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Cross-ownership, R&D Spillovers, and Antitrust Policy*

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Abstract

This paper considers cost-reducing R&D investment with spillovers in a Cournot oligopoly with minority shareholdings. We find that, with high market concentration and sufficiently convex demand, there is no scope for cross-ownership to improve welfare regardless of spillover levels. Otherwise, there *is* scope for cross-ownership provided that spillovers are sufficiently large. The socially optimal degree of cross-ownership increases with the number of firms, with the elasticity of demand and of the innovation function, and with the extent of spillover effects. In terms of consumer surplus standard, the scope for cross-ownership is greatly reduced even under low market concentration.

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1 Introduction

In many industries, minority shareholdings are prevalent in the form of cross-shareholding agreements among firms or common ownership by investment funds. The tendency of such arrangements to reduce price competition has been documented in the airline and banking industries (Azar et al. 2015, 2016), and it has raised antitrust concerns. However, cross-ownership arrangements (COAs) may have a beneficial effect on investment provided there are positive spillovers across firms. The reason is that COAs help to internalize the spillover externality, which is especially important for highly innovative industries. To what extent, and by what means, should antitrust authorities limit the "partial" mergers that result from cross-ownership in innovative industries? In this paper we provide a welfare analysis of COAs—when firms compete in quantities and invest in cost reduction—in the presence of spillovers; we also derive some implications for competition policy. The analysis may help elucidate whether the documented increase in cross-ownership arrangements has outrun its social value.

We consider a general symmetric model of cross-ownership; this model allows for a range of corporate control and for distinguishing between stock acquisitions made by investors and those made by other firms. In our benchmark model, we consider *simultaneous* R&D and output decisions. That approach aids tractability while helping to capture the imperfect observability of firms' R&D investment levels.[1] We test the robustness of results by way of a two-stage specification. The model subsumes earlier contributions to the literature that were based on linear or constant elasticity of demand and on specific innovation functions (Dasgupta and Stiglitz 1980; Spence 1984; d'Aspremont and Jacquemin 1988; Kamien et al. 1992). Perhaps the work closest to ours in spirit is the paper by Leahy and Neary (1997). More recently, Spulber (2013) studies how competition affects the incentives to innovate depending on the degree of appropriability of intellectual property.

Our paper seeks to answer the following questions: How do R&D and output levels vary with minority shareholdings? What are the key determinants of the socially optimal extent of cross-ownership? How is that optimal level affected by structural parameters (demand and cost conditions, industry technological opportunity, extent of spillovers) and

[1] Simultaneous models—in the presence or absence of R&D spillovers—are analyzed by, among others, Dasgupta and Stiglitz (1980), Levin and Reiss (1988), Ziss (1994), Leahy and Neary (1997), Cabral (2000), and Vives (2008).

by the competition authority's objective (to maximize total or rather consumer surplus)?

The main results can be summarized as follows. If demand is not too convex, then increasing the partial ownership interest in rivals will increase (resp. decrease) both R&D and output when spillovers are high (resp. low); for intermediate levels of spillovers, an increase in such ownership interest will increase R&D but reduce output. These are testable predictions. We identify the key determinants of a welfare-optimal degree of cross-ownership: the curvature of demand, the degree of market concentration, and the extent of spillovers. A sufficient (but not necessary) condition for the *absence* of minority shareholdings to be optimal—from either the total surplus (TS) or consumer surplus (CS) perspective, and for any extent of spillover—is that the relative degree of convexity of demand be greater than the inverse of the Herfindahl–Hirschman index (HHI). Otherwise, the range of spillovers is typically partitioned into three regions: one optimally with no cross-ownership for low levels of spillovers; one optimally with positive cross-ownership (by TS and CS standards) for high levels of spillovers; and one optimally with positive cross-ownership (by the TS standard only) in an intermediate region. We remark that the consumer surplus standard is always more stringent than the total surplus standard. We also find that, if the effectiveness of R&D is independent of the degree of cross-ownership, then under the CS standard there is a "bang bang" solution: either independent ownership or cartelization is optimal. Numerical results reveal that the (TS-based) socially optimal extent of cross-ownership is increasing in the number of firms, in the elasticity of demand and of the innovation function, and in the level of spillover effects. Qualitatively similar results hold for the CS-based optimal extent, except that the scope for minority shareholdings is much reduced.

The context analyzed here is of more than theoretical interest. Minority shareholdings are widespread in many industries (e.g., automobiles, airlines, financial, energy, and steel) and have attracted increasing antitrust attention (see EC 2013). There is growing interest among competition authorities in assessing the competitive effects of partial stock acquisitions. This increased attention stems mainly from two factors: (i) the rapid growth of private equity investment firms, which often hold partial ownership interests in competing firms (Wilkinson and White 2007); and (ii) some notorious cases, such as Ryanair's acquisition of Aer Lingus's stock and the Renault–Nissan alliance (under which

Renault owns 44.3% of Nissan even as Nissan owns 15% of Renault).[2] These factors have triggered a debate in Europe over the possibly anticompetitive effects of partial ownership. Yet the European Commission (EC) is not authorized to examine the acquisition of minority shareholdings,[3] and it has proposed extending the scope of its merger regulations so that it can intervene in cases involving minority shareholdings among competitors or in a vertical relationship (EC 2014).[4] In Canada and the United States, cross-ownership is scrutinized under prevailing merger control rules. More specifically, minority shareholdings in the latter country are often examined with reference to the Clayton Act and the Hart–Scott–Rodino Act.[5] However, there is an exception to antitrust scrutiny if the participation is "solely for investment" purposes, although it is subject to interpretation whether institutional investors can hold as much as 15% without needing to notify. These US provisions are important because common ownership by investment funds is widespread in many sectors.[6]

The extant literature, most of which focuses on the potential benefits of cooperative R&D or on how innovation is affected by mergers, has largely ignored the topic of how innovation is affected by minority shareholdings—despite clear evidence that antitrust policy attends closely to innovation. During the period 2008–2014, 36% of the mergers challenged by the US Department of Justice or the US Federal Trade Commission were characterized as harmful to innovation; of the challenged mergers, 76% were in high–

[2] As Gilo (2000) delineates, four other cases that have attracted considerable interest are as follows: (i) a $150 million investment by Microsoft in the nonvoting stock of Apple in 1997; (ii) the purchase by Northwest Airlines (fourth-largest US airline) of 14% of the common stock of Continental Airlines (the fifth-largest) while agreeing to limit its voting power (however, an antitrust suit forced Northwest to sell back nearly half of its purchased stake); (iii) the purchase by TCI (largest US cable operator) of a 9% stake in Time Warner (the second-largest); (iv) the acquisition by Gillette of more than a fifth of the nonvoting stock (and more than a tenth of the debt) of Wilkinson Sword, one of its main competitors.

[3] In some European countries (e.g., Austria, Germany, the United Kingdom), national merger control rules give competition authorities the scope to examine minority shareholdings. Currently, the EC can consider the effects on competition only of (pre-existing) minority shareholdings in the context of a notified merger (and in which the merging firms each have stakes in a third firm).

[4] The European Commission (EC 2014) has proposed a "targeted transparency" system under which the EC and its member states must be notified of potentially harmful acquisitions. Included in this category would be acquisitions of a minority shareholding—in a competitor or vertically related company—when either the acquired shareholding amounts to 20% *or* ranges between 5% and 20% but allows the acquirer "a de-facto blocking minority, a seat on the board of directors, or access to commercially sensitive information of the target" (p. 13).

[5] Section 7 of the Clayton Act prohibits acquisitions (of any part) of a company's stock that "may" substantially lessen competition either by (a) enabling the acquirer to manipulate, directly or indirectly, prices or output or by (b) reducing its own incentives to compete. Although there is no clearly established threshold, acquisitions of less than 25%—but of at least 15%—have been adjudged to be in violation (Salop and O'Brien 2000).

[6] See Azar et al. (2015, 2016).

R&D intensity industries. The anticompetitive effects of minority shareholdings tend to be weaker than those of a merger; at the same time, minority shareholdings seldom yield the efficiencies (e.g., rationalization, fewer duplicated costs) that may arise from a merger. The commonly held view is that, overall, minority shareholdings tend to lessen competition. Nonetheless, the evidence of spillover-induced underinvestment in R&D suggests that cross-ownership could be beneficial.[7]

The paper proceeds as follows. We review the literature in Section 2. In Section 3, we describe the different types of minority shareholdings that can be analyzed via our model, which is presented in Section 4. That section characterizes the equilibrium responses of output and R&D in response to a change in the degree of cross-ownership. In Section 5, we examine the socially optimal degree of cross-ownership and then illustrate the results with three leading specifications from the literature: the d'Aspremont–Jacquemin and Kamien–Muller–Zang models, and a constant elasticity model as in Dasgupta and Stiglitz (1980). Section 6 extends our model to allow for strategic R&D commitments in a two-stage game. Section 7 explores an alternative interpretation of our model when cooperation in R&D extends to the product market; with regard to this case there is empirical evidence, antitrust case study evidence, and also experimental evidence. We conclude in Section 8. Unless noted otherwise, all proofs are given in Appendix A. Appendix B provides details on our analysis of the three model specifications considered. Note that we offer application software (available on the Web), which the reader can use to conduct simulations with the models.

2 Review of the literature

Previous literature has analyzed the anticompetitive effects of cross-ownership (Bresnahan and Salop 1986; Reynolds and Snapp 1986). These researchers show that the presence of partial ownership interests in a Cournot industry may result in less output and higher prices (even if those interests are relatively small). This is because the competitive decisions of one firm—with stakes in a competitor's profit—will take those stakes into account by reducing output (or raising the price) so as to increase that competitor's

[7] For example, Bloom et al. (2013) report underinvestment in R&D (because of spillovers) in a panel of US firms from 1981 to 2001. These authors find that (i) the effects of technology spillovers are much greater than those of product market spillovers and (ii) the socially optimal level of R&D is 2–3 times as high as the observed level of R&D.

profit and hence its own financial profit. Azar et al. (2015) document the substantial common ownership interests of institutional investors (e.g., BlackRock, Vanguard, State Street, Fidelity) in competing technology firms (Apple, Microsoft), pharmacies (CVS, Walgreens), and banks (JP Morgan Chase, Bank of America, Citigroup). In that study of how passive investments by institutional investors affect market outcomes in the US airline industry, the authors find that ticket prices are about 10% higher on the average route than they would be with no cross-ownership (or if strategy decisions were made without regard to the investors' minority shareholdings). Similar results are obtained for the banking industry (Azar et al. 2016).

There is an extensive literature on the effects of cooperation and competition in R&D with spillovers, starting from the seminal articles of Brander and Spencer (1984), Spence (1984), Katz (1986), and d'Aspremont and Jacquemin (1988). Leahy and Neary (1997) present a general analysis of the effects of strategic behavior and cooperative R&D in the presence of price and output competition; they also study optimal public policy toward R&D in the form of subsidies.[8] One of this literature's primary objectives is to examine underprovision of R&D and the welfare effects of moving from a noncooperative to a cooperative regime in R&D. For example, d'Aspremont and Jacquemin (1988) show that, when spillovers are high enough, R&D cooperation (with subsequent competition at the output stage) leads to increased output, innovation, and welfare. Cooperative R&D enables firms to internalize their externalities and thus preserves the incentives to invest in R&D.

We shall identify the conditions under which minority shareholdings may increase total surplus, and even consumer surplus, in industries where R&D investment is important and spillovers are significant. Farrell and Shapiro (1990) show that passive financial stakes may be welfare increasing in asymmetric oligopolies; here we demonstrate the possibility in a symmetric oligopoly. There is some evidence that common ownership improves efficiency. He and Huang (2014), using data on US public firms from 1980 to 2010, estimate the effect of common ownership on market performance and report that firms increase their market share (up to 3.2%) through common ownership.[9]

[8] Suzumura (1992) extends the analysis to multiple firms and general demand and cost functions in Cournot competition. Ziss (1994) does likewise but also considers product differentiation and price competition. Kamien et al. (1992) analyze the effects of R&D cartelization and joint research ventures. For a survey, see Gilbert (2006).

[9] This result is consistent with Giannetti and Laeven (2009), who show that stock acquisitions by

The results that we derive complement those in the extant literature. For instance, Leahy and Neary (1997) find that R&D cooperation leads both to more output and to more R&D when spillovers are positive. Yet we show that, when there are minority shareholdings, this result holds only when spillovers are high enough. We also identify conditions under which a cartelized Research Joint Venture (RJV) is optimal, generalizing Kamien et al. (1992) and finding that this result depends on the innovation function having little curvature.

The empirical literature finds a negative relationship between spillovers and patent protection levels in a range of industries. That is, industries with low patent protection tend to have higher spillover levels than do industries with high patent protection (Griliches 1990).[10] The result is that the intensity of the spillover effects is heterogeneous across industries. Spulber (2013) shows that competitive pressure may decrease the incentives to innovate when intellectual property (IP) is not fully appropriable but this will not happen when IP is appropriable. Our results are therefore consistent with Spulber (2013) since the intensity of competition is inversely linked to the extent of cross-ownership, and we find that for high levels of spillovers an increase in cross-ownership is beneficial for innovation while it is detrimental for low spillover levels.

3 Minority shareholdings

We may consider two types of acquisitions: when investors acquire firms' shares, called *common ownership* or (partial) *cross-ownership by investors*; and when firms acquire other firms' shares, called *cross-ownership by firms*. We discuss two cases of cross-ownership by investors (common ownership) and one case of cross-ownership by firms. In each case we show that, when the stakes are symmetric, the firm-i manager's problem is to maximize

$$\phi_i = \pi_i + \lambda \sum_{k \neq i} \pi_k, \tag{1}$$

where the value of λ depends on the type of ownership and corresponds to what Edgeworth (1881) termed the coefficient of "effective sympathy" among firms. The analysis is

pension funds enhance firm valuation.

[10] Galasso and Schankerman (2015) find that patent rights block knowledge spillovers and downstream innovation in industries such as computers, electronics, and medical instruments (but not in others such as drugs or chemicals).

developed in Appendix A.

3.1 Cross-ownership by investors (common ownership)

In this situation, firms' stakes are held only by investors—for example, large institutional investors such as pension or mutual funds, which now have stakes in nearly three fourths of all publicly traded US firms. Consider an industry with n firms and n investors;[11] we let i and j index (respectively) investors and firms. The share of firm j owned by investor i is υ_{ij}, and the parameter ζ_{ij} captures the extent of i's control over firm j (Salop and O'Brien 2000). The total (portfolio) profit of owner i is $\pi^i = \sum_k \upsilon_{ik}\pi_k$, where π_k are the profits of portfolio firm k. The manager of firm j takes into account shareholders' incentives (through the control weights ζ_{ij}) and maximizes

$$\phi_j = \pi_j + \sum_{k \neq j}^{n} \lambda_{jk}\pi_k,$$

where

$$\lambda_{jk} \equiv \frac{\sum_{i=1}^{n} \zeta_{ij}\upsilon_{ik}}{\sum_{i=1}^{n} \zeta_{ij}\upsilon_{ij}}.$$

We next discuss two important cases: silent financial interests (a.k.a. passive investments) and proportional control.[12]

Silent Financial Interest (SFI). In this case, each owner (i.e., the majority or dominant shareholder) i retains full control of the acquiring firm and is entitled to a share of the acquired firm's profits—but exerts no influence over the latter's decisions. If i owns j, then (i) $\zeta_{ij} = 1$ and $\zeta_{ik} = 0$ for $k \neq j$ and (ii) firm i maximizes $\sum_{k=1}^{n} \upsilon_{ik}\pi_k$. We consider the symmetric case, in which each investor i receives a share α in the acquired firms; hence $\upsilon_{ij} = 1-(n-1)\alpha$ and $\upsilon_{ik} = \alpha$ for $k \neq j$. Then $\lambda^{\text{SFI}} \equiv \alpha/(1-(n-1)\alpha)$. The upper bound of cross-ownership is $\alpha = 1/n$, in which case $\lambda^{\text{SFI}} = 1$ and n identical firms will maximize total joint profit.

Proportional Control (PC). Under proportional control, the firm's manager accounts

[11] We make this assumption solely to simplify notation. In fact, we need only that the total number of investors, say I, be no less than the total number of firms (n). Then the expressions that follow for λ^{SFI} and λ^{PC} hold if we replace n with I. However, this assumption also has the benefit of facilitating comparisons with the case of cross-ownership by firms.

[12] Other governance structures are total control, partial control, fiduciary obligation, Coasian joint control, and one-way control. For a discussion of these structures, see Salop and O'Brien (2000).

for shareholders' own-firm interests in *other* firms in proportion to their respective stakes. Suppose that each investor acquires a share α of those other firms. Then, to compute λ_{jk} for a given $k \neq j$, we first note that if i is the majority shareholder of j then $\zeta_{ij} = 1 - (n-1)\alpha$ and $\upsilon_{ik} = \alpha$. Yet if instead $i+1$ were the majority shareholder of k, then that investor has control over j equal to $\zeta_{(i+1)j} = \alpha$ and receives an own-firm profit share of $\upsilon_{(i+1)k} = 1 - (n-1)\alpha$. Finally, there are $n-2$ investors who are minority shareholders of j and k; for these investors, the *combination* of their profit shares (and control) is equal to α. Thus we obtain

$$\lambda^{\mathrm{PC}} \equiv \frac{2[1-(n-1)\alpha]\alpha + (n-2)\alpha^2}{[1-(n-1)\alpha]^2 + (n-1)\alpha^2}.$$

As with any silent financial interest, here $\lambda^{\mathrm{PC}} = 1$ when $\alpha = 1/n$. If $\alpha < 1/n$, then λ is increasing in both n and α.

3.2 Cross-ownership by firms

In this situation, firms may acquire their rivals' stock in the form of passive investments with no control rights (e.g., nonvoting shares; see Gilo et al. 2006). This setting features a complex, chain-effect interaction between the profits of firms. Here α_{jk} denotes firm j's ownership stake in firm k, and the strategy decisions are made by the controlling shareholder; thus the profit of firm j is given by $\phi_j = \pi_j + \sum_{k \neq j} \alpha_{jk}\phi_k$. We can derive the profit of each firm by solving for a fixed point of a matrix equation. In the symmetric case, $\alpha_{jk} = \alpha_{kj} = \alpha$ for all $j \neq k$ and $\alpha_{jj} = 0$ for all j. It can be shown that firm j will maximize $\pi_j + \lambda \sum_{k \neq j} \pi_k$, where $\lambda^{\mathrm{PCO}} \equiv \alpha/[1-(n-2)\alpha]$ (we use PCO to index partial cross-ownership). It follows that the upper bound of cross-ownership is $\alpha = 1/(n-1)$, in which case λ^{PCO} tends to 1 as α approaches $1/(n-1)$. Just as in the two previous cases, λ is increasing in the number of firms and in the firms' stakes.

3.3 Comparative statics on the degree of cross-ownership (λ)

Table 1 summarizes the value of λ according to the type of cross-ownership. We can see that more firms and higher investment stakes are both positively associated with λ. In addition, it is straightforward to show that $\lambda^{\mathrm{PC}} > \lambda^{\mathrm{SFI}} > \lambda^{\mathrm{PCO}}$. The implication is that, in order to attain the same degree of cross-ownership (and for a given number of firms),

the symmetric investment stake with proportional control must be *lower* than with silent financial interests, which in turn must be lower than with partial cross-ownership by firms: $\alpha^{\text{PC}} < \alpha^{\text{SFI}} < \alpha^{\text{PCO}}$.

Table 1: Comparative Statics on λ

	Common Ownership, Silent Financial Interests	Common Ownership, Proportional Control	Cross-ownership (by firms)
λ	$\frac{\alpha}{1-(n-1)\alpha}$	$\frac{2\alpha[1-(n-1)\alpha]+(n-2)\alpha^2}{[1-(n-1)\alpha]^2+(n-1)\alpha^2}$	$\frac{\alpha}{1-(n-2)\alpha}$
$\partial\lambda/\partial n$	$+$	$+$	$+$
$\partial\lambda/\partial\alpha$	$+$	$+$	$+$

4 Framework and equilibrium

We consider an industry consisting of $n \geq 2$ identical firms, where each firm $i = 1, \ldots, n$ chooses simultaneously their R&D intensity (x_i) and production quantity (q_i). Firms produce a homogeneous good characterized by a smooth inverse demand function $f(Q)$, where $Q = \sum_i q_i$. We make the following three assumptions.

A.1. $f(Q)$ is twice continuously differentiable, where (i) $f'(Q) < 0$ for all $Q \geq 0$ such that $f(Q) > 0$ and (ii) the elasticity of the *slope* of the inverse demand function,

$$\delta(Q) \equiv \frac{Qf''(Q)}{f'(Q)},$$

is constant and equal to δ.

The parameter δ is the curvature (relative degree of concavity) of the inverse demand function, so demand is concave for $\delta > 0$ and is convex for $\delta < 0$. Furthermore, demand is log-concave for $1 + \delta > 0$ and is log-convex for $1 + \delta < 0$. If $1 + \delta = 0$, then demand is both log-concave *and* log-convex.[13] Assumption A.1 is always satisfied by inverse demand functions that are linear or constantly elastic. In particular, the family of inverse demand

[13] We remark that δ is also related to the marginal consumer surplus from increasing output—that is, to $\text{MS} = -f'(Q)Q$. After setting ϵ_{MS} as the elasticity of the *inverse* marginal consumer surplus function (so that $\epsilon_{\text{MS}} = \text{MS}/(\text{MS}'Q)$), Weyl and Fabinger (2013) argue that ϵ_{MS} measures the curvature of the logarithm of demand. Under A.1, we can write $1/\epsilon_{\text{MS}} = 1 + \delta$.

functions for which $\delta(Q)$ is constant can be represented as

$$f(Q) = \begin{cases} a - bQ^{\delta+1} & \text{if } \delta \neq -1, \\ a - b\log Q & \text{if } \delta = -1; \end{cases}$$

here a is a nonnegative constant and $b > 0$ (resp., $b < 0$) if $\delta \geq -1$ (resp., $\delta < -1$).

A.2. The marginal production cost or innovation function of firm i, or c_i, is independent of output and is decreasing in both own and rivals' R&D as follows: $c_i = c(x_i + \beta \sum_{j \neq i} x_j)$, where $c' < 0$, $c'' \geq 0$, and $0 \leq \beta \leq 1$ for $i \neq j$.

A.3. The cost of investment is given by the function $\Gamma(x_i)$, where $\Gamma(0) = 0$, $\Gamma' > 0$, and $\Gamma'' \geq 0$.

The parameter β represents the spillover level of the R&D activity. Since we focus on symmetric firms, we assume symmetric spillover levels; moreover, R&D outcomes are imperfectly appropriable to an extent that varies between 0 and 1.

Firm i's profit is given by

$$\pi_i = f(Q)q_i - c\left(x_i + \beta \sum_{j \neq i} x_j\right) q_i - \Gamma(x_i),$$

and the objective function for firm i is $\phi_i = \pi_i + \lambda \sum_{k \neq i} \pi_k$ (i.e., equation (1)). The model represents distinct scenarios depending on the values of β and λ. When $\lambda \in (0, 1)$ and $\beta \in [0, 1)$, firms compete in the presence of partial ownership interests and the R&D outcomes are again imperfectly appropriable. When $\lambda \in (0, 1)$ and $\beta = 1$, firms form a Research Joint Venture under which all R&D outcomes are fully shared among RJV members and the duplication of R&D efforts is avoided. When $\lambda = \beta = 1$, firms form a "cartelized" RJV. If $\lambda = 0$ then there are no minority interests.

For markets with cross-shareholdings, a modified HHI is proposed by Bresnahan and Salop (1986). This index corresponds to the market share–weighted Lerner index in a Cournot market, and we write MHHI $= \left(\sum_i s_i L_i\right)\eta$. Here s_i and L_i are (respectively) the market share and Lerner index of firm i; the term η denotes the demand price elasticity).[14]

[14] Azar et al. (2015) use the MHHI (in terms of control and share rights) to measure anticompetitive incentives stemming from financial interests in the US airline industry. These authors find that, in year 2013, the market concentration generated by such financial interests was more than 10 times greater than the HHI changes above which mergers are likely to generate antitrust concerns.

In our case it is easy to see that, for a given common marginal cost, $(p-c)/p = \text{MHHI}/\eta$ at a symmetric Cournot equilibrium; here $\text{MHHI} = \Lambda/n$ for $\Lambda = 1 + \lambda(n-1)$, which is monotone in λ. When $\lambda = 0$ we have the standard HHI for a symmetric solution, $1/n$, and if $\lambda = 1$ then the modified HHI is equal to 1.

Now we consider symmetric solutions. Let $B = 1+\beta(n-1)$; then Bx is the "effective" investment that lowers costs for a firm. Let $\tau = 1 + \lambda(n-1)\beta$. Then $-c'(Bx)q\tau$ is the marginal effect of investment by a firm on its internalized profit ϕ_i. A symmetric interior equilibrium $(Q^* = nq^*, x^*)$ must solve the first-order necessary conditions for the maximization of ϕ_i:

$$\frac{f(Q^*) - c(Bx^*)}{f(Q^*)} = \frac{\text{MHHI}}{\eta(Q^*)}; \tag{2}$$

$$-c'(Bx^*)\frac{Q^*\tau}{n} = \Gamma'(x^*). \tag{3}$$

Here $\eta(Q^*) = -f(Q^*)/(Q^* f'(Q^*))$ is the elasticity of demand. Equation (2) is the modified Cournot–Lerner pricing formula; expression (3) equates the marginal benefit and marginal cost of investment by a firm with its internalized profit ϕ_i. Note that both MHHI and τ are increasing in λ and therefore respectively exert pressure to reduce output (or increase prices and margins) and to increase investment.

Let second-order derivatives and cross-derivatives be defined, at symmetric solutions, by $\partial_{zz}\phi_i \equiv \partial^2\phi_i/\partial z_i^2$, $\partial_{z_{i,j}}\phi_i \equiv \partial^2\phi_i/\partial z_i \partial z_j$, $\partial_{hz}\phi_i \equiv \partial^2\phi_i/\partial h \partial z_i$ (with $h = \beta$, λ, and $z = q, x$), and $\partial_{xq}\phi_i \equiv \partial^2\phi_i/\partial x_i \partial q_i$ $(i \neq j;\ i,j = 1,2,\ldots,n)$.[15] We assume that the following stability conditions hold:

$$\Delta_q \equiv \partial_{qq}\phi_i + (n-1)\partial_{q_{i,j}}\phi_i < 0,$$

$$\Delta_x \equiv \partial_{xx}\phi_i + (n-1)\partial_{x_{i,j}}\phi_i < 0,$$

and

$$\Delta \equiv \Delta_q\Delta_x - (\partial_{xq}\phi_i)^2\tau B > 0. \tag{4}$$

Together these conditions imply that (2) and (3) both have a unique solution. It is noteworthy that $\Delta_x < 0$ requires that at least one of c'' and Γ'' be positive (see Table 7 in Appendix A). If $\Delta(Q^*, x^*) > 0$ then we say that the equilibrium is *regular*; the ratio-

[15] See Table 7 in Appendix A for the full expressions of these variables.

Table 2: Model Specifications

	AJ	KMZ	CE
Demand	$f(Q) = a - bQ$	$f(Q) = a - bQ$	$f(Q) = \sigma Q^{-\varepsilon}$
	$\delta = 0$; $a, b > 0$	$\delta = 0$; $a, b > 0$	$\delta = -(1+\varepsilon)$; $a = 0$, $b = -\sigma < 0$
$c(\cdot)$	$\bar{c} - x_i - \beta \sum_{j \neq i} x_j$	$\bar{c} - \left[(2/\gamma)(x_i + \beta \sum_{j \neq i} x_j)\right]^{1/2}$	$\kappa\big(x_i + \beta \sum_{j \neq i} x_j\big)^{-\alpha}$
$\Gamma(x)$	$(\gamma/2)x^2$	x	x

nale for this terminology will become clear in the comparative statics analysis to follow. In particular, we assume that there is a unique regular symmetric interior equilibrium (Q^*, x^*). The focus of our paper is on characterizing that equilibrium.

