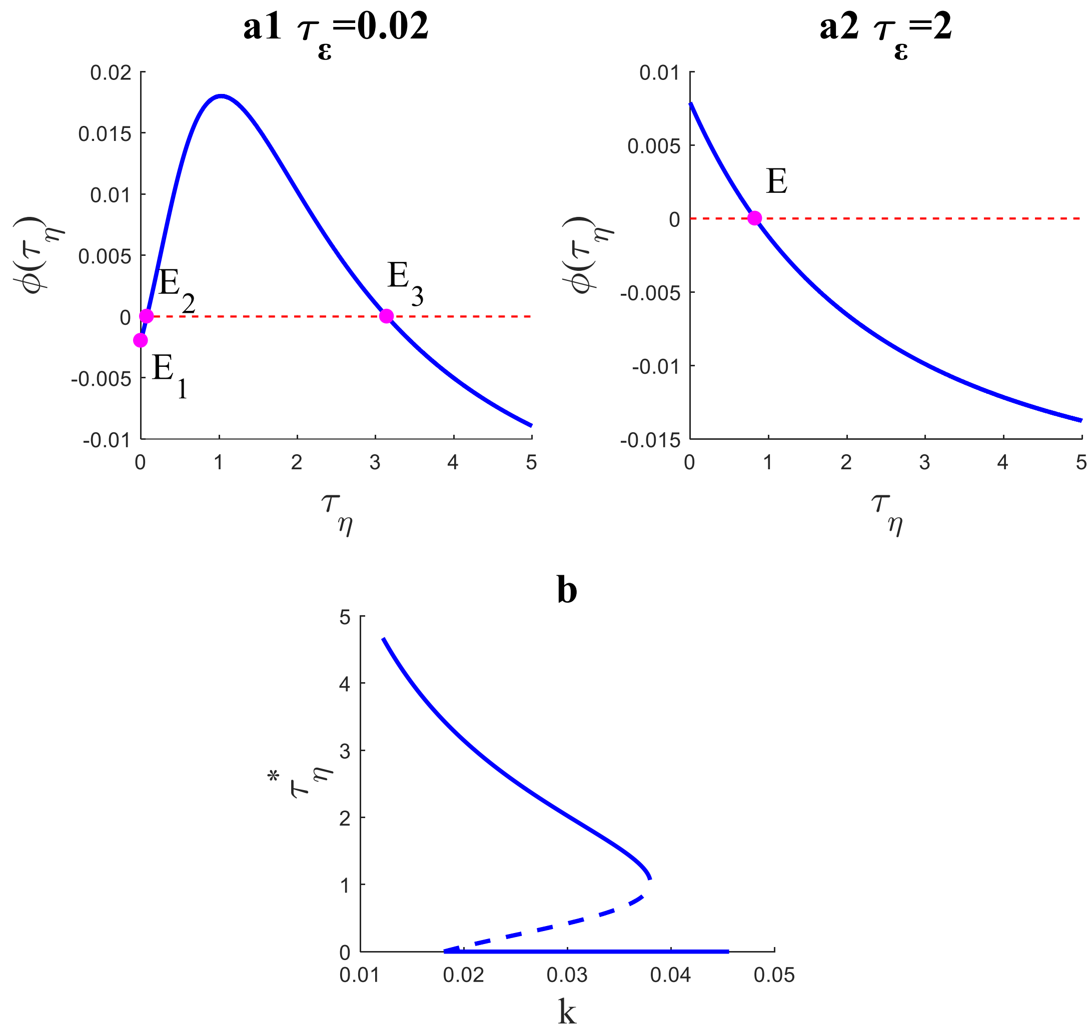
This figure plots price informativeness (negatively measured by α), return volatility ($\sigma(\tilde{V}-\tilde{p})$), and price momentum ($m$) against investors' sophistication level τ_η. In the top panels, we set $\tau_\varepsilon = 0.02$, while in the bottom panels, we set $\tau_\varepsilon = 2$. The other parameters are set as follows: $\tau_v = \tau_\xi = \tau_e = \tau_u = \gamma = 1$. The dashed lines plot the values in a standard REE economy (i.e., $\tau_\eta = \infty$).

