Probability and Time Trade-off *

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Abstract

Probability and time are integral dimensions of virtually any decision. To treat them together, we consider the prospect of receiving outcome $x$ with a probability $p$ at time $t$. We characterize the preferences by means of three functions: a value function, a psychological distance function, and a probabilistic discount rate function. The concavity of the psychological distance function explains the effects of both the common ratio and the common difference. A decreasing probabilistic discount rate contributes to the common difference effect and it accounts for the magnitude effect.

The discount rate and the risk premium depend on the shape of these three functions.

Keywords: Risk and time preferences. Probabilistic discount rate. Subendurance. Psychological distance.

1 Introduction

Time and probability are fundamental attributes of virtually any decision. For example, saving or investment decisions involve the consideration of risky future returns. Consumption decisions often involve the purchase of a future and uncertain reward. In the health domain, treatment or prevention decisions also involve uncertain consequences that occur in the future. Understanding how subjects treat delayed and uncertain outcomes may have

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*A previous paper with the title “Time and Risk Integrated” has been split into two papers, the present one (theory part) and an empirical paper (Baucells et al., 2009). “Time and Risk Integrated” was part of Antonio Villasís doctoral thesis.

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implications for public policy related to important issues such as drug addiction, environmental preservation and consumer sovereignty. In all these situations, decision makers face a trade-off between an immediate and/or sure reward and a delayed and/or uncertain reward. Managers, consumers or policy makers may fail to make efficient choices. A deeper understanding of their preferences can be a step in helping to guide their decisions. In this paper we provide a simple axiomatic model of preferences that directly formalizes the connection between probability and time.

Decision making under risk involves the trade-off between outcome and probability. The standard normative model in this domain is Expected Utility (Von-Neumann and Morgenstern, 1944), although more accurate behavioral models have been proposed in the literature, such as prospect theory (Kahneman and Tversky, 1979) or others [see Wu et al. (2004) for an overview]. Decisions involving time require the trade-off between outcome and delay. Here, discounted utility (Fishburn and Rubinstein, 1982) is the benchmark. Again, more accurate behavioral models that incorporate decreasing impatience have been developed previously [Loewenstein and Prelec (1992) and Laibson (1997), see Frederick et al. (2002) for an overview]. Indeed, time and risk preferences are combined in normative economic applications, usually in the form of expected discounted utility (Gollier, 2001).

The relatively narrow line of research that focuses on the preferences for outcomes that can be simultaneously delayed and uncertain was initiated by Rotter (1954). Accordingly, the similarities between choice under risk and inter-temporal choice are recognized and it was proposed that individuals encode delay in a probabilistic form. More recently, the similarities between decision models in the probability and time domains have been further discussed in the literature (among others by Prelec and Loewenstein (1991), Quiggin and Horowitz (1995), Ebert and Prelec (2007) and Halevy (2008)).

Here, we focus on the trade-off between delay and risk, and we formalize the psychological intuition that time and probability are both measures of psychological distance. Throughout the paper, we study simple prospects that can be described by a triple \((x, p, t)\) (that is, an
amount x which will be received with probability p at time t), otherwise the payoff is zero. Our framework is that of Fishburn and Rubinstein (1982), to which we add the probability dimension. The theoretical structure we propose is adequate to interpret the results of jointly eliciting risk and time preferences. The representation we derive involves a value function, v, together with a psychological distance function, d, and the probabilistic discount rate function r.

Time preference can be thought of as a primitive or it can be considered the product of broader considerations. In the latter case, several sources of impatience and time preference have been discussed. Risk is one such source, although others could involve consumption planning (Epstein and Hynes, 1983) or anticipating changes in future preferences (Loewenstein and Angner, 2003). Our approach does not directly address these sources of time preference. Rather, we characterize the shape of risk and time preferences when time and risk are traded-off in a particular way.

We introduce a preference condition called subendurance: Most subjects prefer a 50/50 chance of $100 today to having $100 for sure a year from now; but this preference is reversed if we increase the amount to $1000. Subendurance describes the growing willingness to wait for the increase in probability as the outcome gets larger. When this trade-off does not change with the size of x we consider isoendurance to occur.