4.1 Model specification examples

We will consider the well-known R&D model specifications—with linear (and therefore log-concave) demand—of d'Aspremont–Jacquemin (AJ) and Kamien–Muller–Zang (KMZ); we also consider a constant elasticity (CE) model with log-convex demand that is similar to the Dasgupta and Stiglitz (1980) model but with spillover effects. Table 2 summarizes these model specifications, and Table 8 (in Appendix B) gives sufficient second-order conditions and also the regularity condition for each model specification. Table 9 (in Appendix B) presents the equilibrium values of output and R&D that are obtained by solving equations (2) and (3).

4.2 Comparative statics on the degree of cross-ownership (λ)

We are interested in how output and R&D respond, in equilibrium, to a change in λ. The sign of the derivatives $\frac{\partial q^*}{\partial \lambda}$ and $\frac{\partial x^*}{\partial \lambda}$ can be ambiguous. For a given x, the extent of cross-ownership λ has a negative effect on output: $\partial_{\lambda q}\phi_i = f'(Q)q(n-1) < 0$. This is the well-known effect of reducing output so as to increase price when the profit of rivals is being taken into account. For a given q, however, λ has a positive effect on investment: $\partial_{\lambda x}\phi_i = -\beta q(n-1)c'(xB) > 0$. This is the internalizing effect of spillovers with a higher λ, and its strength depends directly on the size (β) of those spillovers. The total impact of λ on the equilibrium values of per-firm output and R&D will depend on which of the two previous effects dominates. What is clear is that, if $\partial x^*/\partial \lambda \leq 0$, then $\partial q^*/\partial \lambda < 0$ because $\partial_{xq}\phi_i > 0$. That is, an increase in R&D investment is necessary (but

not sufficient) for output to rise with increasing λ. When β is small, the positive effect on investment is small and so the negative effect on output dominates. Then q^* decreases with λ and, as a result, firms invest less also when λ increases—given that the benefit to firms from investing in R&D decreases proportionally with output.

We shall use R_{I} to denote the region in which $\partial q^*/\partial\lambda < 0$ and $\partial x^*/\partial\lambda \leq 0$. If β is sufficiently high, then the positive effect on R&D reduces significantly the unit cost of production, which in turn stimulates output. Two effects are present in this case. On the one hand, firms want to reduce output in order to increase competitors' profit and hence their own financial profit. On the other hand, firms now have incentives to produce more because they are more efficient. If the first effect dominates, then $\partial q^*/\partial\lambda < 0$ and $\partial x^*/\partial\lambda > 0$ (we label this region R_{II}). But if the second effect dominates, then $\partial q^*/\partial\lambda > 0$ and $\partial x^*/\partial\lambda > 0$ (region R_{III}). Which of these two cases arises in equilibrium will depend on the extent of the spillovers. We find that, whereas R_{I} always exists, regions R_{II} and R_{III} might not exist.

We next derive the conditions and threshold values (in terms of β) that define the boundaries of the regions characterizing the signs of $\partial x^*/\partial\lambda$ (Lemma 1) and $\partial q^*/\partial\lambda$ (Lemma 2).

LEMMA 1 *At equilibrium,*

$$\operatorname{sign}\left\{\frac{\partial x^*}{\partial\lambda}\right\} = \operatorname{sign}\{\beta(1+n+\delta\Lambda)-1\}.$$

COROLLARY 1 *For any fixed λ and for any $\beta \in [0,1]$, only R_{I} exists (with $\partial x^*/\partial\lambda \leq 0$) if and only if demand is convex enough—that is, iff $\delta \leq -n/\Lambda$.*[16] *This statement holds for any λ in the interval $[0,1]$ provided that $\delta \leq -n$.*

We can interpret the critical spillover threshold for β in terms of the cost pass-through coefficient (i.e., the rate at which the price changes with marginal cost). This threshold is equal to the industry-wide per-firm cost pass-through coefficient ($P'(c)/n$) multiplied by the internalized cost-reducing effect of a unit increase in R&D expenditures by a firm (τ); formally, we have $\operatorname{sign}\{\frac{\partial x^*}{\partial\lambda}\} = \operatorname{sign}\{\beta - P'(c)\tau/n\}$. Note that the threshold is decreasing in the pass-through coefficient because firms are less interested in reducing costs when

[16] When $\delta > -(n+1)/\Lambda$, there exists a positive threshold of spillover above which $\partial x^*/\partial\lambda > 0$; however, that threshold exceeds unity unless $\delta > -n/\Lambda$.

doing so translates, in effect, into lower prices.[17]

A consequence of Lemma 1 is that the threshold for spillovers to induce $\partial x^*/\partial\lambda \leq 0$ is decreasing (resp. increasing) in λ when demand is concave (resp. convex)—that is, when $\delta > 0$ (resp. $\delta < 0$).[18]

If demand is extremely convex, then increases in cross-ownership are so restrictive of output that they induce $\partial x^*/\partial\lambda < 0$, in which case only R_{I} exists for any β. And since MHHI $= \Lambda/n$, the applicable condition is that $\delta \leq -(\mathrm{MHHI})^{-1}$. Corollary 1 implies that the degree of demand convexity required for only R_{I} to exist is decreasing in the concentration measured by MHHI; in other words, the condition is less restrictive in markets that are more concentrated. The corollary implies also that R_{II} can exist only when quantities are strategic *substitutes*. Indeed, if quantities are instead strategic *complements* (i.e.., if $\partial_{q_{i,j}}\phi_i > 0$, which holds when $\delta < -n(1+\lambda)/\Lambda$), then the condition $\delta < -n/\Lambda$ also holds and only R_{I} exists. When δ is such that $-n(1+\lambda)/\Lambda < \delta < -n/\Lambda$, quantities are strategic substitutes (as e.g. when demand is log-concave) but again only R_{I} exists. If $\delta > -n/\Lambda$, then quantities are strategic substitutes and so R_{II} exists (see Figure 8 in Appendix A).

As regards the comparative statics on output, totally differentiating the first-order condition (FOC) with respect to λ yields

$$\mathrm{sign}\left\{\frac{\partial q^*}{\partial\lambda}\right\} = \mathrm{sign}\left\{\partial_{\lambda q}\phi_i + (\partial_{xq}\phi_i)B\frac{\partial x^*}{\partial\lambda}\right\}; \tag{5}$$

here $B = 1+\beta(n-1)$ captures the effect, on each firm's marginal cost, of a unit increase in R&D by all firms. At equilibrium, the impact on output of a higher degree of cross-ownership depends directly on its effect on marginal profit with respect to output ($\partial_{\lambda q}\phi_i$) and indirectly through its effect on the R&D effort of each firm at equilibrium. Recall that, since $\partial_{xq}\phi_i > 0$, it follows that if $\partial x^*/\partial\lambda \leq 0$ then $\partial q^*/\partial\lambda < 0$ (R_{I}). By Lemma 1 we know that, if spillovers are sufficiently high and demand is not too convex, then $\partial x^*/\partial\lambda > 0$; however, the sign of $\partial q^*/\partial\lambda$ can be negative (R_{II}) or positive (R_{III}).

We derive an *inverse* measure of R&D effectiveness in terms of the model's basic

[17] Let $P(c) \equiv f(nq^*(c))$; then $P'(c) = f'(nq^*)n\left(\frac{dq^*}{dc}\right) = \frac{n}{\Lambda(1+\delta)+n}$. Since the stability condition $\Delta_q < 0$ holds when $\Lambda(1+\delta)+n > 0$, it follows that $P'(c) > 0$. Furthermore, the pass-through increases with the number of firms when demand is log-concave ($\delta > -1$). See, for example, Weyl and Fabinger (2013).

[18] So for $\delta > 0$, if $\partial x^*/\partial\lambda > 0$ for some λ then that inequality must hold also for larger values of λ. Analogously: for $\delta < 0$, if $\partial x^*/\partial\lambda < 0$ for some λ then that inequality holds also for larger values of λ.

elasticities. This measure H is a function of β and provides the appropriate threshold for the positive effect of minority shareholdings on R&D investments to dominate its negative effect on output. Let $\chi(Bx^*) \equiv -c''(Bx^*)Bx^*/c'(Bx^*) \geq 0$ be the elasticity of the slope of the innovation function (i.e., the relative convexity of $c(\cdot)$) evaluated at the effective R&D, Bx^*; and let $y(x^*) \equiv \Gamma''(x^*)x^*/\Gamma'(x^*) \geq 0$ be the elasticity of the slope of the investment cost function. Our regularity assumptions imply that either $c'' > 0$ or $\Gamma'' > 0$ (or both). If $\Gamma''(x^*) > 0$, let $\xi(Q^*, x^*) \equiv -\beta(c'(Bx^*))^2/(f'(Q^*)\Gamma''(x^*)) > 0$ measure the relative effectiveness of R&D,[19] weighted by β. Then H can be written as

$$H = \frac{1}{\xi(Q^*, x^*)}\left(1 + \frac{\chi(Bx^*)}{y(x^*)}\right),$$

evaluated at the equilibrium (Q^*, x^*) for $\beta > 0$. Note that $\lim_{\beta \to 0} H = \infty$.

LEMMA 2 *Let $B = 1 + \beta(n-1)$. At equilibrium,*

$$\text{sign}\left\{\frac{\partial q^*}{\partial \lambda}\right\} = \text{sign}\{B - H\}. \tag{6}$$

Just as it does for β, the term H provides the appropriate threshold for B, or the effect (on each firm's marginal cost) of a unit increase in R&D by all firms. Therefore, if $B > H$ then the positive effect of minority shareholdings on R&D investments dominates its negative effect on output. At equilibrium, a higher degree of cross-ownership increases output. The values of H, χ, and y for each model specification are presented in Table 3. Note that H is independent of λ under the AJ and KMZ models but is strictly increasing in λ under the CE model. As we shall discuss later, the relationship between H and λ has important consequences for the optimal welfare policy.[20]

Table 3: H, χ,and y

	AJ	KMZ	CE
H	$\frac{b\gamma}{\beta}$	$\frac{b\gamma B}{\beta}$	$\frac{B}{\beta}\left(\frac{\alpha+1}{\alpha}\right)\frac{\varepsilon}{n-\varepsilon\Lambda}\tau$
χ	0	$\frac{1}{2}$	$\alpha + 1$
y	1	0	0

[19] As defined by Leahy and Neary (1997, Sec. V, p. 654).

[20] The *sign* of the effect of changes in the degree of cross-ownership on equilibrium values can be computed explicitly (see Lemma 10 in Appendix B).

We introduce the following mild assumption on $H\colon (0,1] \to \mathbb{R}^+$ (considered as a function of β).

A.4. The slope of $H(\beta)$ is less than $n-1$.

Under Assumption A.4, the equation $B = H(\beta)$ has a unique positive solution (since $\lim_{\beta\to 0} H = \infty$). Denote that solution by β'; then, for $\beta > \beta'$ we have that $\partial q^*/\partial\lambda > 0$. Assumption A.4 seems not to be restrictive in light of the model specifications typically used in the literature. In AJ, KMZ, and CE, for example, an even stronger condition holds—namely, that $H(\beta)/B$ is strictly decreasing in β. Assumption A.4 does not guarantee that $\beta' < 1$, so R_{III} may fail to exist. We have that $\beta' < 1$ if $n > H(1)$. Our next corollary states the results formally.

COROLLARY 2 *Under A.4, if $n > H(1)$ then region R_{III} exists when $\beta > \beta'$ with $\beta' < 1$ (where β' is the unique positive solution to $B - H(\beta) = 0$).*

Using Lemmas 1 and 2—and observing that $\delta > -n/\Lambda$ implies that $1+n+\delta\Lambda > 0$—we obtain the following result.

PROPOSITION 1 *Let $\Lambda = 1 + \lambda(n-1)$. Under assumptions A.1–A.3, if demand is sufficiently convex ($\delta \le -n/\Lambda$) then only region R_{I} exists. Otherwise, assume A.4 and let β' be the unique positive solution of $B = H(\beta)$. Then the following statements hold:*

(i) *if $\beta \le 1/(1+n+\delta\Lambda)$, then $\frac{\partial x^*}{\partial\lambda} \le 0$ and $\frac{\partial q^*}{\partial\lambda} < 0$ (R_{I});*

(ii) *if $1/(1+n+\delta\Lambda) < \beta \le \beta'$, then $\frac{\partial q^*}{\partial\lambda} \le 0$ and $\frac{\partial x^*}{\partial\lambda} > 0$ (R_{II});*

(iii) *if $\beta > \beta'$, then $\frac{\partial q^*}{\partial\lambda} > 0$ and $\frac{\partial x^*}{\partial\lambda} > 0$ (R_{III}).*

This proposition implies that, for demand that is convex enough, the equilibrium is always in R_{I}. Otherwise, the equilibrium is in R_{I} for only a low level of spillovers. It is instructive to compare these results with those reported by Leahy and Neary (1997, Prop. 3), in which there are no minority shareholdings and where R&D cooperation leads to more R&D and output (as in our R_{III}) whenever spillovers are positive. Yet in our case, R_{III} obtains only when spillovers are sufficiently high. Thus the "output cooperation" induced by minority shareholdings requires sufficiently high spillovers in order to increase R&D and output.

Finally, we are interested in analyzing the effect of λ on each firm's profit. We have that

$$\text{sign}\{\pi^{*\prime}(\lambda)\} = \text{sign}\left\{ -\beta c'(Bx^*)\frac{\partial x^*}{\partial \lambda} + f'(Q^*)\frac{\partial q^*}{\partial \lambda}\right\}. \tag{7}$$

Given that $\partial x^*/\partial\lambda > 0$ and $\partial q^*/\partial\lambda < 0$ in R_{II}, we can use (7) to show that—in this region—$\pi^{*\prime}(\lambda) > 0$. The sign of the effect of λ on π^* is less clear in R_{I} (since in that region, $\partial x^*/\partial\lambda < 0$ and $\partial q^*/\partial\lambda < 0$) and in R_{III} (where $\partial x^*/\partial\lambda > 0$ and $\partial q^*/\partial\lambda > 0$). Nevertheless, in Appendix B we prove the following result.

PROPOSITION 2 *At the symmetric equilibrium, the profit per firm* (π^*) *increases with* λ.

According to this proposition, the positive effect on price dominates the negative effect on R&D in R_{I}, and conversely in R_{III}, so that profits in both regions rise with the extent of cross-ownership. Hence firms are incentivized to acquire (minority) shareholdings in the industry—provided the agreements they enter are *binding* ones, because that feature allows them to increase profits.[21] Before proceeding with the welfare analysis, we examine the effect of β on equilibrium values.

4.3 Comparative statics on spillovers (β)

A sufficient (but not necessary) condition for increases in β to raise per-firm R&D and output is that $\partial_{\beta x}\phi_i > 0$. It is not difficult to see that $\text{sign}\{\partial_{\beta x}\phi_i\} = \text{sign}\{\lambda B/\tau - \chi(Bx^*)\}$; here χ is the elasticity of the slope of the innovation function, which is nonnegative. For a positive λ, we have $\partial_{\beta x}\phi_i > 0$ when the curvature (relative convexity) of the innovation function is sufficiently low. The term $\lambda B/\tau$ increases with β, so it suffices that $\chi < \lambda$ (since $B/\tau = 1$ for $\beta = 0$). Our next proposition follows.

PROPOSITION 3 *If the curvature* χ *of the innovation function is sufficiently low* ($\chi < \lambda$ *would be low enough*), *then* $\partial q^*/\partial\beta > 0$ *and* $\partial x^*/\partial\beta > 0$.

We can view the following results as corollaries. In AJ (where $\chi = 0$), stronger spillover effects raise the equilibrium values of output and R&D; in KMZ (where $\chi = 1/2$), the

[21] Farrell and Shapiro (1990), Flath (1991), and Reitman (1994) show that unilateral incentives to implement passive ownership structures may be lacking in Cournot competition with constant marginal costs. However, Gilo et al. (2006) show that cross-ownership arrangements facilitate tacit collusion (in the symmetric case) when the stakes are sufficiently high. For a differentiated product market with two firms, Karle et al. (2011) analyze the incentives of an investor to acquire a controlling or noncontrolling stake in a competitor.

same dynamic is observed when cross-ownership induces a high enough λ ($\lambda > 1/2$) and always in the case of a cartel (for which $\lambda = 1$). In the CE model, $\chi = \alpha + 1 > 1$. In this case, some tedious algebra shows that, for any positive λ, (i) $\partial q^*/\partial\beta > 0$ (with $\partial q^*/\partial\beta = 0$ when $\lambda = 0$) and (ii) x^* increases (resp. decreases) with β for high (resp. low) values of λ.

5 Welfare analysis

Welfare in equilibrium is given by the sum of consumer surplus and industry profits:

$$W(\lambda) = \int_0^{Q^*} f(Q)\, dQ - c(Bx^*)Q^* - n\Gamma(x^*).$$

We are interested in studying the effect of λ on welfare. Using the equilibrium conditions (2) and (3), we can write

$$W'(\lambda) = \left(-\Lambda f'(Q^*)\frac{\partial q^*}{\partial\lambda} - (1-\lambda)\beta(n-1)c'(Bx^*)\frac{\partial x^*}{\partial\lambda}\right)Q^*. \tag{8}$$

An increase in cross-ownership alters equilibrium values of quantities and R&D investments, and each additional unit of output and R&D has social value equal to (respectively) $\Lambda(-f'(Q^*))Q^*$ and $(1-\lambda)\beta(n-1)(-c'(Bx^*))Q^*$. Here Proposition 1 is useful. In R_{I} we have that $W'(\lambda) < 0$ because $\partial x^*/\partial\lambda \leq 0$ and $\partial q^*/\partial\lambda < 0$; in R_{III}, $W'(\lambda) > 0$ because $\partial x^*/\partial\lambda > 0$ and $\partial q^*/\partial\lambda > 0$. In R_{II}, however, the effect of λ on welfare is positive or negative according as whether the positive effect of minority shareholdings on R&D does or does not dominate its negative effect on output level. Moreover, from

$$\mathrm{sign}\{\mathrm{CS}'(\lambda)\} = \mathrm{sign}\left\{\frac{\partial q^*}{\partial\lambda}\right\} \tag{9}$$

it follows that the effect of λ on consumer surplus is positive (i.e., $\mathrm{CS}'(\lambda) > 0$) only in R_{III}. So even as consumers suffer from a higher degree of cross-ownership in R_{I} and R_{II}, it benefits them in R_{III}. One consequence is that optimal antitrust policy will tend to be stricter under the CS standard.

5.1 Socially optimal degree of cross-ownership

Let λ^o_{CS} and λ^o_{TS} denote the optimal degree of cross-ownership under the (respectively) consumer surplus and total surplus standard. Let $\beta'(\lambda)$ denote the dependence of β' on λ. Then it is easy to see that H is increasing in λ if and only if $\beta'(\lambda)$ is also increasing in λ. Recall that H is weakly increasing in λ under the all three model specifications: in AJ and KMZ, H is independent of λ; in the CE model, H is strictly increasing in λ. Furthermore, in these three specifications $W(\lambda)$ is single peaked:[22] a mild additional condition is required in KMZ (as we discuss later). In the CE model, numerical simulations show that—for the parameter range in which the second-order condition (SOC) and the regularity condition are satisfied—$W(\lambda)$ is strictly concave.

We know from Proposition 1 that if demand is convex enough then only R_{I} exists, in which case no cross-ownership is optimal regardless of spillover levels. Otherwise (and under some mild assumptions): if spillovers β are low enough then cross-ownership is also not optimal; and if spillovers are high enough then the degree of cross-ownership should be positive in terms of both total surplus and consumer surplus (i.e., $\lambda^o_{\text{TS}} > 0$ and $\lambda^o_{\text{CS}} > 0$). For intermediate values of β we have that $\lambda^o_{\text{TS}} > \lambda^o_{\text{CS}} = 0$. It follows that more cross-ownership should be allowed under the total surplus standard (i.e., $\lambda^o_{\text{TS}} \geq \lambda^o_{\text{CS}}$). These results are stated formally in our next proposition.

PROPOSITION 4 *Suppose that Assumptions A.1–A.4 hold. Then we have the following statements.*

(i) *If $\delta \leq -n$ (convex enough demand), then $\lambda^o_{\text{TS}} = \lambda^o_{\text{CS}} = 0$.*

(ii) *Otherwise, if H is weakly increasing in λ and $W(\lambda)$ is single peaked, then there are threshold values $\bar{\beta}$ and $\beta'(0)$ (with $\bar{\beta} < \beta'(0)$) such that*

- $\lambda^o_{\text{TS}} = \lambda^o_{\text{CS}} = 0$ *if* $\beta \leq \bar{\beta}$;
- $\lambda^o_{\text{TS}} > \lambda^o_{\text{CS}} = 0$ *if* $\beta \in (\bar{\beta}, \beta'(0))$; *and*
- $\lambda^o_{\text{TS}} \geq \lambda^o_{\text{CS}} > 0$ *if* $\beta > \beta'(0)$.

(iii) *In all cases, $\lambda^o_{\text{TS}} \geq \lambda^o_{\text{CS}}$.*

[22] In other words, $W(\lambda)$ is a function of one variable with only one stationary point that is a maximum (and hence a global maximum).

Figure 1 depicts the critical spillover threshold values stated in Proposition 4.

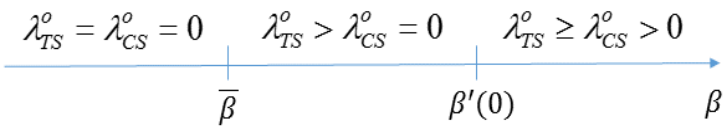

Fig. 1. Critical spillover threshold values when $\delta > -n$.

Remark 1. We have that $\bar{\beta} < 1$ if $n+(n-1)(\delta+n) > H(1)$ (see Lemma 6 in Appendix A). If $\bar{\beta} \geq 1$ then $\lambda^o_{\text{TS}} = \lambda^o_{\text{CS}} = 0$ for all $\beta \leq 1$.

Remark 2. Our single-peakedness assumption on $W(\lambda)$ ensures that $\bar{\beta}$ is the minimum threshold above which total surplus increases with λ (i.e., for which $\beta \leq \bar{\beta}$ implies $\lambda^o_{\text{TS}} = 0$).

Remark 3. The assumption that H is weakly increasing in λ ensures that $\beta < \beta'(0)$ implies $\lambda^o_{\text{CS}} = 0$ and that $\lambda^o_{\text{TS}} \geq \lambda^o_{\text{CS}}$. In the particular case where $\beta = \beta'(0)$ we have that $\lambda^o_{\text{TS}} \geq \lambda^o_{\text{CS}} \geq 0$ (see the proof of Proposition 4 in Appendix A).

If we relax the assumptions that $W(\lambda)$ be single peaked and that H be monotonic in λ, then we can provide a weaker characterization of the regions where cross-ownership is socially optimal (Proposition 5) and can also characterize the extreme solution regions where $\lambda^o_{\text{CS}} = 0$ or $\lambda^o_{\text{CS}} = \lambda^o_{\text{TS}} = 1$ (Proposition 6).

PROPOSITION 5 *Let A.1–A.4 hold. If $\delta > -(1+n)/n$, then there exist threshold values $\underline{\beta} < \bar{\beta} < \beta'(0)$ (where $\underline{\beta} = \inf\{1/(1+n+\Lambda\delta) : \lambda \in [0,1]\}$) such that*: (i) $\lambda^o_{\text{CS}} = \lambda^o_{\text{TS}} = 0$ *for* $\beta \leq \underline{\beta}$; (ii) $\lambda^o_{\text{TS}} > 0$ *for* $\beta > \bar{\beta}$; *and* (iii) $\lambda^o_{\text{CS}} > 0$ *for* $\beta > \beta'(0)$.[23]

[23] Here is a sketch of the argument behind the proposition. From Proposition 1 it now follows that, when $\beta \leq \underline{\beta}$, only R_{I} exists because $\delta > -(1+n)/n$ implies that $1+n+\delta\Lambda > 0$ and $\delta > -n$. The threshold $\underline{\beta}$ depends on the sign of δ. If demand is concave ($\delta > 0$), then $\underline{\beta} = 1/(1+n(1+\delta))$; if demand is convex ($\delta < 0$), then $\underline{\beta} = 1/(1+n+\delta)$. In both cases, $\underline{\beta}$ decreases with n (and tends to 0 with n). Note that in AJ and KMZ, demand is linear; hence $\underline{\beta} = 1/(1+n)$. Under CE, $\delta < 0$ when $\varepsilon > -1$ and so $\underline{\beta} = 1/(n-\varepsilon)$; in contrast, $\underline{\beta} = 1/(1-n\varepsilon)$ when $\varepsilon < -1$. Parts (ii) and (iii) follow as in Proposition 4: part (ii) because if $\beta > \bar{\beta}$ then $W'(0) > 0$ and so $\lambda^o_{\text{TS}} > 0$; and part (iii) because if $\beta > \beta'(0)$ then $\partial q^*/\partial\lambda|_{\lambda=0} > 0$ and $\lambda^o_{\text{CS}} > 0$. (See Appendix B for details.)

PROPOSITION 6 *Under A.1–A.4, the following statements hold*:

(i) $\beta \leq \beta'_{\min}$ *implies* $\lambda^o_{\text{CS}} = 0$; *and*

(ii) $\beta > \beta'_{\max}$ *implies* $\lambda^o_{\text{CS}} = \lambda^o_{\text{TS}} = 1$ *provided that* $\beta'_{\max} \leq 1$.[24]

We can now make the following claims. (a) If H is strictly *decreasing in* λ, then $\beta'(\lambda)$ also is: $\beta'_{\min} = \beta'(1)$ and $\beta'_{\max} = \beta'(0)$. (b) If H is strictly *increasing in* λ, then $\beta'(\lambda)$ also is: $\beta'_{\min} = \beta'(0)$ and $\beta'_{\max} = \beta'(1)$. (c) If H is *independent of* λ, then β' also is: $\beta'_{\min} = \beta'_{\max} = \beta'$.

Proposition 6 determines when the monopoly outcome ($\lambda = 1$) is optimal in terms of both consumer and total surplus (in those cases, we are in R_{III} and welfare is increasing in λ). In AJ and KMZ, the term H is independent of λ; thus case (c) applies and, as a result, the consumer surplus solution is bang-bang under either model specification. In both specifications it is clear that if $\lambda^o_{\text{CS}} > 0$ then necessarily $\lambda^o_{\text{TS}} = \lambda^o_{\text{CS}} = 1$. In the CE model, however, H and β' are strictly increasing in λ and so case (b) applies; hence solutions of the form $\lambda^o_{\text{TS}} > \lambda^o_{\text{CS}} > 0$ are possible.[25]

The scope for a Research Joint Venture.

An RJV can be understood as a situation where spillovers are fully internalized (i.e., $\beta = 1$). If the RJV is "cartelized" then also $\lambda = 1$. This arrangement can be optimal only if R_{III} exists for β large (with $\beta'_{\max} \leq 1$) and if $\partial q^*/\partial\beta > 0$ and $\partial x^*/\partial\beta > 0$ (which, by Proposition 3, requires that $\chi < 1$). Our next corollary states the result.