Figure 2: Trading Volume, Disagreement, and Perceived Risk



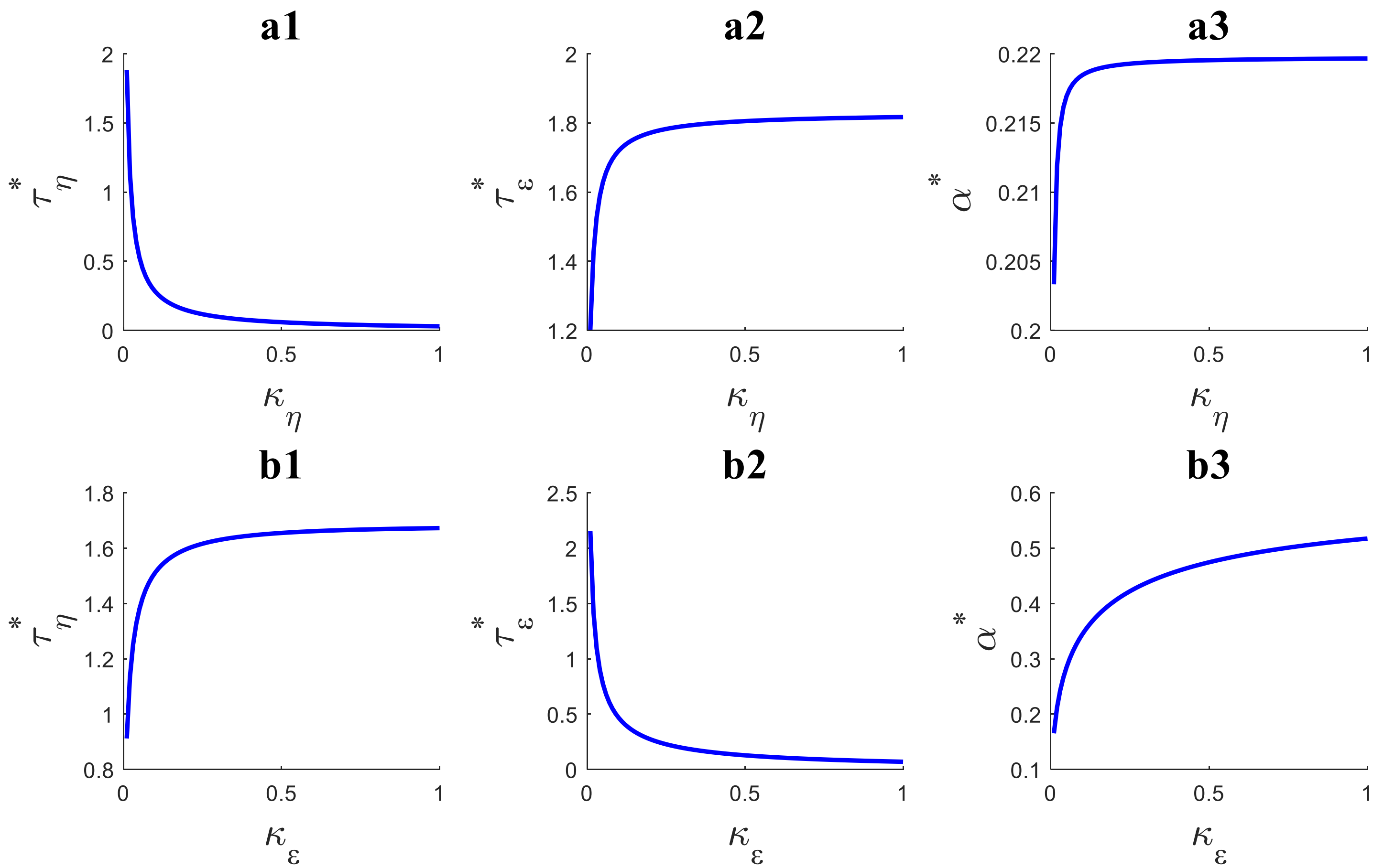
This figure plots trading volume, disagreement, and perceived risk against investors' sophistication level τ_η. In the top panels, we set $\tau_\varepsilon = 0.02$, while in the bottom panels, we set $\tau_\varepsilon = 2$. The other parameters are set as follows: $\tau_v = \tau_\xi = \tau_e = \tau_u = \gamma = 1$. The dashed red lines plot the values in a standard REE economy (i.e., $\tau_\eta = \infty$).