Subendurance is captured by the probabilistic discount rate function that is non-increasing in x. We argue that the probabilistic discount rate provides a new tool for research into time preference to help understand why time is so fragile in the mind of decision makers (Ebert and Prelec, 2007). Indeed, in a natural way subendurance gives rise to the magnitude effect, i.e. discounting rates that decrease with the absolute size of the outcome. However, the probabilistic discount rate function goes beyond the magnitude effect and it allows economic models to incorporate factors that affect the ”uncertainty of time”, such as contract enforcement, trust or institutional stability, among others.

The remainder of the paper is organized as follows: Section 2 presents the main behav-
ioral findings in the combined risk and time domain; in section 3 we develop axioms and a functional representation for the joint evaluation of simple risk and time prospects, a section that introduces the psychological distance function; in section 4 we incorporated behavioral preference conditions that mean the probabilistic discount rate becomes non-increasing and the psychological distance function concave, and we explore the connections between subendurance, the common ratio effect and the common difference effect; in Section 5 we work out how the shape of our functions $v$, $d$ and $r$ is linked to the intuitive notions of decreasing impatience, the magnitude effect, or the risk premium; section 6 shows a parameter-free elicitation procedure to obtain the functions $v$, $d$ and $r$; and finally, section 7 concludes with suggestions for future research.

2 Motivation from Experimental Evidence

Table 1 presents the experimental evidence on how subjects resolve the trade-offs in the $(x, p, t)$ domain.\footnote{In Choices 4-6, we ignore whether the authors employed real incentives while in the rest of choices real incentives were used. Of course, there is abundant evidence of the common difference effect (Choices 4-5) using real incentives (Horowitz, 1991; Loewenstein and Prelec, 1992). Indeed, some subjects were selected at random and one of their choices was played out for real money. Except for Choice 7, the modal preference was significantly higher than 50\% using a Binomial test and hence, all the preference reversals were statistically significant. In all these experiments the timing of resolution is set at $\hat{t} = t$, except for Choice 6, in which it was not stipulated.} We witness five preference reversals, the focus of this section.

Reversal 1-2: Choice 1 presents a trade-off between a smaller more likely reward and a larger less likely reward. The modal preference favors the smaller more likely reward, although if the probabilities are reduced by a common ratio, as in Choice 2, then the preference shifts towards the larger less likely reward. This reversal is known as the common ratio effect, which violates what we later define as proportionality and does not respect the expected utility model (Allais, 1953). The common ratio is the focus of considerable literature on rank-dependent utility.

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Table 1: In bold, the modal preference in the choice between Prospect A and B. Rows 1-3 are taken from Baucells and Heukamp (2010, Table 1). Rows 4-6 are taken from Keren and Roelofsma (1995, Table 1). In 1995, one fl (Dutch Gulden) was equivalent to $0.6. Rows 7-8 are from Baucells et al. (2009). The last column indicates the number of subjects involved.

<table>
<thead>
<tr>
<th>Prospect A v. Prospect B</th>
<th>Response</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>(9 €, fixed, now) v. (12 €, 80% Pr, now)</td>
<td>58% v. 42%</td>
<td>142</td>
</tr>
<tr>
<td>(9 €, with 10% Pr, now) v. (12 €, 8% Pr, now)</td>
<td>22% v. 78%</td>
<td>65</td>
</tr>
<tr>
<td>(9 €, fixed, 3 months) v. (12 €, 80% Pr, 3 months)</td>
<td>43% v. 57%</td>
<td>221</td>
</tr>
<tr>
<td>(100 fl, fixed, now) v. (110 fl, fixed, 4 weeks)</td>
<td>82% v. 18%</td>
<td>60</td>
</tr>
<tr>
<td>(100 fl, fixed, 26 weeks) v. (110 fl, fixed, 30 weeks)</td>
<td>37% v. 63%</td>
<td>60</td>
</tr>
<tr>
<td>(100 fl, 50% Pr, now) v. (110 fl, 50% Pr, 4 weeks)</td>
<td>39% v. 61%</td>
<td>100</td>
</tr>
<tr>
<td>(5 €, 90% Pr, now) v. (5 €, fixed, 1 month)</td>
<td>57% v. 43%</td>
<td>79</td>
</tr>
<tr>
<td>(100 €, 90% Pr, now) v. (100 €, fixed, 1 month)</td>
<td>19% v. 81%</td>
<td>79</td>
</tr>
</tbody>
</table>