COROLLARY 3 *Again assume that A.1–A.4 hold. If* $\beta'_{\max} \leq 1$ *and if the innovation function's curvature is not too large* ($\chi < 1$), *then a cartelized RJV* ($\lambda = \beta = 1$) *is optimal in terms of consumer surplus and welfare.*

The assumptions of the corollary are fulfilled in the AJ and KMZ models, where $\gamma b < n$ and $\gamma b < 1$ are needed (respectively) to ensure that β'_{AJ} and β'_{KMZ} are less

[24] This proposition is proved by noting that $\beta'(\lambda)$ is a continuous function on $[0, 1]$ and so achieves a maximum ($\beta'_{\max}$) and a minimum ($\beta'_{\min}$) within that interval. If $\beta \leq \beta'_{\min}$, then $\partial q^*/\partial\lambda < 0$ for all $\lambda > 0$ and so $\lambda^o_{\text{CS}} = 0$; if $\beta > \beta'_{\max}$, then $\partial q^*/\partial\lambda > 0$ for all λ. Since $\partial q^*/\partial\lambda > 0$ implies $\partial x^*/\partial\lambda > 0$ by equation (5), it follows that $W'(\lambda) > 0$ for all λ by equation (8). Therefore, $\lambda^o_{\text{CS}} = \lambda^o_{\text{TS}} = 1$ provided that $\beta'_{\max} \leq 1$.

[25] In the proof of Proposition 4 we show that, in the CE case, CS is globally concave in λ when $B > H(\beta)|_{\lambda=0}$. Letting λ^* denote the unique value for λ such that $\partial q^*/\partial\lambda = 0$, we have that $\lambda^*_{\text{CS}} = 1$ when $\lambda^* > 1$; hence $\lambda^*_{\text{CS}} = \min\{\lambda^*, 1\}$, which yields $\lambda^*_{\text{TS}} > \lambda^*_{\text{CS}} > 0$.

than unity; recall that $\chi = 0$ in AJ and $\chi = 1/2$ in KMZ. In CE, $\lambda = 1$ is never socially optimal because $\beta'_{\text{CE}}(1) < 1$ only if $\varepsilon < \alpha/(1+2\alpha)$—which would contradict the regularity condition (see Table 8 in Appendix B).

Under some different conditions, an RJV with no cross-ownership ($\lambda = 0$ and $\beta = 1$) can be socially optimal in all three models (see Proposition 7 in Appendix B). To determine when an RJV with no cross-ownership is socially optimal, we use that—when $W(\lambda)$ is single peaked—$\bar{\beta}$ is the minimum threshold above which allowing some cross-ownership *increases* welfare (Proposition 4), so no cross-ownership is optimal if $\bar{\beta} \geq 1$. Satisfying that inequality requires $\gamma b \geq n^2$ in AJ, $\gamma b \geq n$ in KMZ, and a more involved condition in CE. In contrast with the AJ model, in both KMZ and CE we find that if $\lambda = 0$ then greater R&D spillovers reduce R&D expenditures ($\partial x^*/\partial\beta < 0$) while having no effect on output ($\partial q^*/\partial\beta = 0$). Although R&D expenditures are lower with higher β, the production costs of all firms are also lower. In both cases, the greater R&D spillover's negative effect on R&D expenditures is dominated by its positive effect on the innovation function; as a result, $\beta = 1$ is also socially optimal.

5.2 Comparative statics by model

We are interested in the comparative statics of the regions determining the scope for cross-ownership as described in Proposition 4. We are also interested in the comparative statics on λ^o_{CS} and λ^o_{TS} in the specified models.

Comparative statics on β'. Table 4 summarizes the comparative static results on $\beta'(0)$ in the models with respect to basic parameters. The threshold $\beta'(0)$ is weakly decreasing in n and is strictly decreasing in demand elasticity; however, the effect of the innovation function's elasticity is ambiguous. In terms of consumer surplus, in AJ it is optimal to suppress minority shareholdings for any level of spillovers when firm entry is insufficient—that is, when $n < \gamma b$ (since then $\beta'_{\text{AJ}} > 1$); in CE, suppression is optimal when $n < \varepsilon(2\alpha + 1)/\alpha$ (since $\beta'_{\text{CE}} > 1$ for $n < \varepsilon(2\alpha + 1)\Lambda/\alpha$).[26]

Table 10 (in Appendix B) reports the spillover thresholds for AJ, KMZ and CE models. To obtain some further insights into the comparative statics on the spillover

[26] In AJ, $\partial H/\partial n = 0$ and so increasing the number of firms reduces the β' threshold ($\partial\beta'_{\text{AJ}}/\partial n < 0$); in KMZ, however, firm entry has no effect on β' ($\partial\beta'_{\text{KMZ}}/\partial n = 0$). In the CE model, the direction of the effect of entry on the β' threshold depends on the value of λ. In particular, β'_{CE} decreases (resp. increases) with n when λ is low (resp. high).

Table 4: Comparative Statics on $\beta'(0)$

	AJ	KMZ	CE
Number of firms (n)	$-$	0	$-$
Elasticity of demand (b^{-1}, ε^{-1})	$-$	$-$	$-$
Elasticity of innovation function (γ^{-1}, α)		$-$	$+$ iff $\varepsilon > 1$
Slope of investment cost function (γ)	$+$		

threshold $\bar{\beta}$ and on the socially optimal degree of cross-ownership, we conducted some numerical simulations.[27] The results are described next.

Results I: Comparative statics on $\bar{\beta}$. First, $\bar{\beta}$ decreases with n in AJ and KMZ, and it also decreases in the CE model (according to numerical simulations). Second, for those models the simulations also reveal that $\bar{\beta}$ increases with the slope of demand and with the investment cost—and that $\bar{\beta}$ may take values greater than 1 when there are only a few firms in the market.[28] Third, in the CE model we find that $\bar{\beta}$ decreases with the (curvature) elasticity of the innovation function, α, and with the elasticity (and curvature) of demand, ε^{-1}. In this model, too, $\bar{\beta}$ may take values greater than 1 when there are few firms in the market. (See Appendix B.2 for more details.)

Therefore, for highly concentrated markets, no cross-ownership should be allowed for a wide range of spillovers. The reason is that the incentives for firms to "free ride" are stronger when the number of firms increases because each firm can then appropriate the R&D efforts of a greater number of participants.

Results II: Comparative statics on the socially optimal degree of cross-ownership. Our simulations generate three main findings. First, the socially optimal level of cross-ownership increases with the size of the spillovers, with the number of firms, and with the elasticity of demand and of the innovation function. Second, if the objective is to maximize consumer surplus, then the comparative statics are qualitatively similar but the scope for minority shareholdings is much lower. Third, increasing the number of firms may not in itself be sufficient for consumers to benefit from cross-ownership; in fact, this is the case in KMZ.

[27] Values for parameters are chosen so that the regularity condition and the SOCs are satisfied.

[28] In particular, from Table 10 (in Appendix B) it is straightforward to show that, in a duopoly, $\bar{\beta} > 1$ when $\gamma b > 4$ in AJ, when $\gamma b > 2$ in KMZ, and when $\alpha > 2\varepsilon/(\varepsilon^2 - 7\varepsilon + 6)$ in CE.

Table 5 summarizes the comparative statics results from our numerical simulations. Please note that we have made available an *application program for readers to perform their own simulations.*[29]

Table 5: Effect of Parameters on $\bar{\beta}$, λ^o_{TS}, and λ^o_{CS}

	$\bar{\beta}$			λ^o_{TS}			λ^o_{CS}		
	AJ	KMZ	CE	AJ	KMZ	CE	AJ	KMZ	CE
Number of firms (n)	$-$	$-$	$-$	$+$	$+$	$+$	$\langle+\rangle$	0	$(+)$
Elasticity of demand (b^{-1}, ε^{-1})	$-$	$-$	$-$	$+$	$+$	$+$	$\langle+\rangle$	$\langle+\rangle$	$[+]$
Elasticity of innovation function (γ^{-1}, α)		$-$	$-$		$+$	$+$		$\langle+\rangle$	$[+]$
Slope of investment cost function (γ)	$+$			$-$			$\langle-\rangle$		
Degree of spillover (β)				$+$	$+$	$+$	$(+)$	$(+)^*$	$[+]$

Key: $\langle+\rangle$, the parameter enlarges the region where $\lambda^o_{\text{CS}} = 1$; $\langle+\rangle$, the parameter reduces the region where $\lambda^o_{\text{CS}} = 1$; $(+)$, the effect is positive only if both β and n are sufficiently large (otherwise there is no effect); $(+)^*$, the effect is positive only if the parameter is sufficiently large and γb is sufficiently small (otherwise there is no effect); $[+]$, the effect is positive when n is sufficiently large (otherwise there is no effect).

We next provide graphical descriptions of the simulation results, first in the CE model and then in the AJ and KMZ models.

Constant elasticity model (Figure 2)**.** When the number of firms is small (less than five, in our example), it is never optimal to allow minority ownership interests (since then the equilibrium is in R_{I}). As the spillover effects and the number of firms increase, λ^o_{TS} also increases; however, any increase in λ^o_{CS} is considerably smaller. The equilibrium is then in R_{II}, where firms benefit and consumers suffer from a higher degree of cross-ownership (because output is lower). Even so, the overall effect on welfare of increasing λ is positive because the positive effect on x^* dominates the negative impact on q^*. Finally, we discover that raising λ *slightly* may be optimal from the consumer's standpoint when the number of firms in the market is sufficiently large (since then the equilibrium is in R_{III}).

Table 6 gives the socially optimal value for λ^o_{TS} in AJ and KMZ.[30]

[29] www.angelluislopez.net

[30] See Appendix B for a proof that welfare is single peaked and for the derivation of the value for λ^*_{TS} in AJ—and also in KMZ if a mild condition (which, as compared with the regularity condition, is slightly stricter in duopoly and looser in an oligopoly of three or more firms) holds. In particular, $\gamma b > 0.62$ ensures that welfare is single peaked for any number of firms under the KMZ model specification.

Table 6: Optimal Degree of Cross-ownership in AJ and KMZ

	λ^o_{TS}
AJ	$\min\left\{\max\left\{0, \frac{[(n+2)(n-1)\beta-(n-2)]\beta-b\gamma}{(n-1)[2(\beta-1)\beta+b\gamma]}\right\}, 1\right\}$
KMZ	$\min\left\{\max\left\{0, \frac{[(n+2)(n-1)\beta-b\gamma(n-1)-(n-2)]\beta-b\gamma}{(n-1)\{[2\beta+b\gamma(n-1)-2]\beta+b\gamma\}}\right\}, 1\right\}$

Optimal degree of cross-ownership in terms of total surplus and consumer surplus

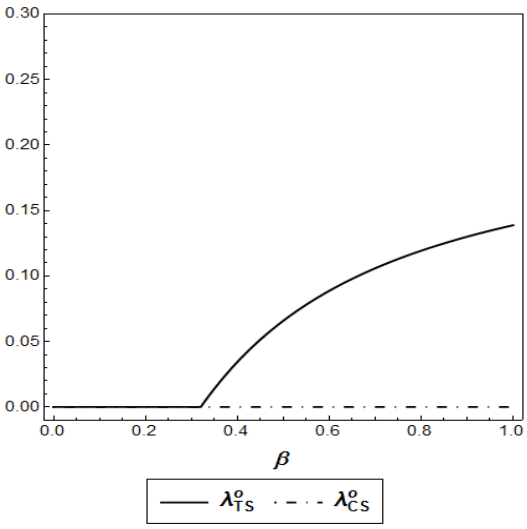

Fig. 2a. Constant elasticity model.
($\alpha = 0.1$, $\varepsilon = 0.8$, $\sigma = \kappa = 1$, $n = 8$)

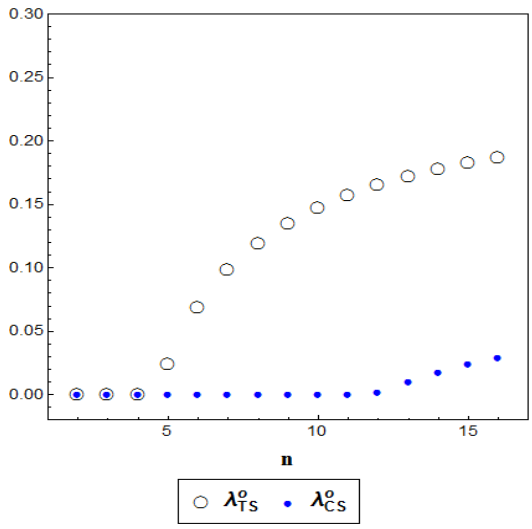

Fig. 2b. Constant elasticity model.
($\alpha = 0.1$, $\varepsilon = 0.8$, $\sigma = \kappa = 1$, $\beta = 0.8$)

AJ model (Figure 3)**.** Figure 3a is a snapshot of the application program. We see how, in the simulation considered (with $\beta = 0.5$ and $n = 6$), price increases whereas cost decreases with λ—and, correspondingly, how output per firm decreases while R&D per firm increases (two lower panels of the figure). The welfare translation of the increase in λ is given in the upper right panel, with decreasing consumer surplus and increasing per-firm profit that results in an interior solution for welfare of $\lambda^o_{\text{TS}} > 0$. The upper left panel plots λ^o_{TS} increasing smoothly with β and λ^o_{CS} increasing in a bang-bang fashion to reach $\beta > 0.82$, where $\lambda^o_{\text{TS}} = \lambda^o_{\text{CS}} = 1$. Figure 3b shows that λ^o_{TS} increases with the number of firms, although λ^o_{CS} does so weakly and only if n is sufficiently large (our example, where $\beta = 0.8$, requires $n > 6$). We have $\lambda^o_{\text{TS}} = \lambda^o_{\text{CS}} = 1$.

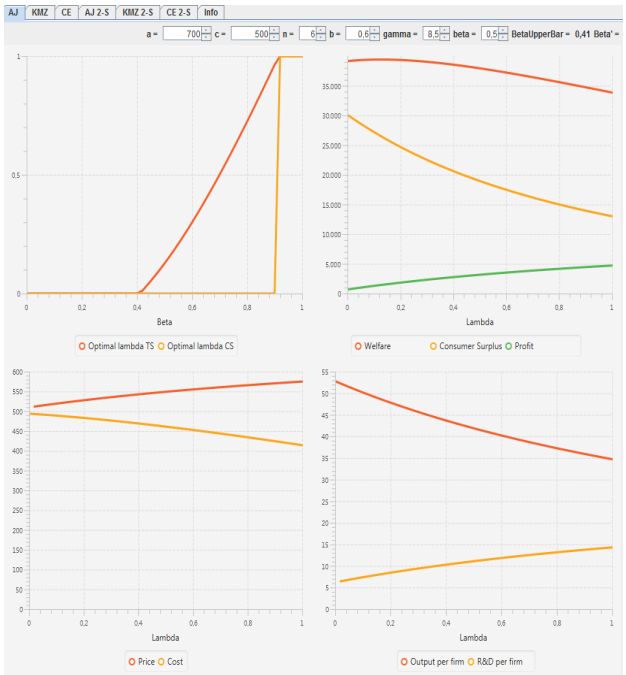

Fig. 3a. AJ model specification. Snapshot of the Application. ($\gamma = 8.5$, $\beta = 0.5$, $b = 0.6$.)

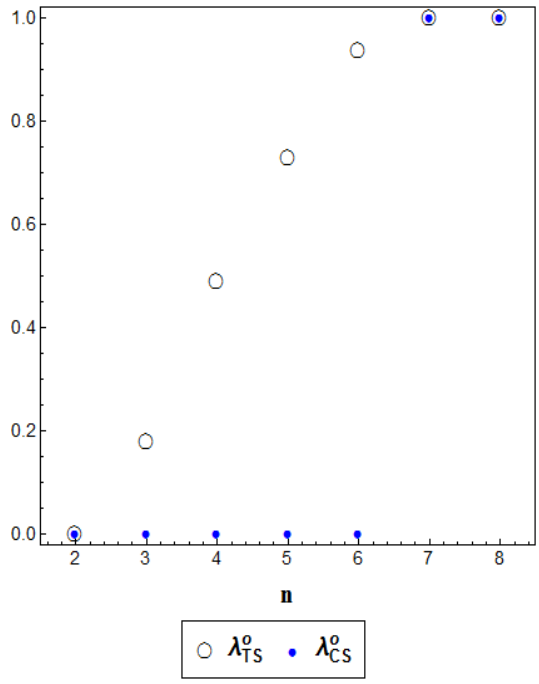

Fig. 3b. AJ model specification. ($\gamma = 7$, $\beta = 0.8$, $b = 0.6$.)

KMZ model (Figure 4)**.** Figure 4a plots the socially optimal degree of cross-ownership for different values of β when $\gamma b = 0.9$ (so that $\gamma b < 1$ and the assumptions of Corollary 3 are fulfilled); in this case, $\lambda^o_{\text{TS}} = 1$ when $\beta > \beta'_{\text{KMZ}} = 0.9$. Considering values of γb that are greater than 1, we find that λ^o_{TS} still increases with β; yet as expected, $\lambda^o_{\text{CS}} = 0$ even when $\beta = 1$. Figure 4b considers the effect of increasing n when $\beta = 0.8$. In KMZ, increasing the number of firms affects neither β' nor (as a result) $\text{sign}\{\text{CS}'(\lambda)\}$. Therefore, consumer surplus in KMZ is constantly decreasing in the degree of cross-ownership whenever $\beta < \beta'_{\text{KMZ}}$.

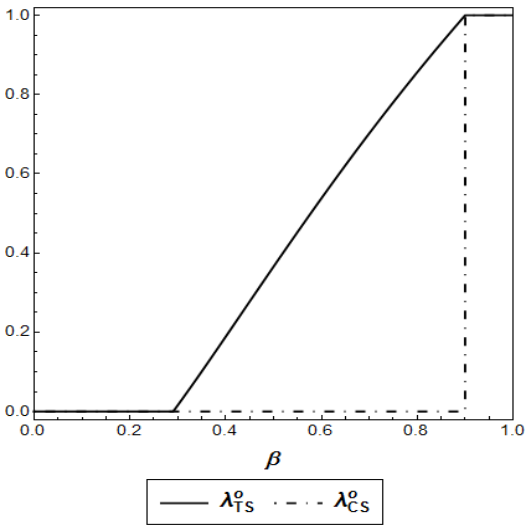

Fig. 4a. KMZ model specification.
($\gamma = 3$, $n = 6$ and $b = 0.3$.)

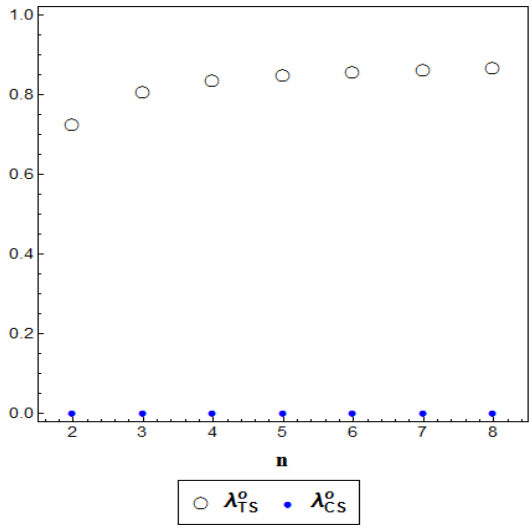

Fig. 4b. KMZ model specification.
($\gamma = 3$, $\beta = 0.8$, $b = 0.3$.)

6 Two-stage model

In this section we consider two-stage competition. In the first stage, every firm i commits to investing an amount x_i into R&D. In the second stage—and for given observable level of R&D expenditures—firms compete in the product market. In each stage we solve for the model's subgame-perfect equilibrium in terms of λ, or the degree of cross-ownership.

6.1 Equilibrium and strategic effects

Let $\mathbf{x} = [x_1, x_2, \ldots, x_n]$ be the first-stage R&D profile and let $\mathbf{q} = [q_1, q_2, \ldots, q_n]$ be the second-stage output profile. Let $q_i^*(\mathbf{x})$ denote firm i's (interior) output equilibrium value of the second-stage game associated with the R&D profile $\mathbf{x}$. Then, for all i, we have

$$\frac{\partial}{\partial q_i}\phi_i(\mathbf{q}^*(\mathbf{x}), \mathbf{x}, \lambda) = 0. \tag{10}$$

In the first stage, the first-order necessary conditions for an interior equilibrium are (for $i \neq j$ and $i, j = 1, 2, \ldots, n$)

$$\frac{\partial}{\partial x_i}\phi_i(\mathbf{q}^*(\mathbf{x}), \mathbf{x}, \lambda) + \sum_{j \neq i} \frac{\partial}{\partial q_j}\phi_i(\mathbf{q}^*(\mathbf{x}), \mathbf{x}, \lambda)\frac{\partial}{\partial x_i}q_j^*(\mathbf{x}) = 0. \tag{11}$$

The equilibrium R&D profile $\mathbf{x}^*$ is characterized by the system of equations (10) and (11)—provided the second-order conditions hold. Let $\mathbf{q}^* = \mathbf{q}^*(\mathbf{x}^*)$; then $\{\mathbf{x}^*, \mathbf{q}^*\}$ is the subgame-perfect equilibrium path of the two-stage game. The second term in equation (11) is the *strategic* effect on profits of investment. Building on Suzumura (1992) and Leahy and Neary (1997), we can show (using A.1) that $\partial q_j^*/\partial x_i$—when evaluated at a symmetric equilibrium, where $q_i^* = q^*$ and $x_i^* = x^*$ for all i—is given (for $\lambda < 1$) by

$$\frac{\partial q_j^*}{\partial x_i} = \frac{-c'(Bx^*)}{f'(Q^*)}\left[\frac{1}{n(1-\lambda)}\right]\left[\frac{2n+\Lambda\delta}{n+\Lambda(\delta+1)}\right](\tilde{\beta}(\lambda)-\beta); \tag{12}$$

here

$$\tilde{\beta}(\lambda) = \frac{n(1+\lambda)+\Lambda\delta}{2n+\Lambda\delta}.$$

Note that the threshold $\tilde{\beta}$ depends only on λ, n, and δ. The inequality $\tilde{\beta}(\lambda) > 0$ holds only if production decisions are strategic substitutes (i.e., only if $\partial_{q_{i,j}}\phi_i < 0$).[31] Note that $\tilde{\beta}(\lambda) < 1$ for $\lambda < 1$ and that $\tilde{\beta}(\lambda) \to 1$ as $\lambda \to 1$.

Evaluating $\partial\phi_i/\partial q_j$ at a symmetric equilibrium, we can rewrite the strategic effect of investment as follows:

$$\psi \equiv \frac{\partial\phi_i}{\partial q_j}\left(\frac{\partial q_j^*}{\partial x_i}\right) = (-c'(Bx^*))q^*\omega(\lambda)(\tilde{\beta}(\lambda)-\beta), \tag{13}$$

where[32]

$$\omega(\lambda) = \frac{\Lambda}{n}\left[\frac{2n+\Lambda\delta}{n+\Lambda(1+\delta)}\right] > 0.$$

Hence we may write the FOC (11) for $\lambda \in [0,1)$ as

$$-c'(Bx^*)\big[\tau + (n-1)\omega(\lambda)(\tilde{\beta}(\lambda)-\beta)\big]\frac{Q^*}{n} - \Gamma'(x^*) = 0. \tag{14}$$

When the stability condition in output is satisfied ($\Delta_q < 0$), we have $\partial q_i^*/\partial x_i > 0$. So if a firm increases its investment in R&D in the first stage, then it will increase its output

[31] This is so when $\delta > -(1+\lambda)n/\Lambda$ (see Table 7 in Appendix A), which holds for all λ and n when $\delta > -2$—in other words, the convexity of inverse demand must not be too high, which in turn implies that marginal revenue is strictly decreasing in output. It is worth noting that, in order for the concavity of ϕ_i with respect to q_i ($\partial_{qq}\phi_i < 0$) at a symmetric equilibrium to be guaranteed for *all* λ, we need the condition $\delta > -2$ (which guarantees strategic substitutability for all λ and n). The concavity condition is $\delta > -2n/\Lambda$, and it is strictest for $\lambda = 1$ (in which case it reduces to $\delta > -2$).

[32] The SOC, $\partial_{qq}\phi_i < 0$, requires that $2n + \Lambda\delta > 0$; the stability condition, $\Delta_q < 0$, requires that $n + \Lambda(1+\delta) > 0$. Therefore, $\omega(\lambda) > 0$.

in the second stage. At the same time, by equation (12) we have that

$$\text{sign}\left\{\frac{\partial q_j^*}{\partial x_i}\right\} = \text{sign}\{\beta - \tilde{\beta}(\lambda)\}$$

and that $\partial q_j^*/\partial x_i > 0$ when quantities are strategic complements (since then $\tilde{\beta} < 0$). In the case of strategic substitutes, $\partial q_j^*/\partial x_i > 0$ only if $\beta > \tilde{\beta}(\lambda)$. When a firm increases the amount invested in R&D, it exerts two opposite effects on the output decision of rival firms. There is a positive effect because rival firms become more efficient owing to the presence of spillovers. Yet there is also a negative effect because the reaction of rivals to firm i's higher quantity is to reduce their own output via competing in the market for strategic substitutes. If spillover effects are strong enough that $\beta > \tilde{\beta}(\lambda)$, then the positive effect outweighs the negative effect; this outcome implies that $\partial q_j^*/\partial x_i > 0$.

We can also conduct comparative statics on the threshold value $\tilde{\beta}(\lambda)$. Under Assumption A.1 and from the expression for $\tilde{\beta}$, it is straightforward to show the following result. This lemma highlights the crucial role played by demand curvature.

LEMMA 3 *For $\lambda < 1$, the threshold $\tilde{\beta}$: decreases (resp. increases) with the number of firms if demand is concave (resp. convex); increases with the degree of cross-ownership ($\partial\tilde{\beta}/\partial\lambda > 0$) if $\delta > -2$; and increases with the curvature of the inverse demand function δ (i.e., $\partial\tilde{\beta}/\partial\delta > 0$).*

Since $\partial\phi_i/\partial q_j < 0$, it follows that the sign of the strategic effect is opposite to the sign of $\partial q_j^*/\partial x_i$; that is,

$$\text{sign}\{\psi\} = -\text{sign}\left\{\frac{\partial q_j^*}{\partial x_i}\right\} = \text{sign}\{\tilde{\beta}(\lambda) - \beta\}.$$

Thus the strategic effect is positive if production decisions are substitutes and if β is below the threshold $\tilde{\beta}$. In this case, there are incentives to overinvest because increasing investment reduces the rival's output. Then, as shown by Leahy and Neary (1997, Prop. 1) for $\lambda = 0$, equations (10) and (14) together imply that output and R&D are higher in the two-stage model than in the static model.[33] It is intuitive that, if $\beta < \tilde{\beta}$, then each

[33] This result is derived under assumptions yielding a unique equilibrium and such that the two models' respective profit functions satisfy the Seade stability condition with respect to R&D—namely, that the marginal profit of each firm with respect to R&D must decrease with a uniform increase in R&D by all firms.

firm expects a higher first-stage investment in R&D to reduce the second-stage output of rival firms. The implication is that $\psi \equiv (\partial\phi_i/\partial q_j)(\partial q_j^*/\partial x_i) > 0$ and so each firm is led to increase their first-stage R&D investments, which in turn boosts output in the second stage ($\partial q_i^*/\partial x_i > 0$). Observe that $\tilde{\beta}(1) = 1$: if there is no RJV ($\beta < 1$) then, for high levels of cross-ownership, the strategic effect is always positive ($\beta < \tilde{\beta}$). In contrast, if β exceeds $\tilde{\beta}$ then the strategic effect is negative; hence both output and R&D are lower in the two-stage model than in the static model.

6.2 Comparative statics on cross-ownership

Next we analyze how the degree of cross-ownership affects the decisions on output and R&D that are made in equilibrium. By using (13) and by totally differentiating the system formed by (10) and (11) before evaluating it at a symmetric equilibrium, we can solve both for $\partial q^*/\partial\lambda$ and for $\partial x^*/\partial\lambda$ under regularity conditions. Let $s(\lambda) = \omega(\lambda)\left(\tilde{\beta}(\lambda) - \beta\right)$. We obtain the following result.