Figure 3: Sophistication Level Equilibrium



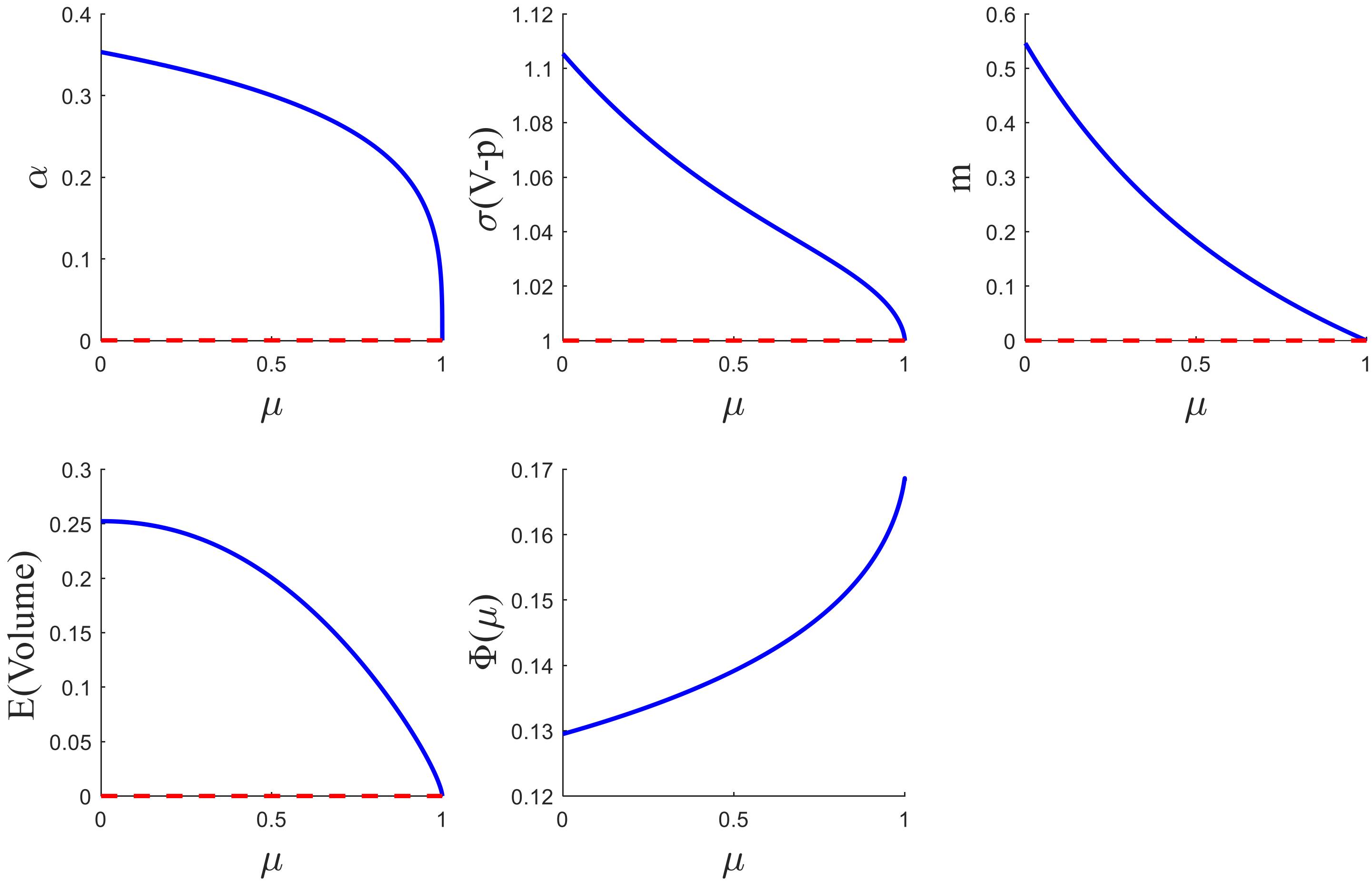
The top panels plots the function $\phi(\tau_\eta) = \frac{\partial B(\tau_\eta;\alpha(\tau_\eta))}{\partial \tau_\eta}$, which determines the equilibrium level of investor sophistication. Investors' cost function of acquiring sophistication is $C(\tau_{\eta_i}) = k\tau_{\eta_i}$. In Panel a1, we set $\tau_\varepsilon = 0.02$, while in Panel a2, we set $\tau_\varepsilon = 2$. In both panels, the other parameters are set as follows: $\tau_v = \tau_\xi = \tau_e = \tau_u = \gamma = 1$ and $k = 0.02$. The bottom panel plots the effect of the cost k of becoming more sophisticated in interpreting market data on the equilibrium values of τ_η^*. The other parameter values are: $\tau_v = \tau_\xi = \tau_e = \tau_u = \gamma = 1$ and $\tau_\varepsilon = 0.02$. The dashed segments in Panel b indicate unstable equilibria.

Figure 4: Endogenous Fundamental Information Acquisition



This figure plots the implications of changing information acquisition cost parameters κ_η and κ_ε in an economy with endogenous fundamental information acquisition. In the top panels, we set κ_ε=0.02, while in the bottom panels, we set κ_η=0.02.The other parameters are set as follows: $\tau_v = \tau_\varepsilon = \tau_\xi = \tau_e = \tau_u = \gamma = 1$.

Figure 5: The Economy with Two Types of Investors



This figure plots the implications of increasing the fraction μ of sophisticated investors in an economy with two types of investors. The other parameters are set as follows: $\tau_v = \tau_\varepsilon = \tau_\xi = \tau_e = \tau_u = \gamma = 1$. The dashed lines plot the values in a standard REE economy (i.e., μ =1).