Reversal 4-5: Choice 4 presents a trade-off between a smaller sooner reward and a larger later reward. The modal preference favors the smaller sooner reward, although if a common delay is added, as in Choice 5, then the preference shifts towards the larger later reward. This reversal is known as the common difference effect, which violates what we later define as stationarity and rules out exponential discounting or constant impatience. Models incorporating decreasing impatience now form part of mainstream economic analysis (Laibson, 1997).

Both the common ratio and the common difference effect overrule the expected discounted utility model, \( V(x, p, t) = pe^{-rt}v(x) \). Thus, it may be appealing to apply \( V(x, p, t) = w(p)f(t)v(x) \), for some subproportional weighting functions \( w \) (Tversky and Wakker, 1995) and for some substationary discount function \( f \) (Loewenstein and Prelec, 1992).

Reversals 1-3 and 4-6: Consider reversal 1-2. If, rather than multiplying probabilities by a common factor, we add a common delay, then preference also shift towards the larger less likely reward, as in Choice 3. Similarly, consider reversal 4-5. If, rather than adding a common delay, we add a common risk, then preferences also shift towards the larger later
reward, as in Choice 6. Reversal 1-3 suggest that *time acts as probability*, while reversal 4-6 suggest that *probability acts as time*.

The implication of reversals 1-3 and 4-6 is that time and probability cannot be separated. Indeed, if \( V(x, p, t) = w(p)f(t)v(x) \), then the term \( w(p) \) would be cancelled out in choices 4 and 6, precluding the reversal; and the term \( f(t) \) would cancel in choices 1 and 3, precluding the reversal (see Proposition 1). Indeed, the proposal time and probability cannot be separated’ suggests a model of the form \( V(x, p, t) = w(p, t)v(x) \).

**Reversal 7-8:** Choice 7 presents a trade-off between an earlier less likely reward and a later more likely reward. This trade-off is central in this paper. The modal preference favors the earlier more likely reward. However, if the reward is larger, as in Choice 8, then the preference shifts towards the delayed more likely outcome. This is a violation of what we later define as *isoendurance*. The preference reversal 7-8 is novel to this paper and will be called *subendurance*. As shown below, the form \( w(p, t)v(x) \) is incompatible with subendurance.

The following proposition summarizes this discussion and motivates our next goal: to provide axiomatic foundations of a utility form compatible with *all* five preference reversals.

**Proposition 1** Let \( V(x, p, t) \) be a monotonic utility representing the preferences for \((x, p, t)\).

None of the following utility forms is compatible with the corresponding preference reversal.

<table>
<thead>
<tr>
<th>Utility Form</th>
<th>Corresponding Reversal</th>
</tr>
</thead>
<tbody>
<tr>
<td>( pv(x, t) )</td>
<td>1-2, common ratio</td>
</tr>
<tr>
<td>( e^{-rt}v(x, p) )</td>
<td>4-5, common difference</td>
</tr>
<tr>
<td>( pe^{-rt}v(x) )</td>
<td>1-2 or 4-5</td>
</tr>
<tr>
<td>( f(t)v(x, p) )</td>
<td>1-3, common ratio using delay</td>
</tr>
<tr>
<td>( w(p)v(x, t) )</td>
<td>4-6, common difference using risk</td>
</tr>
<tr>
<td>( w(p)f(t)v(x) )</td>
<td>1-3 or 4-6</td>
</tr>
<tr>
<td>( w(p, t)v(x) )</td>
<td>7-8, subendurance</td>
</tr>
</tbody>
</table>
3 The PTT Model