LEMMA 4 *In the two-stage model*:

$$\text{sign}\left\{\frac{\partial x^*}{\partial\lambda}\right\} = \text{sign}\big\{(\beta + s'(\lambda))P'(c)^{-1}n - [\tau + (n-1)s(\lambda)]\big\}; \tag{15}$$

$$\text{sign}\left\{\frac{\partial q^*}{\partial\lambda}\right\} = \text{sign}\big\{(\beta + s'(\lambda))B - \beta H(\beta)\big\}. \tag{16}$$

Moreover, if $\partial x^*/\partial\lambda \leq 0$ *then* $\partial q^*/\partial\lambda < 0$.

So once again we find that allowing for some additional degree of cross-ownership will increase output only if it also boosts R&D. In particular, from (15) we obtain that $\partial x^*/\partial\lambda > 0$ if and only if $\beta > \underline{\beta}^{2S}$ (see the proof of this lemma for more on $\underline{\beta}^{2S}$).

We are now in a position to derive the threshold values of spillovers that determine the *sign* of the effect, at equilibrium, of λ on R&D and output. In this we assume that there is a unique positive β, denoted $\beta^{2S\prime}$, that solves the equation $(\beta + s'(\lambda))B = \beta H(\beta)$.[34]

[34] To streamline the discussion, here we shall refer simply to the left-hand side (LHS, $(\beta + s'(\lambda))B$) and the right-hand side (RHS, $\beta H(\beta)$) of this equation. RHS is a constant in AJ (see Table 3), but it increases with β in KMZ and CE. Numerical simulations show that LHS is also increasing in β and that it takes a lower value (than RHS) at $\beta = 0$. In AJ there exists a unique $\beta^{2S'} < 1$ when n is sufficiently large—or when γ and b are sufficiently low—and λ is sufficiently large. In KMZ, RHS increases more slowly than LHS when γ and b are smaller whereas LHS increases more rapidly for higher values of λ. It follows that, for high λ and sufficiently low γ and b, there exists a unique $\beta^{2S'}$ that is nearly (but still less than) 1. In CE, RHS increases faster than LHS and there seems to be no solution, in which case region R_{III} does not exist.

Then we have $\partial q^*/\partial\lambda \leq 0$ for $\beta \in [0, \beta^{2S\prime}]$ and $\partial q^*/\partial\lambda > 0$ for $\beta \in (\beta^{2S\prime}, 1]$. Therefore: R_{I} (where $\partial x^*/\partial\lambda \leq 0$ and $\partial q^*/\partial\lambda < 0$) occurs when $\beta \leq \underline{\beta}^{2S}$; R_{II} (where $\partial q^*/\partial\lambda \leq 0$ and $\partial x^*/\partial\lambda > 0$) occurs for $\beta \in (\underline{\beta}^{2S}, \beta^{2S\prime}]$; and R_{III} (where $\partial q^*/\partial\lambda > 0$ and $\partial x^*/\partial\lambda > 0$) occurs when $\beta > \beta^{2S\prime}$.

These results extend Proposition 1 to the two-stage model. A direct application of (15) and (16) allows us to derive the threshold values for each of the model specifications considered in the paper (see Appendix B.2).

Our findings are comparable to those of Leahy and Neary (1997, Prop. 3). Those authors show that if cooperation does not extend to output (i.e, with collusion only at the R&D level) then the result is reduced output and R&D—unless spillovers are high enough, in which case firms increase both output and R&D. These two results correspond to regions R_{I} and R_{III}, respectively. In addition, we identify region R_{II}: where cooperation driven by minority shareholdings leads to less output and more R&D. Another difference is that, in Leahy and Neary's model, the spillover threshold above which cooperation leads to more output and R&D lies *strictly* between 0 and 1. In contrast, here (as in the simultaneous choice case) there is no guarantee that R_{III} exists; that is, $\beta^{2S\prime}$ may lie above 1.

In the proof of Lemma 7 (see Appendix B.1) we show that, in the two-stage model,

$$W'(\lambda) = \Big\{ -\Lambda f'(Q^*)\frac{\partial q^*}{\partial\lambda} \\ - \big((1-\lambda)\beta - \omega(\lambda)(\tilde{\beta}(\lambda) - \beta)\big)(n-1)c'(Bx^*)\frac{\partial x^*}{\partial\lambda}\Big\}Q^*.$$

Hence the strategic effect of investment, $\omega(\lambda)(\tilde{\beta}(\lambda) - \beta)$, plays an important role in determining the impact of cross-ownership on welfare. When the strategic effect is negative ($\beta > \tilde{\beta}(\lambda)$), the two-stage model behaves like the simultaneous model: $W'(\lambda) < 0$ in R_{I}, $W'(\lambda) > 0$ in R_{III}, and $W'(\lambda)$ either positive or negative (depending on the extent of spillovers) in R_{II}. Yet when the strategic effect is positive and spillovers are sufficiently low (though not necessarily close to zero), $W'(\lambda) < 0$ in R_{II} and $W'(\lambda)$ can be positive or negative in R_{I} and in R_{III}. A consequence of some interest is that, in R_{III}—where $\partial x^*/\partial\lambda > 0$ and $\partial q^*/\partial\lambda > 0$, so consumer surplus increases with λ (indeed, $\lambda = 1$ is

optimal for consumers)—total surplus can be decreasing in λ for sufficiently high λ.[35] Then, in stark contrast to the simultaneous model and owing to the strategic effect of investment, for some spillover values it may be that $\lambda^o_{\text{CS}} = 1 > \lambda^o_{\text{TS}} > 0$. We illustrate this case under the AJ and KMZ model specifications in the simulations that follow (see Figure 6a and Figure 7).[36] Similarly as in the simultaneous case, there is a threshold value $\bar{\beta}^{2S}$ for which $\lambda^o_{\text{TS}} > 0$ if $\beta > \bar{\beta}^{2S}$; the condition under which $\bar{\beta}^{2S} < 1$ is given by Lemma 7 (in Appendix A).[37]

6.3 Simulations

This section presents our simulations of the three considered models.[38] These simulations confirm the qualitative results obtained in the static model, but with two caveats: (i) in the two-stage model, the socially optimal level of cross-ownership tends to be higher when spillovers are high; and (ii) in some cases the consumer surplus standard may call for more cooperation than does the total surplus standard (i.e., $\lambda^o_{\text{CS}} > \lambda^o_{\text{TS}} > 0$). Result (i) indicates that the strictness of antitrust policy (in terms of limiting cross-ownership) should be moderated in the two-stage model when spillovers are high. The reason underlying both results is the strategic effect. When β is high, the strategic effect is negative and so there are incentives to underinvest; then it pays to increase λ in order to stimulate investment and output (result (i)). We have already observed that result (ii) may obtain when the strategic effect is positive (which happens for intermediate levels of β when λ is large, since $\tilde{\beta}(\lambda) \to 1$ as $\lambda \to 1$ and so $\tilde{\beta}(\lambda) > \beta$); the resulting overinvestment increases output (and is good for consumer surplus) but comes at the cost of reducing firms' profits, reducing total surplus, and "overshooting" marginal cost reductions.

Constant elasticity model. As in the simultaneous case, we observe here that if n is small then the equilibrium is in R_{I}, which implies that no cross-ownership is socially optimal. Yet as β and n increase, λ^o_{TS} also increases. This result is consistent with the literature.[39] Note that λ^o_{TS} in the two-stage game is above the static level in a large

[35] For $\beta < 1$, we have $(1-\lambda)\beta - \omega(\lambda)(\tilde{\beta}(\lambda) - \beta)|_{\lambda=1} = -(1-\beta) < 0$.

[36] In CE, as in the simultaneous model, λ^*_{CS} is usually zero or very close to zero.

[37] If the condition holds then $W'(0)|_{\beta=1} > 0$, in which case there exists a sufficiently large spillover value for which some degree of cross-ownership is welfare enhancing. (In Appendix B.2 we compute $\bar{\beta}^{2S}$ for the model specifications considered in this paper.)

[38] Appendix B.2 provides complementary results, explanations, and figures.

[39] For example, in a model with no cross-ownership Spence (1984) used numerical simulations to demon-

region of spillovers. For low values of β and λ, the strategic effect is positive. Then, as stated previously, the two-stage model behaves differently than the static model in that welfare can *increase* with the degree of cross-ownership (in R_{I}). This case is illustrated in Figure 5, where—for low β—λ^o_{TS} in the two-stage model is larger than in the static model. For intermediate values of spillovers, the strategic effect becomes negative (but remains close to zero); for higher spillover values, λ^o_{TS} increases with β more rapidly (i.e., convexly) when the strategic effect is strong.

Comparative static results with respect to α and ε^{-1} are similar to those in the static model. In the CE model, however, λ^o_{CS} is independent of the number of firms and may be positive if spillovers are extensive enough.

Optimal degree of cross-ownership in terms of total surplus and consumer surplus

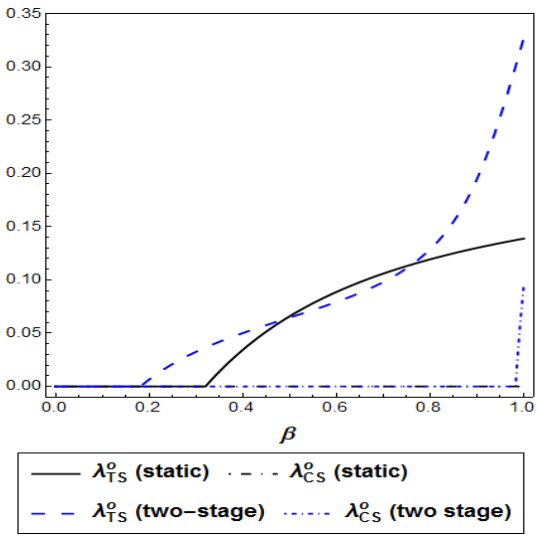

Fig. 5a. Constant elasticity model.

($\alpha = 0.1$, $\varepsilon = 0.8$, $\sigma = \kappa = 1$, $n = 8$.)

AJ model. Figure 6a plots welfare, consumer surplus, profit, price, cost, q^*, and x^* as functions of λ (for $\beta = 0.65$ and $n = 6$). In contrast with the static model, the simulations indicate that prices may be hump-shaped when cost decreases with λ; correspondingly, this figure shows how output per firm is U-shaped when R&D per firm increases (two lower panels). The welfare translation of the increase in λ is given in the upper right panel; it shows U-shaped consumer surplus and increasing profit per firm, with the result of an interior solution for welfare that features a large value of $\lambda^o_{\mathrm{TS}} > 0$. The upper left panel

strate that an increase in β reduces x^* and that, for a given β and $n \geq 2$, the incentives for cost reduction relative to the social optimum decline with n (see Spence 1984, Table I).

shows how, for an intermediate range of spillovers, $\lambda^o_{\text{CS}} = 1 > \lambda^o_{\text{TS}} > 0$. This relation is corroborated in Figure 6b, which also confirms that in the two-stage model it is socially desirable to induce more cooperation that in the static model under the TS standard and also under the CS standard.

Optimal degree of cross-ownership in terms of total surplus and consumer surplus

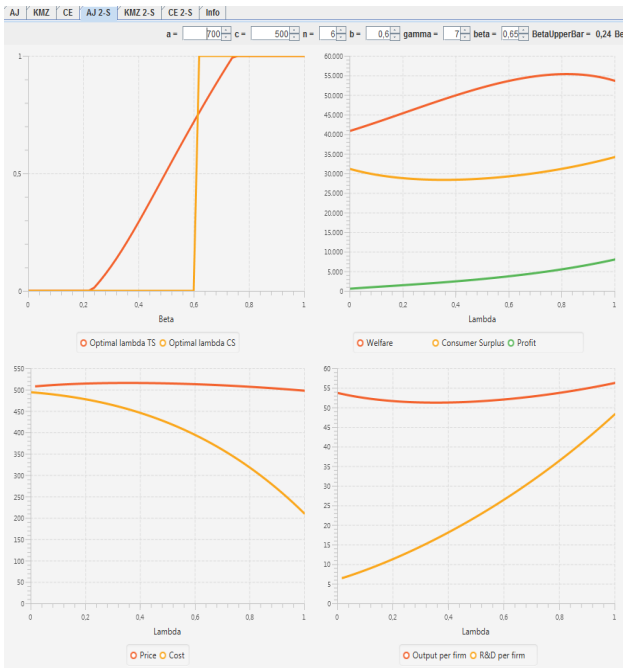

Fig. 6a. AJ model specification. Snapshot of the Application. ($a = 700$, $c = 500$, $\gamma = 7$, $n = 6$ and $b = 0.6$.)

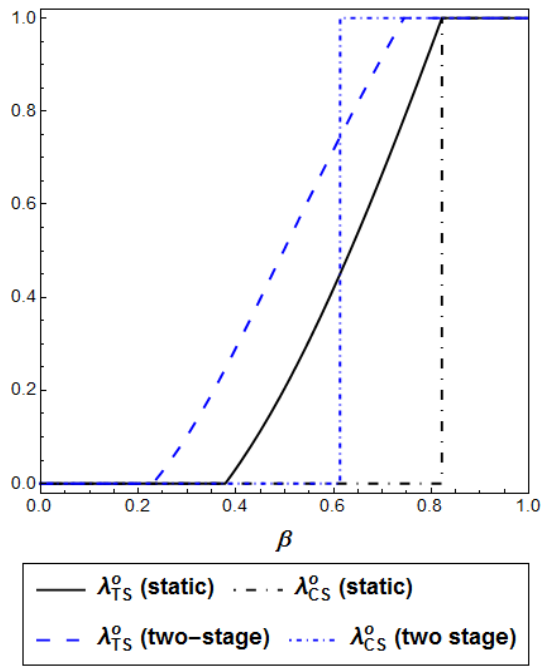

Fig 6b. AJ model specification. ($a = 700$, $c = 500$, $\gamma = 7$, $n = 6$ and $b = 0.6$.)

Satisfying the expression $\lambda^o_{\text{CS}} = 1 > \lambda^o_{\text{TS}} > 0$ becomes possible when the strategic effect is positive and strong enough. Then there is overinvestment in R&D during the first stage, which boosts output in the second stage. The strategic effect becomes positive for intermediate values of β when λ is sufficiently high. For an intermediate level of spillovers, total surplus is not maximized with full cooperation because that would entail too much production (reducing firms' profits). In Figure 6a we see that output increases with increasing λ and also how fast R&D per firm increases with λ. Only when the spillover is large enough ($\beta > 0.75$ in our example) are firms efficient enough to benefit from such high production quantities.

More precisely, since $\beta^{2S\prime}$ decreases with λ, it follows that—for a given β and a sufficiently high λ—we have $\beta > \beta^{2S\prime}$ and so the equilibrium is then in R_{III}, where CS increases with λ.[40] In particular: for $\beta = 0.62$, the equilibrium is in R_{III} when $\lambda > 0.41$ (see Figure 6a). Here the strategic effect is positive since $\tilde{\beta}(\lambda) > 0.62$ for $\lambda > 0.24$. Furthermore, if $\lambda > 0.69$ then the strategic effect is strong enough to *reverse* the sign of the effect of $\partial x^*/\partial\lambda$ on $W'(\lambda)$ (i.e., to make it negative); as a result, in a neighborhood of $\beta = 0.62$ there is a global maximum for $W(\lambda)$: even if the equilibrium is in R_{III} we have that $W'(\lambda) < 0$ for high values of λ, which implies $\lambda^o_{\text{TS}} \in (0, 1)$.

KMZ model. Finally, the optimal degree of cross-ownership in terms of total surplus is increasing in β also under the KMZ model specification. With regard to consumer surplus, numerical simulations suggest that normally no cross-ownership is optimal; however, $\lambda^o_{\text{CS}} = 1$ can be optimal for low n, b, and γ (see Figure 7). As in AJ, we can have $\lambda^o_{\text{CS}} > \lambda^o_{\text{TS}}$ for intermediate spillover values (because of the strategic effect).

Optimal degree of cross-ownership in terms of total surplus and consumer surplus

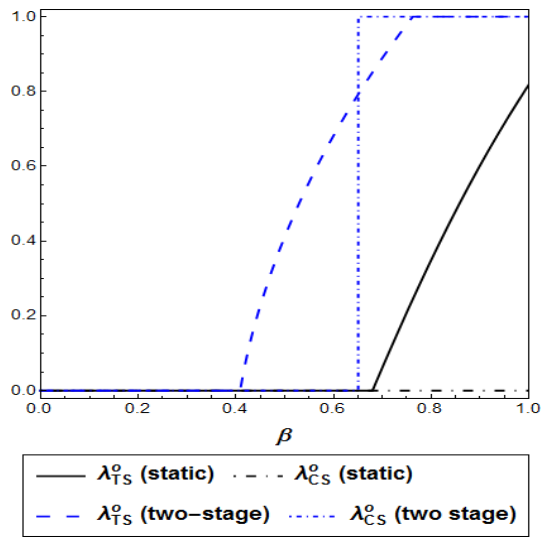

Fig. 7. KMZ model specification.
($a = 700$, $c = 500$, $\gamma = 5.5$, $n = 2$ and $b = 0.2$.)

The pattern of results in our comparative statics analysis of the other parameters in AJ, KMZ, and CE is similar to that for the one-stage game (see Table 5). The only exceptions we have found are as follow. In AJ: although decreasing b enlarges the region

[40] That is, CS is strictly convex in λ and so $\lambda^*_{\text{CS}} = 1$ when CS(1) > CS(0).

where $\lambda^o_{\text{CS}} = 1$ is optimal (as in the static case), λ^o_{CS} can be lower than 1 (for a sufficiently low b) when spillovers are sufficiently high. In KMZ: although λ^o_{CS} is independent of n in the static case, in the two-stage game it can decrease with n when there are few firms in the market.

7 Alternative interpretation: R&D cooperation extending to the product market

The "sympathy coefficient" λ can be viewed also as a measure of the intensity of competition; for example, a low λ may be the result of firms' limited scope for collusion owing to a low discount factor. Note that this parameter has an empirical counterpart in the estimation of market power because it corresponds to a constant elasticity of conjectural variation, which can be used to estimate the degree of industry cooperation.[41] Intermediate degrees of cooperation may arise from the strictness of antitrust policy: in terms of limiting not only cross-shareholdings but also collusion in the product market. The latter scenario is relevant given the long-standing suspicion that R&D cooperation facilitates coordination in the product market. This outcome may reflect the existence of ancillary restraints (or of other channels through which cooperative R&D may lead to coordination in the product market)[42] or the existence of multimarket contacts.[43] There is also growing evidence that R&D cooperation facilitates product market cooperation from empirical studies (Duso et al. 2014; Goeree and Helland 2010), from experiments

[41] Michel (2016) estimates the degree of profit internalization after ownership changes in differentiated product industries. He allows each firm's objective function to depend on other firms' profits by incorporating the parameter λ_{ij}, which is the extent to which brand i accounts for brand j's profits when setting the optimal brand-i price.

[42] As when, for example, an RJV stipulates downstream market division for any patents that may result from the venture or when there are collateral agreements that impose cross-licensing of old patents (or a per-unit output royalty for using new patents)—since these circumstances reduce the incentives of firms to increase their output (Grossman and Shapiro 1986; Brodley 1990). The various channels through which cooperative R&D may facilitate coordination in the product market are analyzed by Martin (1995), Greenlee and Cassiman (1999), Cabral (2000), Lambertini et al. (2002), and Miyagiwa (2009). Rey and Tirole (2013) examine how both independent marketing and joint marketing alliances (e.g., patent pools) can lead to tacit collusion.

[43] See the related evidence in Parker and Röller (1997) for mobile telephony and in Vonortas (2000) for US RJVs.

(Suetens 2008),[44] and from antitrust cases.[45]

Our analysis therefore extends the traditional framework in two directions: no separation between coordination in R&D and output, whether because of cross-ownership or because R&D cooperation naturally extends to product market cooperation; and the presence of intermediate degrees of cooperation in response to the strictness of competition policy. Antitrust authorities affect the parameter λ by limiting cross-shareholdings; we can also interpret λ as a measure of the intensity with which collusion is scrutinized.[46] From a policy perspective, our results highlight the tension between a CS standard as proclaimed by many competition authorities and the fact that R&D cooperation is widely allowed (and even encouraged) by those same public authorities. Whenever cooperation in R&D extends to competition in the product market, policy must in general be much stricter if the aim is to increase consumer surplus.

8 Concluding remarks

In the context of a general symmetric oligopoly Cournot model with cost-reducing R&D investment, spillovers, and symmetric partial ownership interests, we have identified tight conditions—in terms of the curvature of demand, market concentration, and the extent of spillovers—under which cross-ownership is welfare enhancing. We also find that the socially optimal degree of cross-ownership is positively associated with the number of firms, with the elasticity of demand and of the innovation function, and with the extent of spillovers. Yet if the objective is to maximize consumer surplus then (i) the scope for partial ownership interests is greatly reduced and (ii) firm entry need not induce, at the welfare optimum, a higher degree of cross-ownership. We say that an antitrust policy is *strict* to the extent that it limits minority shareholdings, and (alternatively) when it is increasingly activated as cooperation in R&D extends to cooperation in output. The

[44] Suetens (2008) uses a two-stage duopoly model to confirm that cooperation in reducing R&D costs facilitates price collusion. Agents engage in cooperative R&D projects more than once, and they interact repeatedly in the product market. For both small and large spillovers this author finds that cooperativeness in the pricing stage is generally higher when subjects can make binding R&D agreements than in the baseline treatments without the possibility of such agreements.

[45] Goeree and Helland (2010) gather a number of cases in the petroleum industry, the computer industry, the market for semiconductor memory, and the telecommunications sector.

[46] Besanko and Spulber (1989) show that, if collusive behavior is unobservable and if production costs are private information, then the antitrust authority may optimally induce some intermediate degree of collusion among firms.

competition-reducing effect of cross-ownership justifies policy intervention, as forcefully underscored by the empirical work of Azar et al. (2015, 2016). However, some degree of cross-ownership may actually be welfare enhancing, and may even increase consumer surplus, for an industry that exhibits sufficiently large R&D spillovers. In the extreme, it may be socially optimal to form a cartelized RJV when the curvature of the innovation function is not too large. This paper stipulates precise conditions that can be checked to see—in industries with significant R&D spillovers—whether cross-ownership is (or is not) still improving social welfare. Our results imply that competition policy and patent policy are complementary (as in Spulber (2013) but in a very different model) whenever high patent protection goes together with a low level of spillovers. This is so since then with a low (high) level of spillovers we want to tighten (relax) competition policy.

We extend the "simultaneous action" (static) model of R&D investment to a strategic commitment (two-stage) model and find that our results are (with some caveats) robust to this extension. It turns out that, when spillovers are above a given threshold, firms invest less in R&D and produce less in the two-stage than in the static model; hence the strategic effect of investment becomes negative. In this case, the social gains—from a higher degree of cross-ownership that induces firms to invest and approach more nearly the socially optimal production levels—are even greater. We also characterize how these gains are affected by the number of firms, the extent of cross-ownership, and the curvature of the inverse demand function. Numerical simulations corroborate that, when spillovers are high, an antitrust policy should be less strict in the two-stage model than in the static model. In this case, it need no longer be true that the consumer surplus standard calls for reduced cross-ownership: cooperation may be needed to induce the investment required for high output, despite that level of investment possibly being excessive from the total surplus standpoint.

Table 7: Summary of Basic Expressions

$\partial_{qq}\phi_i = (\partial^2\phi_i/\partial q_i^2)\big\vert_{q^*,x^*} =$	$f'(Q^*)(2+\delta\Lambda/n)$
$\partial_{q_{i,j}}\phi_i = (\partial^2\phi_i/\partial q_i\partial q_j)\big\vert_{q^*,x^*} =$	$f'(Q^*)(1+\lambda+\delta\Lambda/n)$
$\partial_{xx}\phi_i = (\partial^2\phi_i/\partial x_i^2)\big\vert_{q^*,x^*} =$	$-(c''(Bx^*)\tilde{\lambda}q^* + \Gamma''(x^*))$
$\partial_{x_{i,j}}\phi_i = (\partial^2\phi_i/\partial x_i\partial x_j)\big\vert_{q^*,x^*} =$	$-c''(Bx^*)\beta q^*\{1+\lambda[1+(n-2)\beta]\}$
$\partial_{xq}\phi_i = (\partial^2\phi_i/\partial x_i\partial q_i)\big\vert_{q^*,x^*} =$	$-c'(Bx^*)$
$\partial_{\lambda q}\phi_i = (\partial^2\phi_i/\partial\lambda\partial q_i)\big\vert_{q^*,x^*} =$	$f'(Q^*)(n-1)q^*$
$\partial_{\lambda x}\phi_i = (\partial^2\phi_i/\partial\lambda\partial x_i)\big\vert_{q^*,x^*} =$	$-\beta(n-1)c'(Bx^*)q^*$
$(\partial_{qq}\phi_i)(\partial_{xx}\phi_i) - (\partial_{xq}\phi_i)^2 =$	$f'(Q^*)(2+\Lambda\delta/n)[c''(Bx^*)(Q^*/n)\tilde{\lambda} + \Gamma''(x^*)] - c'(Bx^*)^2$
$\Delta_q = \partial_{qq}\phi_i + \partial_{q_{i,j}}\phi_i(n-1) =$	$f'(Q^*)\left[n+\Lambda(\delta+1)\right]$
$\Delta_x = \partial_{xx}\phi_i + \partial_{x_{i,j}}\phi_i(n-1) =$	$-(c''(Bx^*)B\tau q^* + \Gamma''(x^*))$
$\Delta(Q^*,x^*) =$	$-\left[c''(Bx^*)B\tau(Q^*/n) + \Gamma''(x^*)\right]\left\{f'(Q^*)\left[\Lambda(1+\delta)+n\right]\right\} - c'(Bx^*)^2\tau B$
$\psi_q =$	$-c'(Bx^*)s(\lambda)$
$\psi_x =$	$-c''(Bx^*)Bs(\lambda)q^*$
$\psi_\lambda =$	$-c'(Bx^*)s'(\lambda)q^*$
$\tilde{\Delta}(Q^*,x^*) =$	$-\left(c''(Bx^*)Bq^*\mu + \Gamma''\right)f'(Q^*)\left[\Lambda(1+\delta)+n\right] - c'(Bx^*)^2\mu B$
$H =$	$-[f'(Q^*)/(\beta c'(Bx^*)^2)](-c''(Bx^*)B\Gamma'(x^*)/c'(Bx^*) + \Gamma''(x^*))$
(one-stage) $\partial x^*/\partial\lambda =$	$[(n-1)(Q^*/n)f'(Q^*)c'(Bx^*)/\Delta]\left\{\beta\left[\Lambda(1+\delta)+n\right]-\tau\right\}$
(one-stage) $\partial q^*/\partial\lambda =$	$[(n-1)(Q^*/n)/\Delta]\left\{c'(Bx^*)^2\beta B + f'(Q^*)\left[c''(Bx^*)(Q^*/n)B\tau + \Gamma''(x^*)\right]\right\}$
	$= [(n-1)(Q^*/n)/\Delta]\,c'(Bx^*)^2\beta\,(B - H(\beta))$
(two-stage) $\partial x^*/\partial\lambda =$	$\xi f'(Q^*)c'(Bx^*)\left\{(\beta+s'(\lambda))\left[\Lambda(1+\delta)+n\right] - \left[\tau+(n-1)s(\lambda)\right]\right\}$
(two-stage) $\partial q^*/\partial\lambda =$	$\xi\left((\beta+s'(\lambda))c'(Bx^*)^2B + f'(Q^*)\left\{c''(Bx^*)(Q^*/n)B\left[\tau+(n-1)s(\lambda)\right]+\Gamma''(x^*)\right\}\right)$

with $B = 1+\beta(n-1)$, $\Lambda = 1+\lambda(n-1)$, $\tau = 1+\lambda(n-1)\beta$, $\tilde{\lambda} = 1+\lambda(n-1)\beta^2$, $\mu \equiv \tau + s(\lambda)(n-1)$ and $\xi \equiv (n-1)(Q^*/n)/\tilde{\Delta}$.

9 Appendix A

9.1 Minority shareholdings

9.1.1 Common-ownership

Letting υ_{ij} be the ownership share of firm j owned by owner/investor i, the total (portfolio) profit of owner i is: $\pi^i = \sum_k \upsilon_{ik}\pi_k$, where π_k are the profits of portfolio firm k. Since each firm takes into account its shareholders' incentives through the control weights ζ_{ij}, firm j maximizes a weighted average of its shareholders' portfolio profits:

$$\sum_{i=1}^{n} \zeta_{ij}\pi^i = \sum_{i=1}^{n} \zeta_{ij} \sum_{k=1}^{n} \upsilon_{ik}\pi_k.$$

A variety of governance structures may be considered by assigning different values to the control rights. Note that the above expression is equivalent to

$$\sum_{i=1}^{n} \zeta_{ij}\upsilon_{ij}\pi_j + \sum_{i=1}^{n} \zeta_{ij} \sum_{k\neq j}^{n} \upsilon_{ik}\pi_k,$$

and dividing by $\sum_{i=1}^{n} \zeta_{ij}\upsilon_{ij}$ we obtain $\pi_j + \sum_{i=1}^{n} \zeta_{ij} \sum_{k\neq j}^{n} \upsilon_{ik}\pi_k / \sum_{i=1}^{n} \zeta_{ij}\upsilon_{ij}$, or, equivalently,

$$\pi_j + \sum_{k\neq j}^{n} \frac{\sum_{i=1}^{n} \zeta_{ij}\upsilon_{ik}}{\sum_{i=1}^{n} \zeta_{ij}\upsilon_{ij}}\pi_k.$$

Thus, the model of common-ownership with control weights can be re-written in the form of maximizing $\phi_j = \pi_j + \sum_{k\neq j}^{n} \lambda_{jk}\pi_k$, where $\lambda_{jk} \equiv \sum_{i=1}^{n} \zeta_{ij}\upsilon_{ik} / \sum_{i=1}^{n} \zeta_{ij}\upsilon_{ij}$.

9.1.2 Cross-ownership by firms

The profit of firm j is given by $\phi_j = \pi_j + \sum_{k\neq j} \alpha_{jk}\phi_k$, where α_{jk} is the firm j's ownership stake in firm k. One can derive the profit for each firm by denoting $\boldsymbol{\phi} = (\phi_1, ..., \phi_n)'$ and $\boldsymbol{\pi} = (\pi_1, ..., \pi_n)'$, and solving the equation: $\boldsymbol{\phi} = \boldsymbol{\pi} + \mathbf{A}\boldsymbol{\phi}$, where $\mathbf{A}$ is the $n \times n$ matrix with the ownership stakes with 0's in the diagonal and α_{jk} off-diagonal. Thus, $\boldsymbol{\phi} = \boldsymbol{\Theta}\boldsymbol{\pi}$, where $\boldsymbol{\Theta} = (\mathbf{I} - \mathbf{A})^{-1}$ is the inverse of the Leontief matrix; its coefficients θ_{jk} represent the effective or imputed stake in firm k's profits received by a "real" equity holder with a 1% direct stake in firm j.[47] We examine the symmetric case: $\alpha_{jk} = \alpha_{kj} = \alpha$ for all

[47] Gilo et al. (2006, Lemma 1, p.85) also show that $\theta_{jj} \geq 1$ for all j, and $0 \leq \theta_{jk} < \theta_{jj}$ for all j and all $k \neq j$.

$j \neq k$, and $\alpha_{jj} = 0$ for all j. The general formula for the coefficients of matrix Θ when stakes are symmetric is, for $\alpha < 1/(n-1)$, $\theta_{jj} = \frac{1-(n-2)\alpha}{[1-(n-1)\alpha](\alpha+1)}$ and $\theta_{jk} = \frac{\alpha}{[1-(n-1)\alpha](\alpha+1)}$ for all j and all $j \neq k$ (see Lemma 8 in Appendix B.1).

Hence, the profit of firm j with symmetric stakes is given by

$$\phi_j = \frac{1-(n-2)\alpha}{[1-(n-1)\alpha](\alpha+1)}\pi_j + \frac{\alpha}{[1-(n-1)\alpha](\alpha+1)}\sum_{k\neq j}\pi_k.$$

Maximizing the above expression is equivalent to maximizing $\pi_j + \lambda\sum_{k\neq j}\pi_k$, where $\lambda = \lambda^{\text{PCO}} \equiv \alpha/[1-(n-2)\alpha]$.

9.2 Simultaneous model

Proof of Lemma 1. If we totally differentiate the two first-order necessary conditions, then after some manipulations we get

$$\frac{\partial q^*}{\partial \lambda} = \frac{1}{\Delta}[(\partial_{\lambda x}\phi_i)(\partial_{xq}\phi_i)B - (\partial_{\lambda q}\phi_i)\Delta_x] \tag{17}$$

$$\frac{\partial x^*}{\partial \lambda} = \frac{1}{\Delta}[(\partial_{\lambda q}\phi_i)(\partial_{xq}\phi_i)\tau - (\partial_{\lambda x}\phi_i)\Delta_q]. \tag{18}$$

Using equation (18) and Table 7 we obtain

$$\frac{\partial x^*}{\partial \lambda} = \frac{c'(Bx^*)f'(Q^*)(n-1)q^*}{\Delta}\left\{\beta\left[\Lambda(1+\delta)+n\right]-\tau\right\}.$$

Since $\Delta > 0$ and $\Delta_q < 0$ (so $\Lambda(1+\delta)+n > 0$):

$$\begin{aligned} sign\left\{\frac{\partial x^*}{\partial \lambda}\right\} &= sign\left\{\beta\left[\Lambda(1+\delta)+n\right]-\tau\right\} \\ &= sign\left\{\beta - \frac{\tau}{\Lambda(1+\delta)+n}\right\} = sign\left\{\beta - P'(c)\frac{\tau}{n}\right\}, \end{aligned}$$

where $P'(c) = n/[\Lambda(1+\delta)+n]$. Finally, by substituting

$$sign\left\{\beta\left[\Lambda(1+\delta)+n\right]-\tau\right\} = sign\left\{\beta(1+\Lambda\delta+n)-1\right\}. \blacksquare$$

Proof of Corollary 1. From Lemma 1 we have that if $\delta \leq -(1+n)/\Lambda$, so $n+1+\delta\Lambda \leq 0$, then $\partial x^*/\partial\lambda < 0$, which, using equation (5), in turn implies that $\partial q^*/\partial\lambda < 0$: for all

β only R_{I} exists. If $\delta > -(n+1)/\Lambda$, then in addition to R_{I}, region R_{II} exists only if $\delta > -n/\Lambda$ also holds. The reason is that when $1+n+\delta\Lambda > 0$, then, from Lemma 1, $\partial x^*/\partial\lambda > 0$ requires that $\beta > 1/(1+n+\delta\Lambda)$. However, $1/(1+n+\delta\Lambda) < 1$ only if $\delta > -n/\Lambda$, in which case there exists some region of feasible spillover values for which $\partial x^*/\partial\lambda > 0$. Note that for a given n, the condition $\delta > -n/\Lambda$ is stricter than the condition $\delta > -(n+1)/\Lambda$. Thus, for $\delta \leq -n/\Lambda$ only R_{I} exists, and since $-n/\Lambda$ increases with λ, the result holds for any λ if $\delta \leq -n$.■

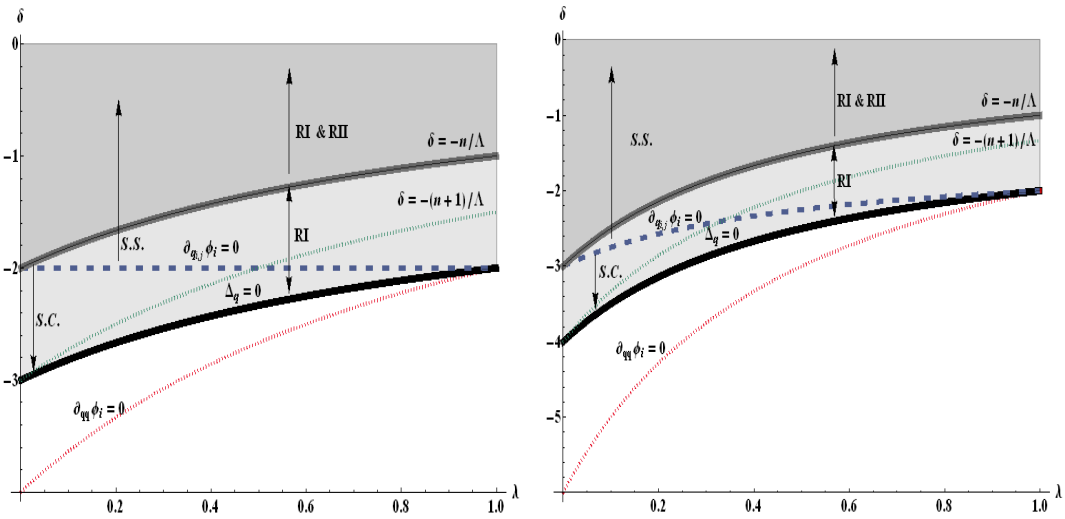

Fig. 8a. $n = 2$.

Fig. 8b. $n = 3$.

Fig. 8. Existence of regions R_{I} and R_{II} with second-order, stability and strategic complements/substitutes output competition conditions.

Proof of Lemma 2. If we totally differentiate the two-first order conditions and solve for $\partial q^*/\partial\lambda$, we obtain

$$\frac{\partial q^*}{\partial\lambda} = \frac{(n-1)(Q^*/n)}{\Delta}\beta\left(c'(Bx^*)\right)^2\left\{B + \frac{f'(Q^*)}{\beta c'(Bx^*)^2}\left[c''(Bx^*)(Q^*/n)B\tau + \Gamma''(x^*)\right]\right\}.$$

Let $H \equiv \frac{\partial_{\lambda q}\phi_i}{\partial_{\lambda x}\phi_i}\frac{\Delta_x}{\partial_{xq}\phi_i} = -[f'(Q^*)/(\beta\left(c'(Bx^*)\right)^2)]\left[c''(Bx^*)(Q^*/n)B\tau + \Gamma''(x^*)\right]$, evaluated at the equilibrium (Q^*, x^*) for $\beta > 0$. From the requirement that either $c'' > 0$ or $\Gamma'' > 0$ (or both) we obtain that $\lim_{\beta\to 0} H = \infty$. The above expression can be rewritten as

$$\frac{\partial q^*}{\partial\lambda} = \frac{(n-1)(Q^*/n)}{\Delta}\beta\left(c'(Bx^*)\right)^2(B-H).■$$

Proof of Corollary 2. Under A.4, $\partial q^*/\partial\lambda > 0$ (so R_{III} exists) if $\beta > \beta'$. We now

show that the condition $n > H(1)$ guarantees that $\beta' < 1$. First, note that $\lim_{\beta \to 0} H = \infty$ (when $c'' > 0$ or $\Gamma'' > 0$), while $B = 1$ at $\beta = 0$. Since $H(\beta)$ has slope less than $n-1$ and $dB/d\beta = n-1$, by continuity there exists only one value for $\beta (= \beta')$ at which $H(\beta) = B$. If the condition $H(\beta) - B < 0$ holds at $\beta = 1$ (which is equivalent to the condition $n > H(1)$), then necessarily H intersects B at some β less than 1, thus $\beta' < 1$.■

Proof of Proposition 3. By totally differentiating the two FOCs with respect to β, we obtain

$$\frac{\partial q^*}{\partial \beta} = \frac{1}{\Delta}[(\partial_{\beta x}\phi_i)(\partial_{xq}\phi_i) B - (\partial_{\beta q}\phi_i)\Delta_x] \tag{19}$$

$$\frac{\partial x^*}{\partial \beta} = \frac{1}{\Delta}[(\partial_{\beta q}\phi_i)(\partial_{xq}\phi_i)\tau - (\partial_{\beta x}\phi_i)\Delta_q]. \tag{20}$$

Since $\partial_{xq}\phi_i > 0$ and $\partial_{\beta q}\phi_i > 0$, $\Delta > 0$, $\Delta_x < 0$ and $\Delta_q < 0$, the sign of the impact of β on output and R&D in equilibrium depends on the sign of $\partial_{\beta x}\phi_i$. It can be shown that

$$\partial_{\beta x}\phi_i = -c'(Bx^*)\frac{(n-1)q^*}{B}\tau\left(\frac{\lambda B}{\tau} - \chi(Bx^*)\right) \tag{21}$$

and the result follows.■

Proof of Proposition 4. To prove Proposition 4 a few preliminary lemmata (assuming A.1-A.4) are useful.

LEMMA 5 *Suppose that $\delta > -n$, then $W'(\lambda) > 0$ if $\beta > \hat{\beta}(\lambda)$ where $\hat{\beta}$ is the unique positive solution to the equation*

$$H(\beta) - B = [(n-\Lambda)/\Lambda]\,[(n+1+\delta\Lambda)\beta - 1]. \tag{22}$$

Proof. We first derive the condition that determines the spillover threshold value $\hat{\beta}$ above which welfare is increasing in the degree of cross-ownership. By inserting $\partial q^*/\partial\lambda$ and $\partial x^*/\partial\lambda$ (given in Table 7) into (8) we obtain:

$$\begin{aligned} W'(\lambda) &= -\Lambda f'(Q^*)\frac{(n-1)q^*}{\Delta}c'(Bx^*)^2\beta(B - H(\beta))Q^* \\ &\quad -(1-\lambda)\beta(n-1)c'(Bx^*)\frac{(n-1)q^*}{\Delta}f'(Q^*)c'(Bx^*)\left\{\beta\left[\Lambda(1+\delta)+n\right]-\tau\right\}Q^*, \end{aligned}$$

which can be rewritten as:

$$W'(\lambda) = \vartheta_w \left(\Lambda \left(B - H(\beta)\right) + (1-\lambda)(n-1)\left\{\beta\left[\Lambda(1+\delta)+n\right] - \tau\right\}\right),$$

where $\vartheta_w \equiv [(n-1)q^*/\Delta]c'(Bx^*)^2(-f'(Q^*))\beta Q^*$ is positive. Note that $(1-\lambda)(n-1) = n - \Lambda$, thus for $\beta > 0$, $W'(\lambda) > 0$ if

$$H(\beta) - B < \frac{n-\Lambda}{\Lambda}\left[(n+1+\delta\Lambda)\beta - 1\right]. \tag{23}$$

Note that $\lim_{\beta\to 0} H = \infty$ and (by Assumption A.4) the left-hand side of (23) is decreasing in β. The right-hand side of (23) is increasing in β (since $n+1+\delta\Lambda > 0$ holds when R_{II} and R_{III} exist) and finite at $\beta = 0$. Thus, there exists a unique positive threshold $\hat{\beta}$ that solves the equation (22), and for any $\beta > \hat{\beta}$ condition (23) holds, that is, $W'(\lambda) > 0$.■

LEMMA 6 *We have that $\hat{\beta}(\lambda) < \beta'(\lambda)$ for all λ, which implies that $\bar{\beta} < \beta'(0)$ where $\bar{\beta} = \hat{\beta}(0)$. Furthermore, $\bar{\beta} < 1$ if*

$$n + (n-1)(\delta + n) - H(1) > 0. \tag{24}$$

Proof. We first show that $\beta'(\lambda) > \hat{\beta}(\lambda)$ for any λ, and as a result $\beta'(0) > \bar{\beta} = \hat{\beta}(0)$. Suppose that for a given λ, $\hat{\beta} > \beta'$, then from (6) we have that for $\beta \in (\beta', \hat{\beta})$ it holds that $\partial q^*/\partial\lambda > 0$. Thus, from equation (5) it also holds that $\partial x^*/\partial\lambda > 0$, which implies from equation (8) that $W'(\lambda) > 0$. However, from equation (22) we have that $W'(\lambda) < 0$ for $\beta < \hat{\beta}$, a contradiction. Suppose now that $\hat{\beta} = \beta'$, then we can pick β such that $\beta = \hat{\beta} = \beta'$, and as a result $H - B|_{\beta=\beta'} = 0$, thus from equation (22) we have that $\hat{\beta} = \beta' = 1/(n+1+\delta\Lambda)$, which implies that $\partial x^*/\partial\lambda = 0$ (see Table 7), and from equation (5) this in turn implies that $\partial q^*/\partial\lambda < 0$. However, at $\beta = \beta'$, $B - H = 0$, so $\partial q^*/\partial\lambda = 0$ (see Table 7), a contradiction.

The proof of Lemma 5 shows that $W'(\lambda) > 0$ for some λ if the spillover is larger than the threshold value $\hat{\beta}(\lambda)$, where $\hat{\beta}$ is the unique positive solution to the equation (22). Furthermore, $\hat{\beta} < 1$ if condition (23) evaluated at $\beta = 1$ holds since $\lim_{\beta\to 0} H(\beta) = \infty$ and $H(\beta) - B$ decreases with β (by Assumption A.4), while the right-hand side of (23) increases with β (for $\lambda < 1$) and takes finite value at $\beta = 0$. Therefore, by evaluating (23) also at $\lambda = 0$ we obtain the condition that ensures that $\bar{\beta} < 1$, $n+(n-1)(\delta+n) - H(1) >$

0.■

We turn now to prove successively each of the statements of Proposition 4.

i) The result follows from Proposition 1: if $-\delta \geq \text{HHI}^{-1}$, then only R_{I} exists, where $\partial x^*/\partial\lambda < 0$ and $\partial q^*/\partial\lambda < 0$, and thus $CS'(\lambda) < 0$ and $W'(\lambda) < 0$ for all λ.

ii) Next we consider the case in which $-\delta < n$:

ii.1) $\lambda^o_{\text{TS}} = \lambda^o_{\text{CS}} = 0$ if $\beta \leq \bar{\beta}$. First, we have to show that there does not exist $\beta < \bar{\beta}$ such that $W'(\lambda) > 0$ for some positive λ. However, this follows trivially from the assumption that $W(\lambda)$ is single peaked: since for any $\beta < \bar{\beta}$, $W'(0) < 0$, we have that $W'(\lambda) < 0$ for all positive λ, otherwise there would exist another stationary point that is a (local) minimum, a contradiction. Similarly, if $\beta = \bar{\beta}$, then $W'(0) = 0$, and the assumption that $W(\lambda)$ is single peaked guarantees that $W'(\lambda) < 0$ for any positive λ. In addition, if $\beta \leq \bar{\beta}$, then $\lambda^o_{\text{CS}} = 0$: from Lemma 6 we know that $\beta'(\lambda) > \bar{\beta} = \hat{\beta}(0)$ for all λ; since β is assumed to be equal to or lower than $\bar{\beta}$, it follows that $CS'(\lambda) < 0$ for all λ, thus $\lambda^o_{\text{CS}} = 0$.

ii.2) $\lambda^o_{\text{TS}} > \lambda^o_{\text{CS}} = 0$ if $\beta \in \left(\bar{\beta}, \beta'(0)\right)$. Noting again that $\bar{\beta} = \hat{\beta}(0)$, the result that $\lambda^o_{\text{TS}} > 0$ when $\beta > \bar{\beta}$ follows immediately from Lemma 5. In addition, $\beta < \beta'(0)$ yields $\lambda^o_{\text{CS}} = 0$: when H is weakly increasing in λ, $\beta'(\lambda)$ also is, and consequently if $\beta < \beta'(0)$, then $\beta < \beta'(\lambda)$ for all λ, i.e., $\partial q^*/\partial\lambda < 0$ for all λ, thus $\lambda^o_{\text{CS}} = 0$.

ii.3) We first show that $\lambda^o_{\text{TS}} > 0$ and $\lambda^o_{\text{CS}} > 0$ if $\beta > \beta'(0)$. From Lemma 6 it follows that $\beta > \beta'(0) > \bar{\beta}$, which yields $\lambda^o_{\text{TS}} > 0$. From Lemma 2 we know that if for some given λ, $\beta > \beta'(\lambda)$, then $\partial q^*/\partial\lambda > 0$. Hence if $\beta > \beta'(0)$, we have that $\partial q^*/\partial\lambda > 0$ at $\lambda = 0$, which using (9) implies that $CS'(0) > 0$, and therefore $\lambda^o_{\text{CS}} > 0$

Next we show that $\lambda^o_{\text{TS}} \geq \lambda^o_{\text{CS}}$ when H is weakly increasing in λ. Note that $B > H$ (since $\partial q^*/\partial\lambda > 0$) at $\lambda = 0$. When H is weakly increasing in λ, we may face the following three cases:

- There does not exist some $\lambda < 1$ at which $H = B$; as a result $\partial q^*/\partial\lambda > 0$ and, by (5), $\partial x^*/\partial\lambda > 0$ for all λ, which from equation (8) yields $W'(\lambda) > 0$ for all λ; thus $\lambda^o_{\text{TS}} = \lambda^o_{\text{CS}} = 1$.

- There exists an interval subset L of the continuum of values of λ in $(0,1]$ at which $H = B$ but (a) H never crosses B, so there is no λ at which $H > B$, or (b) there exists some λ above which $H > B$. In both cases, in the region of values for λ where $H = B$ we have $\partial q^*/\partial\lambda = 0$ (or, equivalently, $CS'(\lambda) = 0$), while $\partial x^*/\partial\lambda > 0$, consequently $W'(\lambda) > 0$. It follows that if H never crosses B or does it for some $\lambda > 1$, then $\lambda^o_{\text{TS}} = 1$, while any $\lambda \in L$ is optimal in terms of CS (even if L is a singleton) since $\partial q^*/\partial\lambda > 0$ for any λ lower than the lower bound of L, thus $\lambda^o_{\text{TS}} \geq \lambda^o_{\text{CS}}$; by the same token, if $H > B$ for some $\lambda < 1$, then any $\lambda \in L$ is optimal in terms of CS (even if L is a singleton), while λ^o_{TS} is larger or equal than the upper bound of L, thus $\lambda^o_{\text{TS}} \geq \lambda^o_{\text{CS}}$.

iii) In ii.1) $\lambda^o_{\text{TS}} = \lambda^o_{\text{CS}} = 0$, in ii.2) $\lambda^o_{\text{TS}} > \lambda^o_{\text{CS}} = 0$, and in ii.3) $\lambda^o_{\text{TS}} \geq \lambda^o_{\text{CS}} > 0$. Therefore, in the three cases $\lambda^o_{\text{TS}} \geq \lambda^o_{\text{CS}}$.

For the sake of completeness, next we consider the particular case where $\beta = \beta'(0)$.

- Case $\beta = \beta'(0)$. If H (as a function of λ) is increasing at some value of λ, then β' also is. Thus, for any larger value of λ, $\beta < \beta'(\lambda)$, which implies that $\partial q^*/\partial\lambda < 0$; as a result, any smaller value of λ, where $\beta = \beta'$, and therefore $\partial q^*/\partial\lambda = 0$, is optimal in terms of consumer surplus. Similarly, when H (and therefore β') is independent of λ for all λ, we have that $\partial q^*/\partial\lambda = 0$ for all λ since $\beta = \beta'$. Hence CS is independent of λ, and any λ is optimal in terms of CS. In both cases, for all λ where $H = B$ we have $\partial x^*/\partial\lambda > 0$, since $\partial q^*/\partial\lambda = 0$, which implies that $W'(\lambda) > 0$. Therefore, $\lambda^o_{\text{TS}} \geq \lambda^o_{\text{CS}} \geq 0$.■

9.3 Two-stage model

Proof of Lemma 4. Using (13), by totally differentiating the system formed by (10; 11) in a symmetric equilibrium, and solving for $\partial q^*/\partial\lambda$ and $\partial x^*/\partial\lambda$, we obtain

$$\frac{\partial q^*}{\partial\lambda} = \frac{1}{\tilde{\Delta}}\left\{\left[\partial_{\lambda x}\phi_i + (n-1)\psi_\lambda\right]\left(\partial_{xq}\phi_i\right)B - \partial_{\lambda q}\phi_i\left[\Delta_x + \psi_x(n-1)\right]\right\} \tag{25}$$

$$\frac{\partial x^*}{\partial\lambda} = \frac{1}{\tilde{\Delta}}\left\{\partial_{\lambda q}\phi_i\left[\partial_{xq}\phi_i\tau + (n-1)\psi_q\right] - \left[\partial_{\lambda x}\phi_i + (n-1)\psi_\lambda\right]\Delta_q\right\}, \tag{26}$$

where $\psi_z \equiv \partial\psi/\partial z$ with $z = q, x, \lambda$, and

$$\tilde{\Delta}(Q^*, x^*) = \Delta_q \left[\Delta_x + \psi_x(n-1)\right] - \partial_{xq}\phi_i \left[\partial_{xq}\phi_i\tau + \psi_q(n-1)\right] B,$$

which is assumed to be strictly positive.[48] By rewriting equation (26) as follows

$$\frac{\partial x^*}{\partial \lambda} = \xi f'(Q^*)c'(Bx^*)\left\{(\beta + s'(\lambda))\left[\Lambda(1+\delta) + n\right] - \left[\tau + (n-1)s(\lambda)\right]\right\}, \tag{27}$$

where $\xi \equiv (n-1)(Q^*/n)/\tilde{\Delta}$ and $s(\lambda) = \omega(\lambda)(\tilde{\beta}(\lambda) - \beta)$, we get that $sign\left\{\partial x^*/\partial\lambda\right\}$ is given by (15). Let us now turn to the impact of λ on output in equilibrium. Equation (25) can be rewritten as follows

$$\frac{\partial q^*}{\partial \lambda} = \xi\left((\beta + s'(\lambda))c'(Bx^*)^2 B + f'(Q^*)\left\{c''(Bx^*)(Q^*/n)B\left[\tau + (n-1)s(\lambda)\right] + \Gamma''(x^*)\right\}\right). \tag{28}$$

By inserting the first-order necessary condition (11) evaluated at the symmetric equilibrium into the above expression, after some manipulations we get that $sign\left\{\partial q^*/\partial\lambda\right\}$ is given by (16). Finally, note that the first-order condition with respect to output is identical to the one associated to the static case. Therefore, by totally differentiating the FOC with respect to output and solving for $\partial q^*/\partial\lambda$, we obtain again equation (5), which implies that if $\partial x^*/\partial\lambda \leq 0$, then $\partial q^*/\partial\lambda < 0$. From (15), we obtain that $\partial x^*/\partial\lambda > 0$ if and only if

$$\beta > \underline{\beta}^{2S} \equiv \frac{1 - (\omega'(\lambda)\tilde{\beta}(\lambda) + \omega(\lambda)\tilde{\beta}'(\lambda))P'(c)^{-1}n + \omega(\lambda)(n-1)\tilde{\beta}(\lambda)}{(1 + n + \Lambda\delta) + (n-1)\omega(\lambda) - P'(c)^{-1}n\omega'(\lambda)}. \blacksquare$$

LEMMA 7 *Under assumptions A.1.-A.4, in the two-stage model, there is a sufficiently large spillover value ($\bar{\beta}^{2S} < 1$) for which allowing some cross-ownership is socially optimal ($\lambda^o_{\text{TS}} > 0$) if*

$$(1+s'(0))n + (1-s(0))(n-1)((1+s'(0))(1+\delta+n) - [1+(n-1)s(0)] - H(1) > 0. \tag{29}$$

Proof. See Appendix B.1.$\blacksquare$

[48] We show in Appendix B.2.2 that $\tilde{\Delta}(Q^*, x^*) > 0$ is also a necessary condition for having a positive output at equilibrium in AJ.

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Appendix B to Cross-ownership, R&D Spillovers and Antitrust Policy

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In this Appendix we provide some proofs of results in the paper in Section B.1 and analysis and results for the three model specifications in Section B.2

B.1 General model: proofs

In this Section we state Lemma 8 and provide proofs for Proposition 2 and 5, and Lemmata 3 and 7.