3.1 Preference Axioms

We adopted a framework similar to that of Fishburn and Rubinstein (1982), who considered preferences over a reward $x$ received at time $t$. We consider preferences over triplets $(x, p, t)$, which describes the prospect of receiving a reward $x$ at time $t$ with a probability $p$, where otherwise the payoff is zero. For simplicity, we consider that $x$ is unidimensional and of non-negative values. For example, $x$ could be money, the quantity of a divisible good, or the quality dimension of a non-divisible good. The decision maker’s preference increases in $x$, and $0$ is interpreted as the neutral outcome (no gain, no loss). For simplicity, we maintain the model silent with respect to the time, $\hat{t}$, $0 \leq \hat{t} \leq t$, at which the lottery is resolved (Drèze and Modigliani, 1974; Machina, 1984). The axioms we propose hold for some given specification of $\hat{t}$ such as $\hat{t} = 0$. Alternatively, one could set $\hat{t} = t$.

Formally, our choice set is $M = X \times P \times T$, where $X = [0, \infty)$ is a positive reward interval, $P = [0, 1]$ is the unit interval of probability, and $T = [0, \infty]$ is the time interval, which includes ‘never’ ($t = \infty$). Let $M^0$ be the set of trivial prospects, those having $xp/t = 0$ (where the reward is either zero, it is impossible or it is never to be received), and $M^+$ the set of non-trivial prospects where $px/t > 0$. Consider a preference relation $\succeq$ over pairs in $M$ possessing the following conditions.

**A1 Ordering and Continuity.** $\succeq$ is a continuous weak order on $M$.

**A2 Monotonicity.**

- **A2.1 Neutrality.** If $(x, p, t), (y, q, s) \in M^0$, then $(x, p, t) \sim (y, q, s)$.
- **A2.2 Monotone in Outcome.** $(y, p, t) \in M^+, y > x \rightarrow (y, p, t) \succ (x, p, t)$.
- **A2.3 Monotone in Probability.** $(x, q, t) \in M^+, q > p \rightarrow (x, q, t) \succ (x, p, t)$.
- **A2.4 Monotone in Time.** $(x, p, s) \in M^+, s < t \rightarrow (x, p, s) \succ (x, p, t)$.

A1-A2 are standard and guarantee the representation of preferences by a continuous
function $V(x, p, t)$ on $\mathcal{M}$ (Fishburn, 1970). If $(x, p, t) \in \mathcal{M}^0$, then $V$ is constant and without a loss of generality, we set $V(x, p, t) = 0$. For $\mathcal{M}^+$, $V(x, p, t)$ is strictly increasing in $x$ and $p$, and strictly decreasing in $t$. By continuity, $V$ tends to zero whenever $px/t$ tends to zero.

Take the prospect $(x, p, t + \Delta)$, and suppose we improve it by eliminating the delay $\Delta$ but make it worse by multiplying the probability by a factor $\theta$. In addition, we suppose that $\theta$ is such that the decision maker is indifferent to the original prospect and the proposed one, $(x, p\theta, t)$. Thus, an increase of $\Delta$ time units has been exchanged for a reduction of probability by a factor $\theta$. Consider a decision maker who would agree to this same $\Delta$ for $\theta$ exchange independent of the moment in time and the degree of probability. Such a decision maker would agree to what we call the probability-time trade-off axiom.

Let $(x, p, t), (x, q, s) \in \mathcal{M}^+$, $\Delta > 0$, and $\theta \in (0, 1)$.

**A3** Probability-Time trade-off. $(x, p\theta, t) \sim (x, p, t + \Delta) \rightarrow (x, q\theta, s) \sim (x, q, s + \Delta)$.

A3 is a novel concept in the literature and there are certain considerations that may motivate the axiom.

1. **A3 captures the intuition that time is intrinsically uncertain.** In folk wisdom we find sayings such as “a bird in the hand is worth two in the bush” referring to both risk and time. Likewise, when presented with the choice between a larger-later and an smaller-immediate reward, many subjects justify their preference for the immediate-smaller reward since “the future is always uncertain.” Moreover, risk has often been a fundamental explanation for time discounting (Mas-Colell et al., 1995, p.734). A3 is a way of capturing this intuition as the passage of time adds risk, which reduces the probabilities at a rate of $\theta$ for each $\Delta$ time unit. However, it is also possible that subjects see risk and time as two independent entities, but they just happen to trade them off as indicated by A3. Our model is open to both interpretations.