LEMMA 8 *The coefficients of matrix Θ with symmetric stakes are given by*

$$\theta_{jj} = \frac{1-(n-2)\alpha}{[1-(n-1)\alpha]\,(\alpha+1)} \quad \textit{and} \quad \theta_{jk} = \frac{\alpha}{[1-(n-1)\alpha]\,(\alpha+1)}$$

for all j and all $j \neq k$.

Proof. To obtain the coefficients of matrix Θ with symmetric stakes we need to compute the inverse of

$$\mathbf{I}-\mathbf{A} = \begin{pmatrix} 1 & -\alpha & \cdots & -\alpha \\ -\alpha & 1 & \cdots & -\alpha \\ \vdots & \vdots & \ddots & \\ -\alpha & -\alpha & \cdots & 1 \end{pmatrix}_{n\times n}.$$

The problem is the same as that of deriving the direct demand system from the inverse demand system of n differentiated products with quadratic utility function (see Vives, 1999, pp. 145-147). By definition $(\mathbf{I}-\mathbf{A})\mathbf{\Theta} = \mathbf{I}$. Thus, $(\mathbf{I}-\mathbf{A})$ multiplies the first column of $\mathbf{\Theta}$ (call that Θ_1) to give the first column of $\mathbf{I}$ (call that e_1). For $n \geq 2$ firms, we have $(\mathbf{I}-\mathbf{A})\Theta_1 = (\ 1 \quad 0 \quad \cdots \quad 0\ )_{1\times n}$. Let us denote the diagonal entries of $\mathbf{\Theta}$ by θ_d, and the

off-diagonal entries by θ_{-d}. Then, the system of equations $(\mathbf{I}-\mathbf{A})\Theta_1$ reduces to the following system of two equations: $\{\theta_d - \alpha(n-1)\theta_{-d} = 1, -\alpha\theta_d + [1-(n-2)\alpha]\,\theta_{-d} = 0\}$. Solving the system for coefficients θ_d and θ_{-d} yields

$$\theta_d = \frac{1-(n-2)\alpha}{[1-(n-1)\alpha]\,(\alpha+1)} \quad and \quad \theta_{-d} = \frac{\alpha}{[1-(n-1)\alpha]\,(\alpha+1)}. \blacksquare$$

Proof of Proposition 2. Profit per firm as a function of λ at equilibrium is given by

$$\pi^*(\lambda) = (f(Q^*) - c(Bx^*))\,q^* - \Gamma(x^*).$$

By differentiating π^* with respect to λ, we obtain

$$\pi^{*\prime}(\lambda) = f'(Q^*)n\frac{\partial q^*}{\partial\lambda}q^* - c'(Bx^*)B\frac{\partial x^*}{\partial\lambda}q^* + (f(Q^*) - c(Bx^*))\,\frac{\partial q^*}{\partial\lambda} - \Gamma'(x^*)\frac{\partial x^*}{\partial\lambda}.$$

Using that in equilibrium $f(Q^*) - c(Bx^*) = -f'(Q^*)\Lambda q^*$ and $\Gamma'(x^*) = -c'(Bx^*)q^*\tau$, we can rewrite the above expression as

$$\begin{aligned}
\pi^{*\prime}(\lambda) &= f'(Q^*)n\frac{\partial q^*}{\partial\lambda}q^* - c'(Bx^*)B\frac{\partial x^*}{\partial\lambda}q^* - f'(Q^*)\Lambda q^*\frac{\partial q^*}{\partial\lambda} + c'(Bx^*)q^*\tau\frac{\partial x^*}{\partial\lambda} \\
&= f'(Q^*)(n-\Lambda)q^*\frac{\partial q^*}{\partial\lambda} + c'(Bx^*)(\tau - B)q^*\frac{\partial x^*}{\partial\lambda} \\
&= f'(Q^*)(n-1)(1-\lambda)q^*\frac{\partial q^*}{\partial\lambda} - c'(Bx^*)\beta(n-1)(1-\lambda)q^*\frac{\partial x^*}{\partial\lambda},
\end{aligned}$$

or

$$\pi^{*\prime}(\lambda) = (n-1)(1-\lambda)q^*\left(f'(Q^*)\frac{\partial q^*}{\partial\lambda} - \beta c'(Bx^*)\frac{\partial x^*}{\partial\lambda}\right).$$

In RII, we have that $\partial x^*/\partial\lambda > 0$ and $\partial q^*/\partial\lambda < 0$. Hence from the above expression it is clear that $\pi^{*\prime}(\lambda) > 0$. Note also that when $\beta = 0$, the equilibrium is in RI, and therefore $\pi^{*\prime}(\lambda) > 0$ since $\partial q^*/\partial\lambda < 0$. To determine $sign\{\pi^{*\prime}(\lambda)\}$ in RI and $RIII$ for $\beta > 0$, we replace $\partial q^*/\partial\lambda$ and $\partial x^*/\partial\lambda$ with the expressions given in Table 7:

$$\begin{aligned}
\pi^{*\prime}(\lambda) &= (n-1)(1-\lambda)q^*\left[f'(Q^*)\frac{(n-1)q^*}{\Delta}c'(Bx^*)^2\beta\,(B - H(\beta))\right. \\
&\quad \left. -\beta c'(Bx^*)\frac{(n-1)q^*}{\Delta}f'(Q^*)c'(Bx^*)\,\{\beta\,[\Lambda(1+\delta)+n] - \tau\}\right].
\end{aligned}$$

After some manipulations we obtain:

$$\pi^{*\prime}(\lambda) = \vartheta_\pi\,\{\beta\,[\Lambda(1+\delta)+n] - \tau + H(\beta) - B\},$$

where $\vartheta_\pi \equiv (n-1)(1-\lambda)q^* \left[(n-1)q^*/\Delta\right] c'(Bx^*)^2\beta(-f'(Q^*))$ is positive. Therefore,

$$sign\left\{\pi^{*\prime}(\lambda)\right\} = sign\left\{(n+1+\delta\Lambda)\beta - 1 + H(\beta) - B\right\}, \tag{30}$$

so it follows that $\pi^{*\prime}(\lambda) > 0$ if

$$1 - (n+1+\delta\Lambda)\beta < H(\beta) - B, \tag{31}$$

or, equivalently, if

$$2(1-\beta) - \delta\Lambda\beta < H(\beta). \tag{32}$$

From Table 7 and using that in equilibrium $\tau q^* = -\Gamma'(x^*)/c'(Bx^*)$, the regularity condition can be written as

$$-\left(-c''(Bx^*)B\frac{\Gamma'(x^*)}{c'(Bx^*)} + \Gamma''(x^*)\right)\frac{f'(Q^*)}{c'(Bx^*)^2}\left[\Lambda\left(1+\delta\right)+n\right] - \tau B > 0.$$

Noting that (see Table 7)

$$\beta H(\beta) = \frac{-f'(Q^*)}{c'(Bx^*)^2}\left(-\frac{c''(Bx^*)}{c'(Bx^*)}B\Gamma'(x^*) + \Gamma''(x^*)\right),$$

we can rewrite the regularity condition in terms of H as follows: $\left[\Lambda(1+\delta)+n\right]\beta H(\beta) - \tau B > 0$, with $\Lambda(1+\delta)+n > 0$ since $\Delta_q < 0$. Thus, if the equilibrium is regular:

$$H(\beta) > \frac{\tau B}{\left[\Lambda(1+\delta)+n\right]\beta}.$$

Then, we only have to show that: $\tau B/\left\{\left[\Lambda(1+\delta)+n\right]\beta\right\} > 2(1-\beta) - \delta\Lambda\beta$, or, equivalently, that

$$\tilde{g}(\beta) \equiv \tau B > \tilde{h}(\beta) \equiv [2(1-\beta) - \Lambda\delta\beta][\Lambda(1+\delta)+n]\beta$$

holds. Note that $\tilde{g}(0) = 1$, $\tilde{g}'(\beta) > 0$, $\tilde{g}''(\beta) > 0$ for $\beta > 0$ and $\tilde{g}''(0) = 0$. On the other hand, $\tilde{h}(0) = 0$ and

$$\tilde{h}'(\beta) = 2[\Lambda(1+\delta)+n][1-(2+\Lambda\delta)\beta].$$

Furthermore, it can be shown that solving the equation $\tilde{g}(\beta) = \tilde{h}(\beta)$ for β yields the following two roots:

$$\beta_1 = \frac{1}{\Lambda\delta+n+1} \text{ and } \beta_2 = \frac{1}{\Lambda(\delta+1)+1}.$$

Consider RI. If the smallest (positive) root in this region is larger or equal than the spillover threshold that determines RI, then $\tilde{g}(\beta) > \tilde{h}(\beta)$ in RI, and consequently, $\pi^{*\prime}(\lambda) > 0$. First, note that when $\Lambda\delta + n + 1 > 0$ holds, β_1 is indeed the threshold value that determines RI, i.e. for $\beta < \beta_1$, $\partial x^*/\partial\lambda < 0$ (if $\Lambda\delta + n + 1 < 0$, then $\partial x^*/\partial\lambda < 0$ for all λ). Depending on the values of λ, δ and n, one of the following cases may apply:

- If $2 + \Lambda\delta > 0$, i.e., $\delta > -2/\Lambda$, then $\Lambda\delta + n + 1 > 0$ and $\Lambda(\delta + 1) + 1 > 0$: $\beta_1 > 0$ and $\beta_2 > 0$. Furthermore, $\beta_1 < \beta_2$ (for $\lambda < 1$) and $\tilde{h}''(\beta) < 0$. Therefore, $\tilde{g}(\beta) > \tilde{h}(\beta)$ for $0 < \beta < \beta_1$.

- If $2 + \Lambda\delta < 0$ and $\Lambda(\delta + 1) + 1 > 0$, i.e., $-(1 + \Lambda)/\Lambda < \delta < -2/\Lambda$ (so $\Lambda\delta + n + 1 > 0$ also holds), then $\beta_1 > 0$, $\beta_2 > 0$, $\beta_1 < \beta_2$ (for $\lambda < 1$) and $\tilde{h}''(\beta) > 0$: $\tilde{g}(\beta) > \tilde{h}(\beta)$ for $0 < \beta < \beta_1$.

- If $\Lambda(\delta + 1) + 1 < 0$ and $\Lambda\delta + n + 1 > 0$, i.e., $-(n + 1)/\Lambda < \delta < -(1 + \Lambda)/\Lambda$, then $\beta_1 > 0$, $\beta_2 < 0$ and $\tilde{h}''(\beta) > 0$: $\tilde{g}(\beta) > \tilde{h}(\beta)$ for $0 < \beta < \beta_1$.

- If $\Lambda\delta + n + 1 < 0$, i.e., $\delta < -(n + 1)/\Lambda$, then $\beta_1 < 0$, $\beta_2 < 0$, $\beta_1 > \beta_2$ (for $\lambda < 1$) and $\tilde{h}''(\beta) > 0$: $\tilde{g}(\beta) > \tilde{h}(\beta)$ for all β.

- If $\beta = \beta_1$, so $\beta = 1/(1+n+\Lambda\delta)$, then $\partial x^*/\partial\lambda = 0$, so $sign\{\pi^{*\prime}(\lambda)\} = sign\{f'(Q^*)\partial q^*/\partial\lambda\}$, which is positive in RI since in this region: $\partial q^*/\partial\lambda < 0$.

Consider $RIII$. Note that $RIII$ may exist only if $\delta > -n/\Lambda$, in which case $\delta > -(n+1)/\Lambda$, so $\beta_1 > 0$. Furthermore, $\beta' \geq \beta_1$.[46] Next we show that for any $\beta > \beta'$, $\tilde{g}(\beta) > \tilde{h}(\beta)$, and consequently, $\pi^{*\prime}(\lambda) > 0$. Again, depending on the values of λ, δ and n, we may face one of the following cases:

- If $\delta > -2/\Lambda$, then $\beta_1 > 0$, $\beta_2 > 0$, $\beta_1 < \beta_2$ (for $\lambda < 1$) and $\tilde{h}''(\beta) < 0$. Hence, $\tilde{g}(\beta) > \tilde{h}(\beta)$ for $0 < \beta < \beta_1$, $\tilde{g}(\beta) < \tilde{h}(\beta)$ for $\beta_1 < \beta < \beta_2$, and $\tilde{g}(\beta) > \tilde{h}(\beta)$ for $\beta > \beta_2$. Thus, we only have to show that $\beta' > \beta_2$, so that $\tilde{g}(\beta) > \tilde{h}(\beta)$ for any $\beta \geq \beta'$. Note that if $\pi'(\lambda) > 0$ for $\beta = \beta'$, then necessarily $\beta' > \beta_2$ since $\beta' > \beta_1$ and $\pi'(\lambda) < 0$ for $\beta \in (\beta_1, \beta_2)$. Since condition (31) holds at $\beta = \beta'$: $H(\beta') - \left[1 + \beta'(n-1)\right] = 0 > 1 - (n + 1 + \delta\Lambda)\beta$, we thus have $\beta' > \beta_2$, and as a result $\tilde{g}(\beta) > \tilde{h}(\beta)$ for any $\beta > \beta'$.

- If $\delta < -2/\Lambda$, when $-n/\Lambda > -(\Lambda + 1)/\Lambda$ (i.e., $\Lambda > n - 1$), the feasible range is $-n/\Lambda < \delta < -2/\Lambda$, where $\beta_1 > 0$, $\beta_2 > 0$, $\beta_1 < \beta_2$ (for $\lambda < 1$) and $\tilde{h}''(\beta) > 0$. As in the previous case, we can conclude that $\beta' > \beta_2$: for any $\beta > \beta'$, $\tilde{g}(\beta) > \tilde{h}(\beta)$.

[46] Suppose that $\beta' < \beta_1$, then from Lemma 2 we have that $\partial q^*/\partial\lambda > 0$ for $\beta > \beta'$. However, from Lemma 1 we have that $\partial x^*/\partial\lambda < 0$ for $\beta < \beta_1$. Furthermore, if $\partial x^*/\partial\lambda < 0$, then $\partial q^*/\partial\lambda < 0$. Thus, $\partial q^*/\partial\lambda < 0$ for $\beta' < \beta < \beta_1$, a contradiction.

- If $\delta < -2/\Lambda$ but $-n/\Lambda < -(\Lambda+1)/\Lambda$, we can distinguish between two cases: (i) when $-(\Lambda+1)/\Lambda < \delta < -2/\Lambda$, then again we have $\beta_1 > 0$, $\beta_2 > 0$, $\beta_1 < \beta_2$ (for $\lambda < 1$) and $\tilde{h}''(\beta) > 0$, so $\beta' > \beta_2$: for any $\beta > \beta'$, $\tilde{g}(\beta) > \tilde{h}(\beta)$; (ii) when $-n/\Lambda < \delta < -(\Lambda+1)/\Lambda$, in which case $RIII$ does not exist. To see this note that in this case $\beta_1 > 0$, $\beta_2 < 0$, and $\tilde{h}''(\beta) > 0$: $\tilde{g}(\beta) > \tilde{h}(\beta)$ only for $\beta < \beta_1$. If $\beta' < 1$, then condition (31) holds at β', i.e., $\tilde{g}(\beta) > \tilde{h}(\beta)$ for $\beta \geq \beta'$. Therefore, $\beta' < \beta_1$, a contradiction.■

Proof of Proposition 5. If $\delta > -(1+n)/n$, then $1+n+\delta\Lambda > 0$ for all λ. From Lemma 1 we know that when $\beta \leq 1/(1+n+\delta\Lambda)$: $\partial x^*/\partial\lambda \leq 0$. From Lemma 5 we have that $W'(\lambda) > 0$ if $\beta > \hat{\beta}(\lambda)$ where $\hat{\beta}$ is the unique positive solution to the equation (22). Necessarily, $\hat{\beta} > 1/(1+n+\delta\Lambda)$, otherwise for any $\beta \in (\hat{\beta}, 1/(1+n+\delta\Lambda)]$, we have that $\partial x^*/\partial\lambda \leq 0$, which from equation (5) implies that $\partial q^*/\partial\lambda < 0$, which using equation (8) yields $W'(\lambda) < 0$, a contradiction. Since $\hat{\beta}(\lambda) > 1/(1+n+\delta\Lambda)$ for any λ, then $\hat{\beta}(0) = \bar{\beta} > \underline{\beta}$. From Lemma 6 we also know that $\bar{\beta} < \beta'(0)$. Thus, the relationship $\underline{\beta} < \bar{\beta} < \beta'(0)$ is established. Next we prove each of the statements. (i) When $\delta > -(1+n)/n$ not only RI but also RII may exist since $\delta > -n$. If $-(1+n)/n < \delta < 0$, then $\inf\{1/(1+n+\Lambda\delta) : \lambda \in [0,1]\} = 1/(1+n+\delta) > 0$, whereas if $\delta \geq 0$, $\inf\{1/(1+n+\Lambda\delta) : \lambda \in [0,1]\} = 1/[1+n(1+\delta)] > 0$. In both cases, if $\beta \leq \underline{\beta}$, it follows from Proposition 1 that only RI can exist. (ii) Lemma 5 ensures that for some given λ, if $\beta > \hat{\beta}(\lambda)$, then $W'(\lambda) > 0$. As a result, if $\beta > \bar{\beta} = \hat{\beta}(0)$, then $W'(0) > 0$, thus $\lambda^o_{TS} > 0$; (iii) From Lemma 2 we have that if $\beta > \beta'(0)$, then $\partial q^*/\partial\lambda|_{\lambda=0} > 0$, which using (9) implies that $CS'(0) > 0$: $\lambda^o_{CS} > 0$. ■

Proof of Lemma 3. We have

$$\tilde{\beta}(\lambda) = \frac{n(1+\lambda)+\Lambda\delta}{2n+\Lambda\delta}.$$

Then, by differentiating $\tilde{\beta}$ with respect to n we obtain:

$$\frac{\partial\tilde{\beta}}{\partial n} = -\frac{\delta(1-\lambda)^2}{(2n+\delta\Lambda)^2}.$$

Thus, for $\lambda < 1$ and convex demand ($\delta < 0$), $\partial\tilde{\beta}/\partial n > 0$, if demand is concave ($\delta > 0$), $\partial\tilde{\beta}/\partial n < 0$. Let us now differentiate $\tilde{\beta}$ with respect to λ:

$$\frac{\partial\tilde{\beta}}{\partial\lambda} = \frac{n^2(\delta+2)}{(2n+\delta\Lambda)^2},$$

then, $\partial\tilde{\beta}/\partial\lambda > 0$ if $\delta > -2$. Finally, we differentiate $\tilde{\beta}$ with respect to δ:

$$\frac{\partial\tilde{\beta}}{\partial\delta} = \frac{\Lambda n\,(1-\lambda)}{(2n+\delta\Lambda)^2}.$$

Thus, $\partial\tilde{\beta}/\partial\delta > 0$ if $\lambda < 1$.■

Proof of Lemma 7. By differentiating $W(\lambda)$ we have

$$W'(\lambda) = [f(Q^*) - c(Bx^*)]n\frac{\partial q^*}{\partial\lambda} - c'(Bx^*)BQ^*\frac{\partial x^*}{\partial\lambda} - n\Gamma'(x^*)\frac{\partial x^*}{\partial\lambda}.$$

Using the first-order conditions, $f(Q^*) - c(Bx^*) = -f'(Q^*)Q^*\Lambda/n$ and (14) in the above expression, and simplifying, we obtain:

$$W'(\lambda) = \left\{-\Lambda f'(Q^*)\frac{\partial q^*}{\partial\lambda} - [(1-\lambda)\beta - s(\lambda)]\,(n-1)c'(Bx^*)\frac{\partial x^*}{\partial\lambda}\right\}Q^*. \tag{33}$$

If we insert (27) and (28) into (33), after some manipulations we get

$$\begin{aligned} W'(\lambda) \;=\; & \xi Q^*(-f'(Q^*))\left[\Lambda\left(c'(Bx^*)^2(\beta+s'(\lambda))B\right.\right. \\ & +f'(Q^*)\left\{c''(Bx^*)(Q^*/n)B\left[\tau+(n-1)s(\lambda)\right]+\Gamma''(x^*)\right\}\Big) \\ & +c'(Bx^*)^2\left[(1-\lambda)\beta-s(\lambda)\right](n-1)\left\{(\beta+s'(\lambda))\left[\Lambda(1+\delta)+n\right]\right. \\ & \left.\left.-\left[\tau+(n-1)s(\lambda)\right]\right\}\right], \end{aligned} \tag{34}$$

where $\xi \equiv (n-1)(Q^*/n)/\tilde{\Delta}$. Then $W'(0)|_{\beta=1} > 0$ if and only if

$$\begin{aligned} 0 \;<\; & (c'(nx^*))^2\left((1+s'(0)\big|_{\beta=1})n + (1-s(0)|_{\beta=1})(n-1)\left\{(1+s'(0)\big|_{\beta=1})(1+\delta+n)\right.\right. \\ & \left.\left.-\left[1+(n-1)\,s(0)|_{\beta=1}\right]\right\}\right) + f'(Q^*)\left\{c''(nx^*)Q^*\left[1+(n-1)\,s(0)|_{\beta=1}\right]+\Gamma''(x^*)\right\}. \end{aligned} \tag{35}$$

From equation (14) we have that in equilibrium and for $\lambda = 0$ and $\beta = 1$:

$$Q^*|_{\lambda=0,\beta=1} = -\frac{n\Gamma'(x^*)}{c'(nx^*)\left[1+(n-1)\,s(0)|_{\beta=1}\right]}.$$

Substituting $Q^*|_{\lambda=0,\beta=1}$ into (35) and using the definitions for $\chi(Bx^*)$ and $\xi(Q^*,x^*)$, we obtain the condition for the two-period model:

$$(1+s'(0)\big|_{\beta=1})n+(1-s(0)|_{\beta=1})(n-1)\left\{(1+s'(0)\big|_{\beta=1})(1+\delta+n)-\left[1+(n-1)\,s(0)|_{\beta=1}\right]\right\}-H(1) > 0,$$

Table 2: Model Specifications.

	AJ	KMZ	CE
Demand	$f(Q) = a - bQ$	$f(Q) = a - bQ$	$f(Q) = \sigma Q^{-\varepsilon}$
	$\delta = 0;\ a, b > 0$	$\delta = 0;\ a, b > 0$	$\delta = -(1+\varepsilon);\ a = 0, b = -\sigma < 0$
$c(\cdot)$	$\bar{c} - x_i - \beta \sum_{j \neq i} x_j$	$\bar{c} - [(2/\gamma)(x_i + \beta \sum_{j \neq i} x_j)]^{1/2}$	$\kappa(x_i + \beta \sum_{j \neq i} x_j)^{-\alpha}$
$\Gamma(x)$	$(\gamma/2)x^2$	x	x

where

$$s(0) = \frac{(2n+\delta)[(n+\delta)/(2n+\delta) - \beta]}{n(n+1+\delta)}$$

and

$$s'(0) = -\frac{\left[2n^2 + \delta(2n+1) + \delta^2\right](n-1)\beta - \delta^2(n-1) - \delta(2n^2-1) - n(n^2+1)}{(n+1+\delta)^2 n}.$$

Thus, $s'(0)|_{\beta=1} = [1 + \delta - n(n-2)]/(n+1+\delta)^2$. Note that by setting $s = s' = 0$, we obtain the condition for the simultaneous case, that is, condition (24).■

B.2 Examples

In this Section we characterize each of the model specifications considered in the paper: first in the simultaneous and then in the two-stage model. Before characterizing AJ, KMZ and CE, we describe briefly the main assumptions of each model specification.

Model specifications: main assumptions. As shown in Amir (2000) the AJ and the KMZ model specifications are not equivalent for large spillover values (the critical value depends on the innovation function or unit cost of production function and on the number of firms). The difference between the two models lies on the unit cost of production function and the autonomous R&D expenditures. Under the KMZ specification, *the effective R&D investment* for each firm is the sum of its own expenditure x_i and a fixed fraction (β) of the sum of the expenditures of the rest of firms, i.e., $X_i = x_i + \beta \sum_{j \neq i} x_j$. Instead, under the AJ specification, X_i is *the effective cost reduction* for each firm, so $c(\cdot)$ is a linear function. Thus, in AJ decision variables are unit-cost reductions, whereas in KMZ decision variables are the autonomous R&D expenditures.[47] In particular, in KMZ the unit cost of firm i is $\bar{c} - h(x_i + \beta \sum_{j \neq i} x_j)$, where for given $x_i \geq 0$ ($i = 1, ..., n$) the effective cost reductions to firm i, $h(\cdot)$, is a twice differentiable

[47] Furthermore, while in AJ the joint returns to scale (in R&D expenditure and number of firms) are decreasing, constant or increasing when spillover effects are less than, equal to, or greater than $1/(n+1)$, in KMZ the joint returns to scale are always nonincreasing (Proposition 4.1 in Amir (2000)).

Table 8: Second-order conditions and regularity condition.

	AJ	KMZ	CE
$S.O.C$	$\gamma b > 1/2$	$\gamma b > \tau/(2\tilde{\lambda})$	$n > \frac{\Lambda(1+\varepsilon)}{2}$ *and* $\frac{\varepsilon(1+\alpha)}{\alpha} > \frac{n(n-\varepsilon\Lambda)}{\tilde{\lambda}(2n+\Lambda\delta)}$
Regularity Condition	$\gamma b > \tau B/(\Lambda+n)$	$\gamma b > \tau/(\Lambda+n)$	$\varepsilon - \alpha(1-\varepsilon) > 0$

with $\tilde{\lambda} \equiv 1 + \lambda(n-1)\beta^2$.

and concave function with $h(0) = 0$, $h(\cdot) \leq \bar{c}$, and $(\partial/\partial x_i)h(\cdot) > 0$. As in Amir (2000), to allow for a direct comparison between AJ and KMZ, we consider a particular case of the KMZ model: $h = [(2/\gamma)(x_i + \beta \sum_{j \neq i} x_j)]^{1/2}$ with $\gamma > 0$. The CE model considers constant elasticity demand and costs with $\alpha, \kappa > 0$ (see Table 2); α is the unit cost of production (or innovation function) elasticity with respect to the investment in R&D (and no spillover effects). Finally, $\Gamma(x)$ is quadratic in AJ but linear in KMZ and CE.

B.2.1 Simultaneous model

We first derive Table 8, which provides the second-order and regularity conditions for the three model specifications (we also explore the feasible region for the constant elasticity model in Lemma 9). Second, we establish Lemma 10, which determines $sign\{\partial q^*/\partial\lambda\}$ and $sign\{\partial x^*/\partial\lambda\}$ for each model specification. Third, we derive the spillover threshold value $\bar{\beta}$ (Table 10). After that, we conduct a comparative statics analysis on $\bar{\beta}$. Finally, we examine welfare in AJ and KMZ, derive Table 6 and state Proposition 7.

Table 9: Equilibrium Values.

	AJ	KMZ	CE
q^*	$\frac{\gamma(a-\bar{c})}{\gamma b(\Lambda+n)-B\tau}$	$\frac{\gamma(a-\bar{c})}{\gamma b(\Lambda+n)-\tau}$	$\frac{1}{\alpha\kappa\tau}\left[\sigma\left(\tau\alpha/n\right)^{\varepsilon}\kappa^{\varepsilon-1}\left(1-\varepsilon\Lambda/n\right)\right]^{(1+\alpha)/[\varepsilon-\alpha(1-\varepsilon)]}$
x^*	$\frac{\tau(a-\bar{c})}{\gamma b(\Lambda+n)-B\tau}$	$\frac{\gamma\tau^2(a-\bar{c})^2}{2B[\gamma b(\Lambda+n)-\tau]^2}$	$\frac{1}{B}\left[\sigma\left(\tau\alpha/n\right)^{\varepsilon}\kappa^{\varepsilon-1}\left(1-\varepsilon\Lambda/n\right)\right]^{1/[\varepsilon-\alpha(1-\varepsilon)]}$

It is worth noting that in AJ and KMZ the R&D expenditure x^* and output q^* per firm increase with the size of the market (a) and decrease with the level of inefficiency of the technology employed, $\bar{c}$, the slope of inverse demand, b, and the parameter γ (which is the slope of the marginal R&D costs in AJ). In the CE model x^* and q^* also increase with the size of the market, σ. In addition, the costlier the technology employed, κ, the lower is total output, Q^*. However, x^* decreases (respectively, increases) with κ if demand is elastic (inelastic). The last two results hold for any value of λ and β.[48]

[48] The same result is obtained in Dasgupta and Stiglitz (1980) for $\lambda = \beta = 0$ and free entry.

Table 10 provides the spillover thresholds in the examples.

Table 10: Spillover Thresholds $\bar{\beta}$ and $\beta'(0)$.

	$\bar{\beta}$	$\beta'(0)$
AJ	$\dfrac{(n-2)+\sqrt{(n-2)^2+4b\gamma(n+2)(n-1)}}{2(n+2)(n-1)}$	$[-1+\sqrt{1+4b\gamma(n-1)}]/[2(n-1)]$
KMZ	$\dfrac{(n-2)+b\gamma(n-1)+\sqrt{(n-2)^2+b\gamma(n-1)\left[b\gamma(n-1)+6n+4\right]}}{2(n+2)(n-1)}$	γb
CE	is the value above which: $(n-\varepsilon)\alpha\beta\left\{B+(n-1)\left[\beta(n-\varepsilon)-1\right]\right\}-\varepsilon(\alpha+1)B>0$	$\varepsilon(\alpha+1)/[\alpha(n-\varepsilon)]$

Second-order and regularity conditions. To start with, let us rewrite the regularity condition as follows

$$\Delta(Q^*,x^*)=-\left[c''(Bx^*)B\tau(Q^*/n)+\Gamma''(x^*)\right]\left[f'(Q^*)(\Lambda(1+\delta)+n)\right]-(c'(Bx^*))^2\tau B>0. \quad (36)$$

In particular, for $\beta>0$ the above condition can be rewritten as $\Delta(Q^*,x^*)=\left[\Lambda(1+\delta)+n\right]\beta H(\beta)-\tau B>0$. Second-order conditions are: (i) $\partial_{qq}\phi_i=2f'(Q)+\Lambda(Q/n)f''(Q)=f'(Q)(2+\Lambda\delta/n)<0$, so $\partial_{qq}\phi_i<0$ if $\delta>-2n/\Lambda$; (ii) $\partial_{xx}\phi_i<0$, which is trivially satisfied by Assumptions A.2 and A.3; and (iii) $\partial_{qq}\phi_i\left(\partial_{xx}\phi_i\right)-\left(\varphi_{qx}\right)^2>0$, which is equivalent to

$$c'(Bx^*)^2+f'(Q^*)(2+\Lambda\delta/n)\left[c''(Bx^*)(Q^*/n)\tilde{\lambda}+\Gamma''(x^*)\right]<0, \quad (37)$$

where $\tilde{\lambda}=1+\lambda(n-1)\beta^2$. Noting that $\partial_{q_{i,j}}\phi_i=f'(Q^*)(1+\lambda)+f''(Q^*)\Lambda q^*=f'(Q^*)(1+\lambda+\delta\Lambda/n)$, we have that

$$\partial_{qq}\phi_i+\partial_{q_{i,j}}\phi_i(n-1)=f'(Q^*)\left[n+\Lambda(\delta+1)\right]<0,$$

which is satisfied if $\delta>-(n+\Lambda)/\Lambda$. Similarly, noting that $\partial_{xx}\phi_i=-c''(Bx^*)\tilde{\lambda}q^*-\Gamma''(x^*)$ and $\partial_{x_{i,j}}\phi_i=-c''(Bx^*)\beta q^*\left\{1+\lambda\left[1+(n-2)\beta\right]\right\}$, it is straightforward to show that

$$\partial_{xx}\phi_i+\partial_{x_{i,j}}\phi_i(n-1)=-\left[c''(Bx^*)B\tau q^*+\Gamma''(x^*)\right]<0,$$

which is satisfied by Assumptions A.2 and A.3.

In AJ and KMZ it is immediate that $\partial_{qq}\phi_i=-2b<0$. Furthermore, in AJ: $\partial_{qq}\phi_i\left(\partial_{xx}\phi_i\right)-(\varphi_{qx})^2=2b\gamma-1$, since $c''(\cdot)=0$ and $\Gamma''(x)=\gamma$, so $\partial_{xx}\phi_i=-\gamma$ and $\varphi_{qx}=-c'(\cdot)=1$. In KMZ, condition (37) can be written as

$$\left[\frac{1}{\gamma^2}\left(\frac{2}{\gamma}(Bx^*)\right)^{-1}\right]-2b\left[\frac{1}{\gamma^2}\left(\frac{2}{\gamma}(Bx^*)\right)^{-3/2}\right]q^*\tilde{\lambda}<0. \quad (38)$$

From first-order condition (3) we have that in equilibrium

$$q^* = \frac{\Gamma'(x^*)}{-c'(Bx^*)\tau} = \frac{1}{(1/\gamma)\left[(2(Bx^*)/\gamma\right]^{-1/2}\tau}. \tag{39}$$

Inserting the above equation into condition (38), after some manipulations, it reduces to $1 - 2b\gamma\tilde{\lambda}/\tau < 0$. (Note that if $\gamma b > \tau/2$ holds, then the condition $\gamma b > \tau/(2\tilde{\lambda})$ is satisfied.) In AJ and from (36), it is immediate that $\Delta = \gamma b(\Lambda + n) - \tau B$ since $c''(\cdot) = \delta = 0$, $f'(Q) = -b$ and $\Gamma'(x) = \gamma x$. In KMZ we have:

$$\Delta = -\left[\frac{1}{\gamma^2}\left(\frac{2}{\gamma}Bx^*\right)^{-3/2} B\tau \frac{1}{(1/\gamma)(2Bx^*/\gamma)^{-1/2}\tau}\right]\left[-b\left(\Lambda + n\right)\right] - \frac{1}{\gamma^2}\left(\frac{2}{\gamma}Bx^*\right)^{-1}\tau B.$$

Inserting (39) into the above equation, after some manipulations, we obtain

$$\Delta = \frac{1}{\gamma}\left(\frac{2}{\gamma}Bx^*\right)^{-1}\left[Bb(\Lambda + n) - \frac{\tau B}{\gamma}\right].$$

Therefore, in KMZ $\Delta > 0$ if $\gamma b > \tau/(\Lambda + n)$. Regarding the constant elasticity model we have:

LEMMA 9 *(Constant elasticity model) At the equilibrium, for a given $n \geq 2$ and $\lambda \geq 0$, second-order conditions together with the condition of non-negative profits require that*

(i) $\max\{\varepsilon\Lambda, \Lambda(1+\varepsilon)/2\} < n \leq \varepsilon\Lambda(B + \alpha\tau)/(\alpha\tau)$,

(ii) $\varepsilon(1+\alpha)/\alpha > n(n - \varepsilon\Lambda)/\left[\tilde{\lambda}(2n + \Lambda\delta)\right]$, *with* $\tilde{\lambda} \equiv 1 + \lambda(n-1)\beta^2$.

Furthermore, the equilibrium is regular if and only if $(1+\alpha)/\alpha > 1/\varepsilon$.

Proof. From the first-order condition (2) we need that

$$n > \varepsilon\Lambda, \tag{40}$$

otherwise the system (2; 3) will not have a solution. This condition also guarantees that Q^* and x^* are both positive. Notice that $\partial_{qq}\phi_i < 0$ if $(f'(Q^*)/n)(2n + \Lambda\delta) < 0$. Since $\delta = -(1+\varepsilon)$, $\partial_{qq}\phi_i < 0$ if

$$n > \Lambda(1+\varepsilon)/2. \tag{41}$$

Since $\Lambda \in [1, n]$, we have that the latter condition is always satisfied for $\varepsilon < 1$. By construction $\partial_{xx}\phi_i < 0$. Furthermore, second-order condition $\partial_{qq}\phi_i\left(\partial_{xx}\phi_i\right) - \left(\gamma_{qx}\right)^2 > 0$, which is given by (37), reduces to

$$-\frac{\varepsilon\sigma}{n}Q^{*-(\varepsilon+1)}(2n + \Lambda\delta)\left[\alpha(\alpha+1)\kappa(Bx^*)^{-(\alpha+2)}(Q^*/n)\tilde{\lambda}\right] + (\alpha\kappa)^2(Bx^*)^{-2(\alpha+1)} < 0. \tag{42}$$

From the first-order condition (2) we have that at the symmetric equilibrium

$$Q^* = [\sigma(n - \varepsilon\Lambda)/(n\kappa)]^{1/\varepsilon} (Bx^*)^{\alpha/\varepsilon}. \tag{43}$$

By substituting (43) into (42), after some manipulations, we obtain

$$(Bx^*)^{-2(\alpha+1)}\alpha\kappa^2 \left\{ - [\varepsilon/(n - \varepsilon\Lambda)] (2n + \Lambda\delta)(\alpha + 1)\tilde{\lambda}/n + \alpha \right\} < 0.$$

The above condition is satisfied if $\varepsilon(\alpha+1)/\alpha > n(n-\varepsilon\Lambda)/[(2n+\Lambda\delta)\tilde{\lambda}]$, which proves statement (ii) of the Proposition.

From (36) we have that $\Delta > 0$ if

$$\begin{aligned} 0 \;<\; & -\alpha(\alpha+1)\kappa(Bx^*)^{-(\alpha+2)}(Q^*/n)\tau B \left[\varepsilon(1+\varepsilon)\sigma Q^{*-(\varepsilon+2)}\Lambda Q^* - \varepsilon\sigma Q^{*-(\varepsilon+1)}(\Lambda+n)\right] \\ & -(\alpha\kappa)^2(Bx^*)^{-2(\alpha+1)}\tau B, \end{aligned}$$

or,

$$\begin{aligned} 0 \;<\; & Q^{*-(\varepsilon+1)} \left[-\alpha(\alpha+1)\kappa(Bx^*)^{-(\alpha+2)}(Q^*/n)\tau B\right] [\varepsilon(1+\varepsilon)\sigma\Lambda - \varepsilon\sigma(\Lambda+n)] \\ & -(\alpha\kappa)^2(Bx^*)^{-2(\alpha+1)}\tau B. \end{aligned}$$

Substituting (43) in the above expression, we obtain

$$\begin{aligned} 0 \;<\; & -\left[\frac{\sigma(n-\varepsilon\Lambda)}{n\kappa}\right]^{-(\varepsilon+1)/\varepsilon} (Bx^*)^{-(\varepsilon+1)\alpha/\varepsilon}\alpha(\alpha+1)\kappa(Bx^*)^{-(\alpha+2)} \left[\frac{\sigma(n-\varepsilon\Lambda)}{n\kappa}\right]^{1/\varepsilon} (Bx^*)^{\alpha/\varepsilon} \\ & \frac{\tau B}{n}[\varepsilon(1+\varepsilon)\sigma\Lambda - \varepsilon\sigma(\Lambda+n)] - (\alpha\kappa)^2(Bx^*)^{-2(\alpha+1)}\tau B, \end{aligned}$$

rearranging terms yields

$$0 < (Bx^*)^{-2(\alpha+1)} \left[\frac{n\kappa}{\sigma(n-\varepsilon\Lambda)} \left(-\alpha(\alpha+1)\frac{\kappa\tau B}{n}\right) \left(-\varepsilon\sigma n + \varepsilon^2\sigma\Lambda\right) - (\alpha\kappa)^2\tau B\right],$$

or, equivalently,

$$0 < (Bx^*)^{-2(\alpha+1)}\alpha\kappa^2\tau B\left[\varepsilon(\alpha+1) - \alpha\right].$$

Therefore, $\Delta > 0$ holds if $(1+\alpha)/\alpha > 1/\varepsilon$, or, equivalently, $\varepsilon - \alpha(1-\varepsilon) > 0$.

We turn now to deriving the condition under which profits in equilibrium are nonnegative. At the symmetric equilibrium, each firm's profit is given by $\pi(Q^*/n, x^*) = [f(Q^*) - c(Bx^*)](Q^*/n) -$

x^*. Then, $\pi(Q^*/n, x^*) \geq 0$ if $\bar{\pi} \equiv [f(Q^*) - c(Bx^*)]\, Q^*/(x^* n) \geq 1$. Write

$$\psi \equiv \sigma \left(\frac{\tau\alpha}{n}\right)^{\varepsilon} \kappa^{\varepsilon-1} \left(\frac{n - \varepsilon\Lambda}{n}\right).$$

Then $Q^* = [n/(\alpha\kappa\tau)]\, \psi^{(1+\alpha)/[\varepsilon-\alpha(1-\varepsilon)]}$, $x^* = (1/B)\psi^{1/[\varepsilon-\alpha(1-\varepsilon)]}$, and condition $\bar{\pi} \geq 1$ can be expressed as

$$\left[\sigma(\frac{n}{\alpha\kappa\tau})^{-\varepsilon}\psi^{-\varepsilon(1+\alpha)/[\varepsilon-\alpha(1-\varepsilon)]} - \kappa\psi^{-\alpha/[\varepsilon-\alpha(1-\varepsilon)]}\right] \frac{1}{\alpha\kappa\tau}\psi^{(1+\alpha)/[\varepsilon-\alpha(1-\varepsilon)]} B\psi^{-1/[\varepsilon-\alpha(1-\varepsilon)]} \geq 1.$$

Rearranging terms, and replacing ψ into the above expression, we get $[\varepsilon\Lambda/(n - \varepsilon\Lambda)]\, [B/(\alpha\tau)] \geq 1$. It follows that $\bar{\pi} \geq 1$ if

$$\left(\frac{\varepsilon\Lambda}{\alpha\tau}\right)(B + \alpha\tau) \geq n. \tag{44}$$

Combining conditions (40), (41) and (44) yields statement (i).■

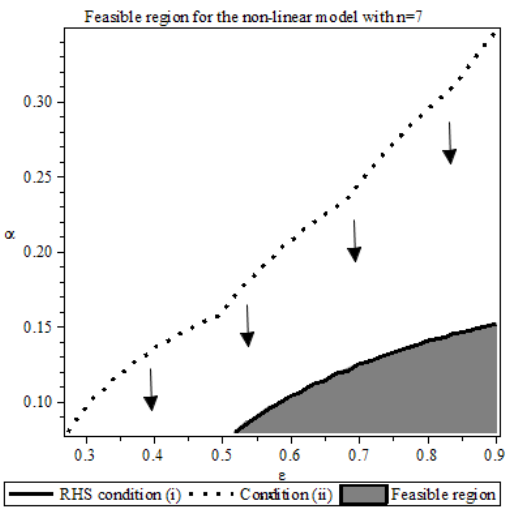

Fig. 9. Feasible region for the CE model with $n = 7$.

Feasible region for the constant elasticity model with $\lambda = 0$. From Lemma 9 we have that $\Delta > 0$ if $(1+\alpha)/\alpha > 1/\varepsilon$. When $\lambda = 0$, the LHS of condition (i) is trivially satisfied for any $n \geq 2$, moreover the RHS of condition (i) can be rewritten as follows $n \leq \rho(\beta) = \varepsilon(1+\alpha-\beta)/(\alpha-\varepsilon\beta)$. Since $\rho' > 0$ (as we are also imposing that $\Delta > 0$), condition $n \leq \rho(\beta)$ will hold for all β if $n \leq \varepsilon(1+\alpha)/\alpha$. Last, condition (ii) with $\lambda = 0$ writes as $\varepsilon(1+\alpha)/\alpha > n(n-\varepsilon)/[2n - (1+\varepsilon)]$. Therefore, at $\lambda = 0$ we only have to consider the RHS of condition (i) and condition (ii). These two conditions are depicted in Fig. 9 for $n = 7$; the grey area are combinations (α, ε) for which

the two conditions are satisfied (these combinations of parameters also satisfy the two conditions for $n \leq 7$).

Determination of $sign\{\partial q^*/\partial\lambda\}$ and $sign\{\partial x^*/\partial\lambda\}$ in AJ, KMZ and CE. Note that $\partial q^*/\partial\lambda$ can be written in the following manner

$$\frac{\partial q^*}{\partial\lambda} = \frac{(n-1)(Q^*/n)}{\Delta}\left\{\left(c'(Bx^*)\right)^2\beta B + f'(Q^*)\left[c''(Bx^*)(Q^*/n)B\tau + \Gamma''(x^*)\right]\right\}, \tag{45}$$

then after some calculations, it is simple to verify that in the simultaneous model:

LEMMA 10 *(i) In AJ: $sign\{\frac{\partial q^*}{\partial\lambda}\} = sign\{\beta(1+\beta(n-1)) - b\gamma\}$ and $sign\{\frac{\partial x^*}{\partial\lambda}\} = sign\{\beta(n+1)-1\}$;*

(ii) In KMZ: $sign\{\frac{\partial q^*}{\partial\lambda}\} = sign\{\beta - \gamma b\}$ and $sign\{\frac{\partial x^*}{\partial\lambda}\} = sign\{\beta(n+1)-1\}$;

(iii) In the CE model: $sign\{\frac{\partial q^*}{\partial\lambda}\} = sign\{\beta\left[\alpha(n-\varepsilon\Lambda) - \lambda(n-1)\varepsilon(\alpha+1)\right] - \varepsilon(\alpha+1)\}$ and $sign\{\frac{\partial x^*}{\partial\lambda}\} = sign\{\beta\left[(n-\varepsilon) - \lambda(n-1)(1+\varepsilon)\right] - 1\}$.

Derivation of $\bar{\beta}$ (Table 10). Note that $\partial x^*/\partial\lambda$ can be written as

$$\frac{\partial x^*}{\partial\lambda} = \frac{(n-1)(Q^*/n)f'(Q^*)c'(Bx^*)}{\Delta}[\beta(\Lambda(1+\delta)+n) - \tau] \tag{46}$$

If we insert equations (45) and (46) into equation (8), after some manipulations we obtain $W'(\lambda) = \left[(n-1)(Q^*)^2/(n\Delta)\right](-f'(Q^*))\digamma$, where

$$\begin{aligned}\digamma &= \Lambda\{\left(c'(Bx^*)\right)^2\beta B + f'(Q^*)[c''(Bx^*)(Q^*/n)B\tau + \Gamma''(x^*)]\} \\ &\quad + \left(c'(Bx^*)\right)^2(1-\lambda)\beta(n-1)\left\{\beta\left[\Lambda(1+\delta)+n\right] - \tau\right\}.\end{aligned}$$

By noting that in AJ: $f' = -b$, $\delta = 0$, $c' = -1$, $c'' = 0$ and $\Gamma'' = \gamma$, it then follows that

$$\begin{aligned}\digamma^{AJ} &= \digamma|_{\lambda=0} = \beta B - b\gamma + \beta(n-1)\left[\beta(1+n)-1\right] \\ &= (n-1)(n+2)\beta^2 - (n-2)\beta - b\gamma.\end{aligned}$$

By solving $\digamma^{AJ} = 0$ for β we obtain the expression for $\bar{\beta}^{AJ}$. Notice that $\bar{\beta}^{AJ} < 1$ if

$$(n-2) + \sqrt{(n-2)^2 + 4b\gamma(n+2)(n-1)} < 2(n+2)(n-1),$$

or

$$(n-2)^2 + 4b\gamma(n+2)(n-1) < \left[2(n+2)(n-1) - (n-2)\right]^2,$$

which can be rewritten as $4b\gamma(n+2)(n-1) < 4n^2(n+2)(n-1)$. Thus, $\bar{\beta}^{AJ} < 1$ if $b\gamma < n^2$. In KMZ we have $c = \bar{c} - \sqrt{(2/\gamma)(x_i + \beta \sum_{j\neq i} x_j)}$, $f' = -b$, $\delta = 0$ and $\Gamma'' = 0$, then

$$\begin{aligned} \digamma^{KMZ} &= \digamma|_{\lambda=0} = \frac{\beta}{2\gamma x^*} + \frac{-bq^*B}{\gamma^2 \left(2Bx^*/\gamma\right)^{3/2}} + \frac{\beta(n-1)\left[\beta(1+n)-1\right]}{2\gamma B x^*} \\ &= \frac{1}{B}\left(\frac{-bq^*B^{1/2}}{\gamma^2\left(2x^*/\gamma\right)^{3/2}} + \frac{\beta}{2\gamma x^*}\left\{B + (n-1)\left[\beta(1+n)-1\right]\right\}\right). \end{aligned}$$

By replacing q^* and x^* into the above expression, after some calculations we get

$$\digamma^{KMZ} = \frac{\left[b\gamma(1+n)-1\right]^2}{\gamma(a-\bar{c})^2}\left(-bB + \frac{\beta}{\gamma}\left\{B + (n-1)\left[\beta(1+n)-1\right]\right\}\right).$$

It is then immediate that: $\digamma^{KMZ} > 0 \Leftrightarrow \beta > \bar{\beta}^{KMZ}$. Notice that $\bar{\beta}^{KMZ} < 1$ if

$$\left\{(n-2)^2 + b\gamma(n-1)\left[b\gamma(n-1) + 2(3n+2)\right]\right\}^{1/2} < 2(n+2)(n-1) - n + 2 - b\gamma(n-1),$$

which can be rewritten as $4n(n+2)(n-1)(-n+b\gamma) < 0$. In the constant elasticity model $f = \sigma Q^{-\varepsilon}$, $c = \kappa(x_i + \beta\sum_{j\neq i} x_j)^{-\alpha}$ and $\Gamma(x) = x$, then

$$\begin{aligned} \digamma^{CE} &= \digamma|_{\lambda=0} = (\alpha\kappa)^2(Bx^*)^{-2(\alpha+1)}\beta B - \varepsilon\sigma(Q^*)^{-\varepsilon-1}\alpha(\alpha+1)\kappa(Bx^*)^{-(\alpha+2)}q^*B \\ &\quad + (\alpha\kappa)^2(Bx^*)^{-2(\alpha+1)}\beta(n-1)\left[\beta(-\varepsilon+n)-1\right]. \end{aligned}$$

By replacing q^* and x^* into the above expression, we obtain

$$\begin{aligned} \digamma^{CE} &= \alpha^2\kappa^2 z^{-2(1+\alpha)}\beta B - \varepsilon\sigma\left[n/(\alpha\kappa)\right]^{-(1+\varepsilon)} z^{-(1+\alpha)(1+\varepsilon)}(\alpha+1)z^{-(\alpha+2)}z^{\alpha+1}B \\ &\quad + \alpha^2\kappa^2 z^{-2(1+\alpha)}\beta(n-1)\left[\beta(-\varepsilon+n)-1\right], \end{aligned} \tag{47}$$

where

$$z \equiv \left[\sigma\left(\frac{\tau\alpha}{n}\right)^{\varepsilon}\kappa^{\varepsilon-1}\left(1-\varepsilon/n\right)\right]^{1/[\varepsilon-\alpha(1-\varepsilon)]}.$$

By noting that $z^{-(\alpha+1)(1+\varepsilon)-(\alpha+2)+(\alpha+1)} = z^{-\varepsilon+\alpha(1-\varepsilon)}z^{-2(1+\alpha)}$ we can re-write equation (47) as follows

$$\digamma^{CE} = z^{-2(1+\alpha)}\alpha\kappa^2\left\{\alpha\beta B + \alpha\beta(n-1)\left[\beta(-\varepsilon+n)-1\right] - \varepsilon(\alpha+1)B/(n-\varepsilon)\right\}.$$

Hence $\digamma^{CE} > 0$ if and only if

$$(n-\varepsilon)\alpha\beta\left\{B + (n-1)\left[\beta(n-\varepsilon)-1\right]\right\} - \varepsilon(\alpha+1)B > 0. \blacksquare$$

Threshold values $\bar{\beta}$, above which some partial ownership interests are socially optimal

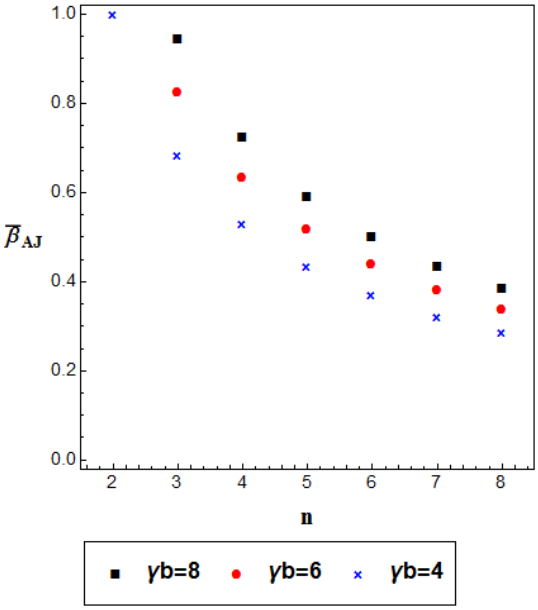

Fig. 10a. AJ model specification.

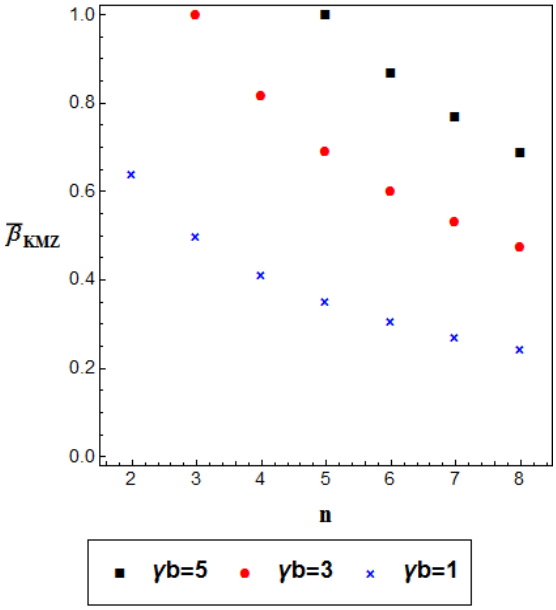

Fig. 10b. KMZ model specification.

Threshold values $\bar{\beta}$, above which some partial ownership interests are socially optimal

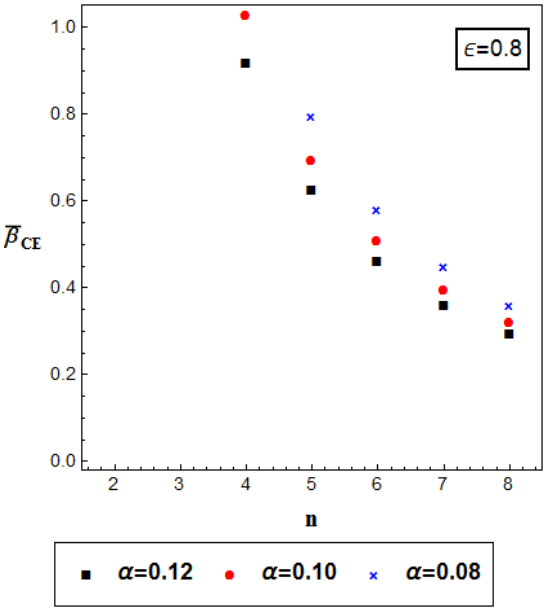

Figure 11a. Constant elasticity model.

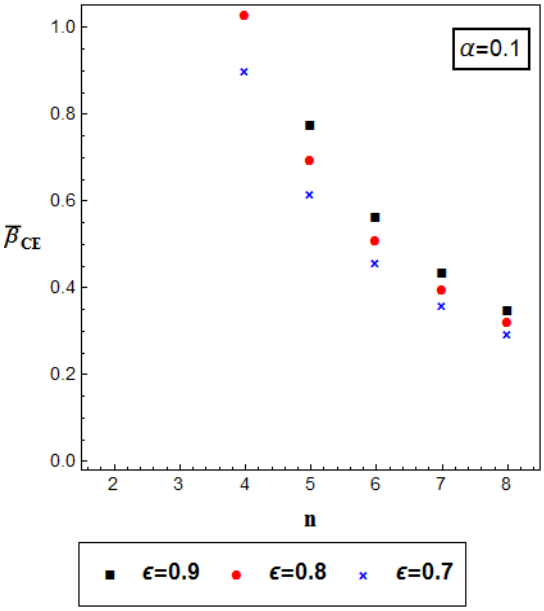

Figure 11b. Constant elasticity model.