2. **A3 is compatible with all the experimental evidence presented in Table 1.** A direct consequence of A3 is that adding time has consequences equivalent to multiplying
probabilities, and vice versa (reversals 1-3 and 4-6). For example, if large delays induce time insensitivity, then large delays also induce probability insensitivity; and if small probabilities induce risk insensitivity, then small probabilities also induce time insensitivity. In fact, A3 creates a strong link between the common ratio and common difference effects (see Section 4). Finally, A3 implies that if subendurance holds for a particular level of probability and time, it also holds at all levels of probability and time.

3. **A3 is useful to define and elicit pure time preference.** The usual way to elicit time preference is via the discount rate. The elicitation of the discount rate uses two reward levels: the smaller sooner and the larger later. But this means that time preference may be confounded with outcome preference. For example, if subjects anchor on the larger reward, they might perceive the smaller sooner reward as a loss. Hence, the discount rate will be a function of both time preference and loss aversion. Using A3, the elicitation of time preference involves just one reward level that is received sooner but less likely or later but more likely. If $\theta$ is the ratio between the less likely and the more likely probabilities, and $\Delta$ the time difference between the later and the sooner rewards, then we define $r$ as the solution to $\theta = e^{-r\Delta}$, or

$$r = \frac{1}{\Delta} \ln \frac{1}{\theta}. \quad (1)$$

According to A3, $r$ may depend on $x$ but it is independent of $p$ and $t$. We call $r_x$ the **probabilistic discount rate** and it can be taken as a measure of pure time preference. Clearly, the probabilistic discount rate is related to, but different from, the usual discount rate.

4. **A3 is empirically sound.** The A3 has been tested directly Baucells et al. (2009). They proposed a choice between prospects $(x, p\theta, t)$ and $(x, p, t + \Delta)$, and using the bisection method, the value of $\Delta$ was adjusted until both prospects were deemed to be approx-
imately indifferent. If \( x \) and \( \theta \) are held constant, then A3 holds if \( \Delta \) is independent of the particular values of \( p \) and \( t \) chosen. The axiom was tested for three levels of \( x \) (€50, €100 and €1000), \( \theta \) was fixed at 1/2 and three pairs \((p, t)\) for each \( x \). The experiment involved real incentives, as some choices were selected at random and played for real (we provide the average values of \( \Delta \) in bold):

<table>
<thead>
<tr>
<th>Prospect A</th>
<th>Prospect B</th>
<th>( \Delta )</th>
<th>( r_x ) [monthly]</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>(50, 1/2, 0) ( \sim ) (50, 1, 10.5)</td>
<td>10.5</td>
<td>0.066</td>
<td>66</td>
<td></td>
</tr>
<tr>
<td>(50, 1/3, 5) ( \sim ) (50, 2/3, 12.4)</td>
<td>7.4</td>
<td>0.094</td>
<td>65</td>
<td></td>
</tr>
<tr>
<td>(50, 1/6, 1) ( \sim ) (50, 1/3, 8.7)</td>
<td>7.7</td>
<td>0.090</td>
<td>61</td>
<td></td>
</tr>
<tr>
<td>(100, 1/6, 0) ( \sim ) (100, 1/3, 10.9)</td>
<td>10.9</td>
<td>0.063</td>
<td>65</td>
<td></td>
</tr>
<tr>
<td>(100, 1/3, 5) ( \sim ) (100, 2/3, 9.8)</td>
<td>8.8</td>
<td>0.079</td>
<td>67</td>
<td></td>
</tr>
<tr>
<td>(100, 1/2, 5) ( \sim ) (100, 1, 17.3)</td>
<td>12.3</td>
<td>0.056</td>
<td>66</td>
<td></td>
</tr>
<tr>
<td>(1000, 1/3, 0) ( \sim ) (1000, 2/3, 15.9)</td>
<td>15.9</td>
<td>0.044</td>
<td>63</td>
<td></td>
</tr>
<tr>
<td>(1000, 1/2, 1) ( \sim ) (1000, 1, 17.5)</td>
<td>16.5</td>
<td>0.042</td>
<td>67</td>
<td></td>
</tr>
<tr>
<td>(1000, 1/6, 1) ( \sim ) (1000, 1/3, 17.5)</td>
<td>12.5</td>
<td>0.056</td>
<td>64</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Using the bisection method, the delay in Prospect B was adjusted until finding approximate indifference between Prospects A and B (Baucells et al., 2009). In all cases, \( \theta = 1/2 \), rewards are in Euros and the time is measured in months. The table includes the probabilistic discount rate calculated as in (1).