Comparative statics on $\bar{\beta}$. Fig. 10a (respectively Fig. 10b) shows the value for $\bar{\beta}$ under the AJ (KMZ) model specification as a function of the number of firms and for different values of γb. As the figure makes clear, $\bar{\beta}^{AJ}$ and $\bar{\beta}^{KMZ}$ decrease with n: when there are more firms in the market, there is more need for minority shareholdings in order to internalize the additional externalities. We also have that $\bar{\beta}^{AJ}$ and $\bar{\beta}^{KMZ}$ decrease with γb, although $\bar{\beta}$ is lower than 1

for lower values of γb in the KMZ model than in the AJ model.

Fig. 11a and Fig. 11b depict $\bar{\beta}^{CE}$ as a function of n and for different values for α and ε. A glance at these figures shows that $\bar{\beta}^{CE}$ decreases again with n (for given ε and α). In addition, Fig. 11a tells us that for given n and ε, $\bar{\beta}^{CE}$ decreases with the elasticity of the innovation function, α, whereas Fig. 11b shows that for given n and α, $\bar{\beta}^{CE}$ increases with ε, so it decreases with the elasticity of demand. We also have that for the (feasible) combination of parameters (α, ε) considered here, $\bar{\beta}^{CE} \geq 1$ when there are two or three firms in the market.

Optimal degree of cross-ownership in terms of total surplus and consumer surplus

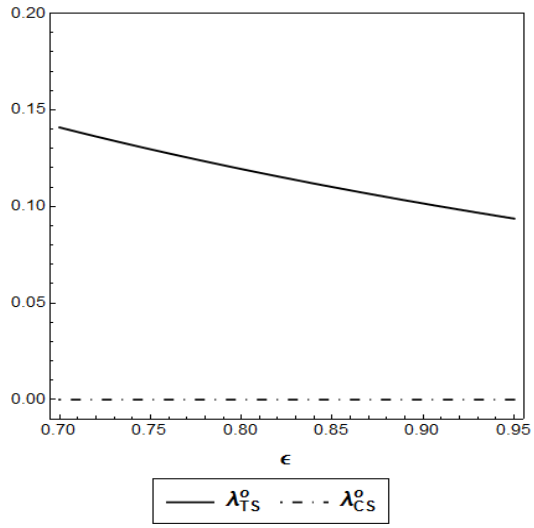

Fig. 12a. Constant elasticity model.
($\alpha = 0.1$, $\sigma = \kappa = 1$, $n = 8$, $\beta = 0.8$.)

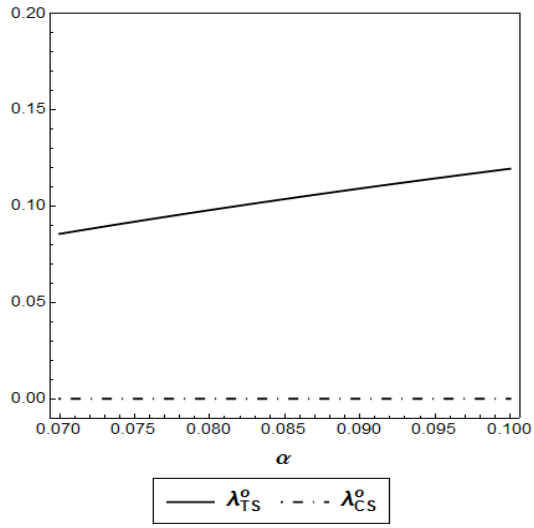

Fig. 12b. Constant elasticity model.
($\varepsilon = 0.8$, $\sigma = \kappa = 1$, $n = 8$, $\beta = 0.8$.)

Fig. 12a and 12b show that the greater is the elasticity of demand, ε^{-1}, or the elasticity of the innovation function, α, the greater should be the degree of cross-ownership if the social planner seeks to maximize total surplus; however, if the objective is to maximize consumer surplus, then for the same parameter range, $\lambda^o_{CS} = 0$.

Welfare in AJ and KMZ, and derivation of Table 6. Here, we show that welfare is a single-peaked function in AJ and KMZ; we also derive λ^o_{TS} under these two model specifications.

Case AJ: By inserting equilibrium values into the welfare function we get

$$W = \frac{1}{2} n\gamma (a - \bar{c})^2 \frac{(2\Lambda + n)\gamma b - \tau^2}{[(\Lambda + n)\gamma b - B\tau]^2}.$$

If we differentiate W with respect to λ we obtain:

$$\frac{dW}{d\lambda} = -\frac{(n-1)(a-\bar{c})\gamma b \left\{\Lambda\gamma b + \beta\left[2\lambda\left(B-n\right) + n - 2 - \beta(n+2)(n-1)\right]\right\}}{\left[\left(\Lambda + n\right) b\gamma - B\tau\right]^2} Q.$$

Note that solving $dW/d\lambda = 0$ for λ yields a unique stationary point, given by $\hat{\lambda}_{AJ}$. By taking the second-order derivative with respect to λ, evaluating it at $\lambda = \hat{\lambda}_{AJ}$, and simplifying, we obtain

$$\left.\frac{d^2W}{d\lambda^2}\right|_{\lambda=\hat{\lambda}_{AJ}} = -\frac{(n-1)^2(a-\bar{c})\gamma b\left[2\left(\beta-1\right)\beta+\gamma b\right]^3}{\left[-(n+2)(n-1)^2\beta^4-6(n-1)\beta^3+Z_1+2Z_2-Z_3\right]^2}Q,$$

where $Z_1 \equiv \left[\left(n^2+4n-1\right)\gamma b+3\left(n-2\right)\right]\beta^2$, $Z_2 \equiv 2\left[\gamma b(1-2n)+1\right]\beta$ and $Z_3 \equiv \gamma b(1-\gamma bn)$. The second-order condition requires that $\gamma b > 1/2$ (see Table 8), then $2(\beta-1)\beta+\gamma b > 0$ for any $\beta \in [0,1]$, and as a result: $d^2W/d\lambda^2\big|_{\lambda=\hat{\lambda}_{AJ}} < 0$. Since $\hat{\lambda}_{AJ}$ is the unique stationary point of W , it follows that $\hat{\lambda}_{AJ}$ is a global maximum. This is the desired λ^o_{TS}.

Case KMZ: By inserting equilibrium values into the welfare function we get

$$W = \frac{1}{2}n\gamma(a-\bar{c})^2\frac{(2\Lambda+n)B\gamma b-\tau^2}{\left[(\Lambda+n)\gamma b-\tau\right]^2 B}.$$

By differentiating W with respect to λ we obtain:

$$\frac{dW}{d\lambda} = -\frac{(n-1)(a-\bar{c})\gamma b\left\{\Lambda B\gamma b+\beta\left[2\lambda\left(B-n\right)+n-2-\beta(n+2)(n-1)\right]\right\}}{B\left[\left(\Lambda+n\right)b\gamma-\tau\right]^2}Q,$$

and by solving $dW/d\lambda = 0$ for λ we get a unique stationary point, given by $\hat{\lambda}_{KMZ}$. The second-order derivative with respect to λ evaluated at $\lambda = \hat{\lambda}_{KMZ}$ yields

$$\left.\frac{d^2W}{d\lambda^2}\right|_{\lambda=\hat{\lambda}_{KMZ}} = \frac{\gamma b(n-1)^2(a-\bar{c})}{B\left[(\Lambda+n)\gamma b-\tau\right]^3}ZQ,$$

where $Z \equiv -\left[\beta n+(1-\beta)\right]n(\gamma b)^2+\left[4\beta(1-\beta)n+(1-\beta)^2-\beta^2n^2\right]\gamma b+\beta B\left[\beta(n+2)-2\right]$. The regulatory condition requires that $\gamma b > \tau/(\Lambda+n)$ (see Table 8), thus $d^2W/d\lambda^2\big|_{\lambda=\hat{\lambda}_{KMZ}} < 0$ whenever $Z < 0$. Since $\hat{\lambda}_{KMZ}$ is the unique stationary point of W, it follows that $\hat{\lambda}_{KMZ}$ is a global maximum whenever $Z < 0$. This is the desired λ^o_{TS}.It is straightforward to show that the regularity condition is stricter than the second-order condition under the KMZ model specification (see Table 8). In addition, the regularity condition becomes stricter as the degree of cross-ownership and the number of firms increase. For $\lambda = 1$, the maximum value of the right-hand side of the regularity condition is $\sqrt{n}(n-1)/\left[4(n-\sqrt{n})\right]$, which for example equals 0.60 for $n = 2$ and 0.68 for $n = 3$. Numerical simulations show that assuming $\gamma b > 0.62$ guarantees that $Z < 0$ holds for any n; thus, $Z < 0$ is a mild condition: it is slightly stricter than the regularity condition in duopoly but softer for oligopoly of three or more firms.■

PROPOSITION 7 *A Research Joint Venture with no cross-ownership ($\lambda = 0$ and $\beta = 1$) is socially optimal in AJ when $\gamma b \geq n^2$, in KMZ when $\gamma b \geq n$, and in CE* (provided that $W(\lambda)$

is single peaked) *when* $\alpha \geq \varepsilon n/[(n-1)\varepsilon^2 + (-1+n-2n^2)\varepsilon + n(n^2+1-n)]$.

Proof. When $W(\lambda)$ is single peaked, $\bar{\beta}$ is the minimum threshold above which allowing some positive λ is welfare enhancing (Proposition 4). Consequently, $\lambda^o_{TS} = 0$ for any $\beta \in [0,1]$ if $\bar{\beta} \geq 1$. From Table 10 we have that $\bar{\beta}_{AJ} \geq 1$ if $\gamma b \geq n^2$ and $\bar{\beta}_{KMZ} \geq 1$ if $\gamma b \geq n$; in both cases $W(\lambda)$ is single peaked (see above). Also, from Table 10 we obtain $\bar{\beta}_{CE}$, and solving $\bar{\beta}_{CE} = 1$ for α, yields the threshold value in terms of n and ε: $\bar{\beta}_{CE} \geq 1$ if $\alpha \geq \varepsilon n/[(n-1)\varepsilon^2 + (-1+n-2n^2)\varepsilon + n(n^2+1-n)]$. Next we show that for $\lambda = 0$, $W'(\beta) > 0$ under AJ, KMZ and CE model specifications, and therefore it is socially optimal to set $\beta = 1$ in the three cases. Using (8) we can write

$$\begin{aligned} \frac{\partial W}{\partial \beta} &= (f(Q^*)n - nc(Bx^*))\frac{\partial q^*}{\partial \beta} - nc'(Bx^*)(n-1)x^*q^* - nc'(Bx^*)B\frac{\partial x^*}{\partial \beta}q^* \\ &\quad -n\Gamma'(x^*)\frac{\partial x^*}{\partial \beta} \\ &= \left[-\Lambda f'(Q^*)\frac{\partial q^*}{\partial \beta} - (1-\lambda)\beta(n-1)c'(Bx^*)\frac{\partial x^*}{\partial \beta} - c'(Bx^*)(n-1)x^*\right]Q^*. \end{aligned} \tag{48}$$

In AJ and for $\lambda = 0$, $\partial q^*/\partial\beta > 0$ and $\partial x^*/\partial\beta > 0$ (see Table 9), thus from (48) it is clear that $\partial W/\partial\beta > 0$. In KMZ and for $\lambda = 0$, $\partial q^*/\partial\beta = 0$ and $\partial x^*/\partial\beta < 0$. Higher R&D spillovers reduce R&D expenditures but also the unit cost of production of all firms. The latter dominates the former:

$$\left.\frac{\partial W}{\partial \beta}\right|_{\lambda=0} = \frac{1}{2}\frac{n(a-\bar{c})^2\gamma(n-1)}{[b\gamma(n+1)-1]^2 B^2} > 0.$$

In CE and for $\lambda = 0$, $\partial q^*/\partial\beta = 0$ and $\partial x^*/\partial\beta < 0$. As in KMZ, welfare is increasing in β:

$$\left.\frac{\partial W}{\partial \beta}\right|_{\lambda=0} = \frac{n\left[\sigma\left(\frac{\alpha}{n}\right)^{\varepsilon}\kappa^{\varepsilon-1}\left(1-\frac{\varepsilon}{n}\right)\right]^{\frac{1}{\varepsilon-\alpha(1-\varepsilon)}}(n-1)}{B^2} > 0. \blacksquare$$

B.2.2 Two-stage model

Next we present equilibrium values of output and R&D together with $sign\{\partial q^*/\partial\lambda\}$ and $sign\{\partial x^*/\partial\lambda\}$ for each model specification. Then, we conduct a comparative statics analysis on $\bar{\beta}$.

AJ model. First-order necessary conditions (10; 14) yields

$$-b\Lambda q^* + a - bnq^* - \bar{c} + Bx^* = 0$$

$$\left[\tau + \frac{\Lambda}{n+\Lambda}(n-1)(1+\lambda-2\beta)\right]q^* - \gamma x^* = 0.$$

Solving the system for equilibrium values gives

$$q^* = \frac{\gamma(a-\bar{c})}{\tilde{\Delta}} \text{ and } x^* = \frac{\left[(n-1)(\frac{\Lambda}{n+\Lambda})(1+\lambda-2\beta)+\tau\right](a-\bar{c})}{\tilde{\Delta}}$$

where

$$\tilde{\Delta} = \frac{\gamma b(\Lambda+n)^2 - B\left[(n-1)\Lambda(1+\lambda-2\beta)+(n+\Lambda)\tau\right]}{\Lambda+n}.$$

In this case, as in the static model, $H(\beta) = b\gamma/\beta$, then using (16) we obtain

$$sign\left\{\frac{\partial q^*}{\partial \lambda}\right\} = sign\left\{(B\beta - b\gamma)(n+\Lambda) + B\left[\frac{1+\lambda-2\beta}{n+\Lambda}(n-1)n+\Lambda\right]\right\}$$

and using (15) we get

$$\begin{aligned} sign\left\{\frac{\partial x^*}{\partial \lambda}\right\} &= sign\{\beta\left[\Lambda+n+(n-1)(\omega(\lambda)-\lambda)\right] \\ &\quad + \left[\frac{1+\lambda-2\beta}{n+\Lambda}(n-1)n+\Lambda\right] - 1 - (n-1)\omega(\lambda)\tilde{\beta}(\lambda)\}, \end{aligned} \tag{49}$$

where we have used that

$$\left[\omega'(\lambda)(\tilde{\beta}(\lambda)-\beta)+\omega(\lambda)\tilde{\beta}'(\lambda)\right](\Lambda+n) = \frac{1+\lambda-2\beta}{n+\Lambda}(n-1)n+\Lambda.$$

KMZ model. The output and R&D values in equilibrium are given by (10; 14):

$$-b\Lambda q^* + a - bnq^* - \bar{c} + \left[\left(\frac{2}{\gamma}\right)Bx^*\right]^{1/2} = 0$$

$$\frac{1}{\gamma}\left[\left(\frac{2}{\gamma}\right)Bx^*\right]^{-1/2}\left[\tau + (n-1)\frac{\Lambda}{n+\Lambda}(1+\lambda-2\beta)\right]q^* - 1 = 0.$$

Solving the system for equilibrium values gives

$$q^* = \frac{\gamma(a-\bar{c})}{\gamma b(\Lambda+n)-\mu} \text{ and } x^* = \frac{1}{2}\frac{(a-\bar{c})^2\mu^2\gamma}{B\left[b\gamma(\Lambda+n)-\mu\right]^2}$$

with

$$\mu \equiv \tau + s(\lambda)(n-1) = (n-1)\frac{\Lambda}{n+\Lambda}(1+\lambda-2\beta)+\tau,$$

where $s(\lambda) = \omega(\lambda)(\tilde{\beta}(\lambda)-\beta)$.

In this case, as in the static model, $H(\beta) = b\gamma B/\beta$, then from (16) we have

$$sign\left\{\frac{\partial q^*}{\partial \lambda}\right\} = sign\left\{(\beta - b\gamma)(n+\Lambda) + \left[\frac{1+\lambda-2\beta}{n+\Lambda}(n-1)n+\Lambda\right]\right\}$$

and $sign\{\frac{\partial x^*}{\partial \lambda}\}$ is again given by (49).

CE model. The output and R&D values in equilibrium are obtained from (10; 14):

$$\sigma Q^{*-\varepsilon}\left(1-\varepsilon\frac{\Lambda}{n}\right)-\kappa(Bx^*)^{-\alpha}=0$$

$$\alpha(Bx^*)^{-\alpha-1}\left[\tau+(n-1)\omega(\lambda)(\tilde{\beta}(\lambda)-\beta)\right]\frac{Q^*}{n}=1.$$

Solving the system for Q^* and x^*, after some manipulations, we get

$$Q^*=\frac{n}{\alpha\kappa\left[(n-1)s(\lambda)+\tau\right]}\left(\sigma\left\{\frac{\left[(n-1)s(\lambda)+\tau\right]\alpha}{n}\right\}^{\varepsilon}\kappa^{\varepsilon-1}\left(1-\frac{\varepsilon\Lambda}{n}\right)\right)^{(1+\alpha)/[\varepsilon-\alpha(1-\varepsilon)]}$$

and

$$x^*=\frac{1}{B}\left(\sigma\left\{\frac{\left[(n-1)s(\lambda)+\tau\right]\alpha}{n}\right\}^{\varepsilon}\kappa^{\varepsilon-1}\left(1-\frac{\varepsilon\Lambda}{n}\right)\right)^{1/[\varepsilon-\alpha(1-\varepsilon)]},$$

where $s(\lambda)=\omega(\lambda)(\tilde{\beta}(\lambda)-\beta)$ with

$$\omega(\lambda)=\frac{\Lambda\left[2n-\Lambda(1+\varepsilon)\right]}{n(n-\varepsilon\Lambda)}\quad\text{and}\quad\tilde{\beta}(\lambda)=\frac{n(1+\lambda)-\Lambda(1+\varepsilon)}{2n-\Lambda(1+\varepsilon)}.$$

It can be shown that in the constant elasticity model:

$$H(\beta)=\frac{B}{\beta}\left(\frac{\alpha+1}{\alpha}\right)\frac{\varepsilon}{n-\varepsilon\Lambda}\left[(n-1)s(\lambda)+\tau\right].$$

Hence, we have

$$sign\left\{\frac{\partial q^*}{\partial\lambda}\right\}=sign\left\{\left[\beta+s'(\lambda)\right]-\frac{\alpha+1}{\alpha}\frac{\varepsilon}{n-\varepsilon\Lambda}\left[(n-1)s(\lambda)+\tau\right]\right\}.$$

And, one can obtain $sign\{\partial x^*/\partial\lambda\}$ by inserting values into (15) with $\delta=-(1+\varepsilon)$.

Threshold values $\bar{\beta}$, above which some partial ownership interests are socially optimal

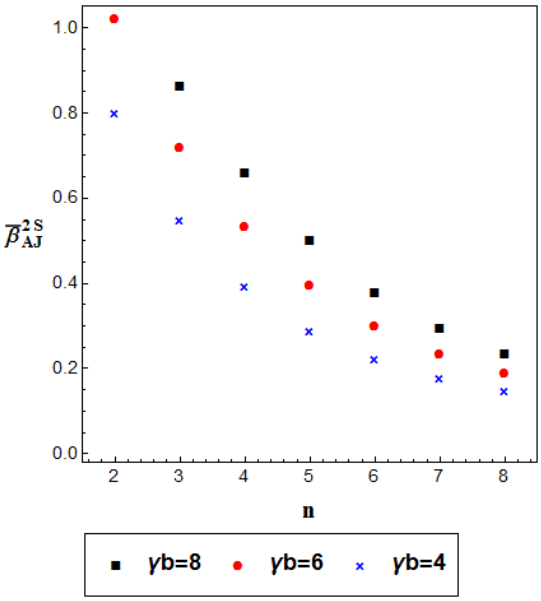

Fig. 13a. AJ model specification.

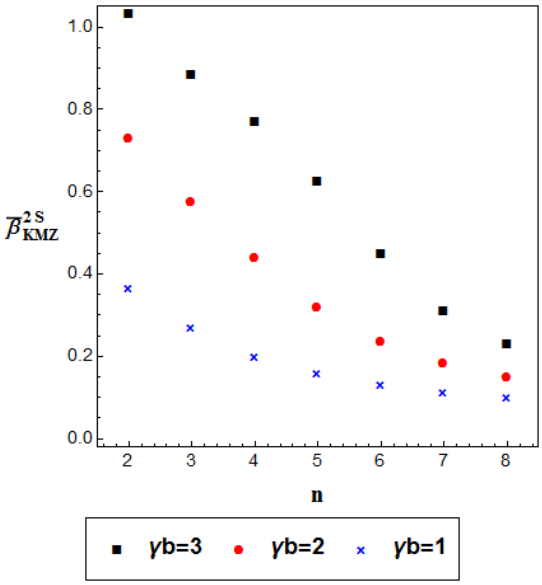

Fig. 13b. KMZ model specification.

Threshold values $\bar{\beta}$, above which some partial ownership interests are socially optimal

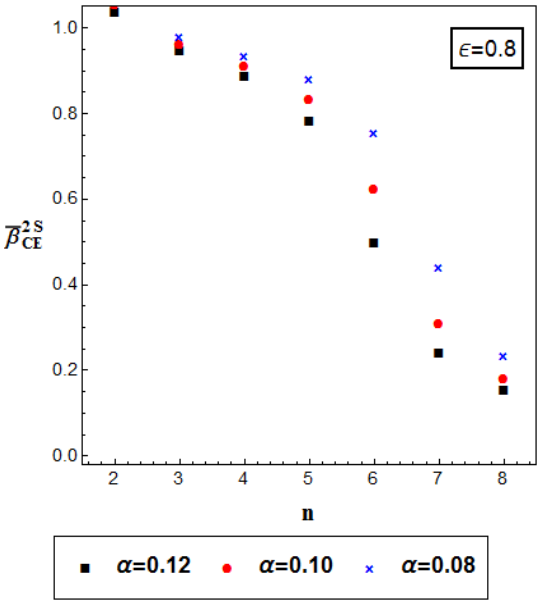

Fig. 14a. Constant elasticity model.

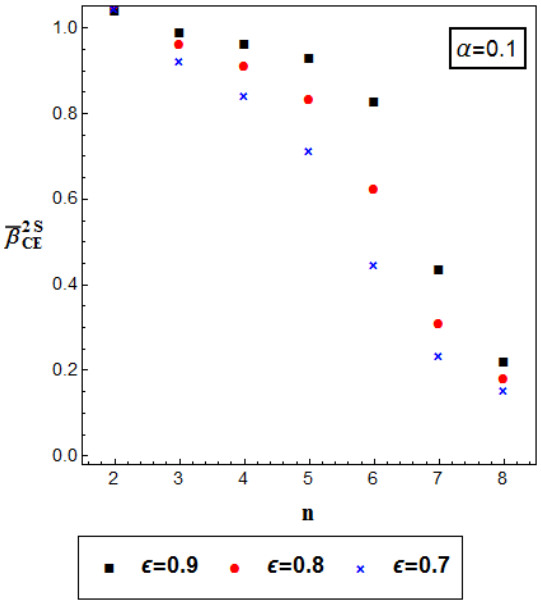

Fig. 14b. Constant elasticity model.

Comparative statics on $\bar{\beta}$. Fig. 13a and 13b depict, respectively, the threshold $\bar{\beta}^{2S}$ under the AJ and KMZ model specifications. Fig. 13b reveals that in KMZ, $\bar{\beta}^{2S}$ tends to be above 1 if we consider the same values as in AJ. In particular, only if γb is low enough, we have that $\bar{\beta}^{2S} < 1$ (this result is in line with the simultaneous model). Also, we observe that under the AJ and KMZ model specifications, $\bar{\beta}^{2S}$ decreases with the number of firms and increases with

γb. Figures 14a (respectively 14b) depict the threshold $\bar{\beta}^{2S}$ for the CE model and for a given ε (α) and different values of n and α (ε). As in the simultaneous model, the threshold value decreases with n, the elasticity of the innovation function, α, and the elasticity of demand ε^{-1}.

Comparative statics on λ^o_{TS} and λ^o_{CS}. Fig. 5b, 6c and 7b show, respectively, optimal lambdas in CE, AJ, and KMZ as functions of the number of firms. We see that under the three model specifications, λ^o_{TS} increases with n when n is sufficiently large, whereas λ^o_{CS} only increases with n (and when n is sufficiently large) in AJ.

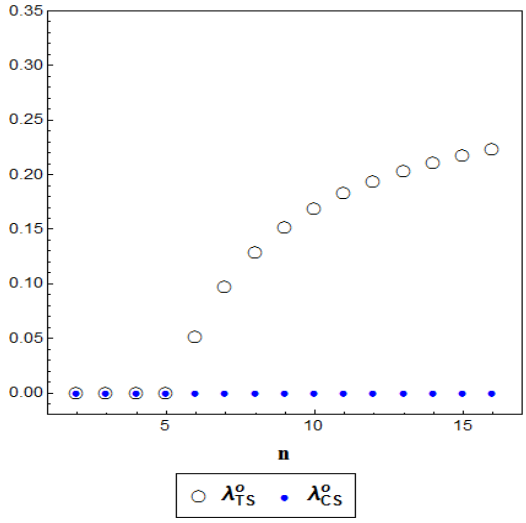

Fig. 5b. Constant elasticity model.
($\alpha = 0.1$, $\varepsilon = 0.8$, $\sigma = \kappa = 1$, $\beta = 0.8$.)

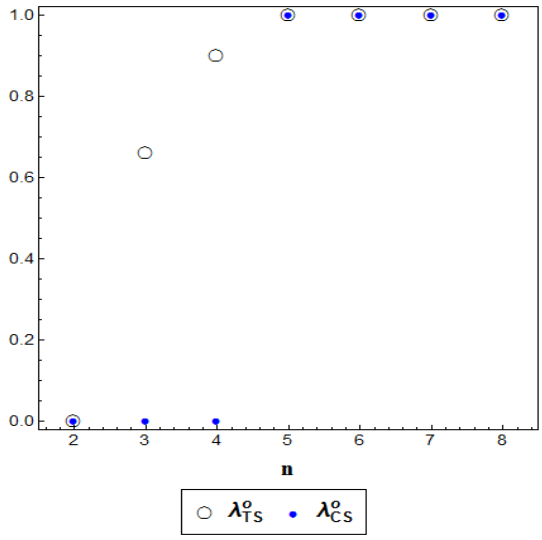

Fig. 6c. AJ model specification.
($a = 700$, $c = 500$, $\gamma = 7$, $\beta = 0.8$ and $b = 0.6$.)

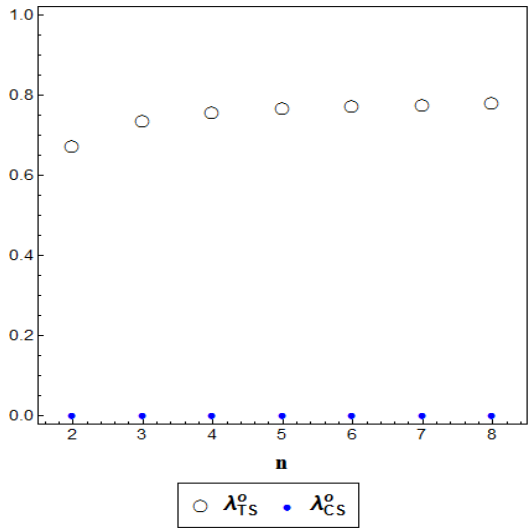

Fig. 7b. KMZ model specification.

($a = 700$, $c = 500$, $\gamma = 5$, $\beta = 0.8$ and $b = 0.3$.)