In choices 1, 2, and 3, where the reward is fixed at €50, the three values of \( \Delta \) should be similar. This produces three pairwise comparisons. Choices 5, 6, and 7, with €100 rewards, produces another three pairwise comparisons. Finally, choices 8, 9, and 10, with €1000 rewards, produces three more pairwise comparisons. In a between subjects test, seven of the nine pair comparisons do not reject the null hypothesis that they are equal. The exceptions are the comparisons of \( \Delta \) in choices 1-2 (p-value 0.042), and in choices 8-9 (p-value 0.037). The evidence is not as strong in the within subject comparison. Still, in choice pairs 2-3, 4-6, and 7-8 we cannot reject the hypothesis that the delays are the same. This evidence suggests that the axiom is behaviorally plausible, at least in some parts of the domain.
3.2 The PTT Representation

By A1-A3, if $\theta = e^{-rt}$, then any future prospect $(x,p,t)$, $t > 0$, is indifferent to the less likely present prospect $(x,p\theta,0)$. In other words, $V(x,p,t) = V(x,pe^{-rt},0)$ and only the outcome and probability components of $V$ become relevant (see proof of Proposition 2). It is convenient to obtain a representation where $V$ is multiplicative separable in these two components. To this end, we resort to a standard separability condition (Fishburn and Rubinstein, 1982, A6), to be imposed on immediate prospects.

A4 Outcome-Probability Separability. For all $x,y,z \in X$ and $p,q,u \in P$,

$$(x,p,0) \sim (y,q,0), (y,u,0) \sim (z,p,0) \rightarrow (x,u,0) \sim (z,q,0).$$

The combination of A1-A4 yields our main result.

**Proposition 2** The PTT Model: A1-A4 hold if and only if for all $(x,p,t) \in M$,

$$V(x,p,t) = w(pe^{-rt})v(x) \quad (2)$$

for some continuous and strictly positive function $r_x$, and some continuous, positive and strictly increasing functions $v(x)$ and $w(p)$, having $v(0) = w(0) = 0$ and $w(1) = 1$. $r_x$ is such that $V(x,p,t)$ is monotonic in $x$. Moreover, for any strictly positive numbers $\kappa$ and $\eta$, the functions $\hat{v} = \kappa v^\eta$ and $\hat{w} = w^\eta$ can equally represent the preferences.

The uniqueness part of Proposition 2 is typical of a setup with simple prospects and a multiplicative form (Fishburn and Rubinstein, 1982). We could extend the domain of the rewards to include the negative domain. With simple prospects, there are no preference conditions that link the two domains. Hence, the representation in the negative domain would be an independent duplicate, involving a weighting function for negative outcomes, $w^-$, and the functions $v$ and $r$ would be defined for negative values. Finally, we could scale $v$ in the negative domain and $w^-$ independently of the scaling in the positive domain.
Note that if the function $r_x$ is non-increasing in $x$, then $V$ is certain to increase in $x$. Indeed, larger outcomes are discounted less and become even more attractive. In Sub-section 4.1 we will show that subendurance guarantees that $r_x$ is non-increasing.

3.3 Psychological Distance

Both risk and time distance the decision maker from the outcome (Todorov et al., 2007). *Risk distance* can be measured by $\ln 1/p$ and conveniently, a certain outcome ($p = 1$) is at zero risk distance, while an impossible outcome is at infinite risk distance. *Time distance* is measured directly by $t$.

In the PTT model, risk and time distance can be combined, but only after we correctly adjust their units. The probabilistic discount rate achieves this conversion, with $a = \ln 1/p + r_x t$ being the combined distance. If $r_x$ is not constant, then psychologically, the speed of time varies with the size of the outcome. For instance, if $r_x$ is decreasing then time runs faster for smaller rewards than for larger rewards: one month seems an eternity if we are to receive a sweet but it represents an instant if we are to receive one million dollars. In addition, subjects may not perceive the distance $a$ linearly, and it is possible that they apply some non-linear transformation $d(a)$. In fact, the PTT model can be written according to this interpretation. We define $d(a) = \ln(1/w(e^{-a}))$ as the *psychological distance function*, and rewrite the PTT model as follows:

$$V(x,p,t) = e^{-d(a)}v(x), \quad a = \ln 1/p + r_x t,$$

where $d(a)$ is increasing and continuous, with $d(0) = 0$ and $d(a)$ tending to infinity as $a$ goes to infinity. Of course, the function $d$ could have been chosen as the primitive, defining the weighting function as $w(p) = e^{-d(\ln 1/p)}$.

A third way to represent the PTT model is by means of a discount function. Let $f(a) = w(e^{-a}) = e^{-d(a)}$ be the *discount function* and rewrite the PTT model as

$$V(x,p,t) = f(a)v(x), \quad a = \ln 1/p + r_x t,$$
where $f$ is a decreasing and continuous function, with $f(0) = 1$ and tending to zero when the distance increases towards infinity. If we take $f$ as the primitive, then we can define $w$ and $d$ as $w(p) = f(\ln(1/p))$ and $d(a) = \ln(1/f(a))$, respectively. Note that $f$ is not a function of time directly but rather, of time measured in units of probability (i.e., a function of $r_x t$). Hence, exponential $f$ does not imply exponential discounting, as will become clear in Sub-section 4.3.

4 Subendurance, Subproportionality and Substationarity

In Table 1 evidence was introduced for subendurance, subproportionality (common ratio) and substationarity (common difference). In this section, we will formally define these three preference conditions, derive their implications on the PTT functional representation and explore their interconnections.

4.1 Subendurance

The trade-off between probability and time might depend on the magnitude of $x$. The preference reversal 7-8 reported in Table 1 shows that as the stakes increase, subjects are willing to support larger delays. In line with this experimental evidence, we propose that subjects are only prepared to endure a longer delay for larger outcomes. We also consider the more restrictive case where the trade-off is not affected by the size of the reward.

Let $(x, p, t) \in M^+, y > x, \theta \in (0, 1)$ and $\Delta \in (0, \infty)$.

A5 Subendurance. $(x, p\theta, t) \sim (x, p, t + \Delta) \rightarrow (y, p\theta, t) \preceq (y, p, t + \Delta)$.

A5i Isoendurance. $(x, p\theta, t) \sim (x, p, t + \Delta) \rightarrow (y, p\theta, t) \sim (y, p, t + \Delta)$.

Conditions A5 and A5i are directly connected with the probabilistic discount rate.
Proposition 3  Under A1-A4,

a) Subendurance $\leftrightarrow r_x$ is non-increasing in $x$.

b) Isoendurance $\leftrightarrow r_x$ is constant.

To check for subendurance or isoendurance, we take the nine data points from Table 2 and try two regression models: $r_x$ against $1/x$, and $\ln r_x$ against $\ln x$. The fit is quite good (correlation = 0.84 in both cases) and the relative size of the slope refutes the null hypothesis of isoendurance in favor of subendurance (one tailed p-value = 0.01 and 0.009, respectively). The monthly probabilistic discount rate is estimated by

$$r_x = 0.046 + \frac{1.9}{x} \quad \text{or} \quad r_x = \frac{0.16}{x^{0.18}}, \quad (5)$$

Using the first specification, for outcomes the size of 50, 100, and 1000 Euros, the passage of one year is equivalent to multiplying probabilities by a factor of $e^{-r_x t} = 37\%, 46\%$ and 56\%, respectively. This probability drops quite rapidly as we reduce the size of the reward. If $r$ were a mortality rate, then these would be the probabilities of survival. Because the survival probability changes with the size of the outcome, it is not possible to interpret time as having a common mortality rate, as in Halevy (2008). The probabilistic discount rate may also be affected by factors that make time more uncertain, such as the way time is described in the choice task, the age of the subject or the stability of the economic environment.

Simon (1978) said that in “a world where attention is a major scarce resource, information may be an expensive luxury, for it may turn our attention from what is important to what is unimportant” (see also Hogarth and Einhorn (1990) who built Venture Theory based on a similar intuition). Subendurance could be considered a mental habit of not thinking hard about small consequences in the future, reflecting the large degree of uncertainty associated with claiming (and receiving) small outcomes in the distant future.
4.2 Diminishing Sensitivity to Risk and Time Distance

In the domain of risk preference, the independence axiom postulates that a reduction of probabilities by a common factor should not alter the preference between two prospects. By contrast, the common ratio effect shows that for small probabilities, subjects become more insensitive to probability ratios, and pay more attention to the size of the outcome. Subproportionality is a preference condition that accounts for this behavior (Kahneman and Tversky, 1979; Prelec and Loewenstein, 1991).

Let \((x, p, t) \in M^+, y > x, q \in (0, p)\) and \(\theta \in (0, 1)\).

**A6** **Subproportionality.** \((x, p, t) \sim (y, p\theta, t) \rightarrow (x, q, t) \preceq (y, q\theta, t)\).

**A6i** **Proportionality.** \((x, p, t) \sim (y, p\theta, t) \rightarrow (x, q, t) \sim (y, q\theta, t)\).

A6 (A6i) is directly connected with diminishing (constant) sensitivity to risk distance.

**Proposition 4** Under A1-A4,

a) Subproportionality \(\leftrightarrow d\) is concave.

b) Proportionality \(\leftrightarrow d\) is linear.

Diminishing sensitivity to outcomes has been the cornerstone of economic analysis. This is captured in the concavity of \(v\). Intuitively, a windfall of $10,000 feels better than the same $10,000 added to a windfall of $100,000. The same principle of diminishing sensitivity applies to risk distance via the concavity of \(d\). A difference between, say, 80% and 100% is felt much more than a difference between 20% and 25%, which in turn is felt more than a difference between 4% and 5%. Similarly, for time distance, a delay of one hour feels longer than a delay of one hour added to a four hour delay.

In the domain of time preference, one can consider substationarity as a relaxation of stationarity that accounts for the common difference effect (Prelec and Loewenstein, 1991).

Let \((x, p, t) \in M^+, y > x, s \in (t, \infty)\) and \(\Delta \in (0, \infty)\).

**A7** **Substationarity.** \((x, p, t) \sim (y, p, t + \Delta) \rightarrow (x, p, s) \preceq (y, p, s + \Delta)\).
A7i Stationarity. \( (x, p, t) \sim (y, p, t + \Delta) \rightarrow (x, p, s) \sim (y, p, s + \Delta) \).

Under subendurance, the common ration and the common difference effects become equivalent.

**Proposition 5** Under A1-A4 and subendurance,

\[ \text{Subproportionality} \leftrightarrow \text{Substationarity} \leftrightarrow d \text{ concave}. \]

It can be shown that concave \( d \) is equivalent to subproportional \( w \), \( w(p\theta)/w(p) \) is decreasing in \( p \); and to substationary \( f \), \( f(t+\Delta)/f(t) \) is increasing in \( t \). Clearly, a subproportional \( w \) is equivalent to subproportional preferences. Moreover, under subendurance, a substationary \( f \) is equivalent to substationary preferences.

To help fix ideas, suppose the psychological distance function takes the power form,

\[ d(a) = a^\gamma, 0 < \gamma \leq 1, \text{ and } \]
\[ V(x, p, t) = e^{-(\ln 1/p+r_xt)^\gamma} v(x). \]

We assume \( r_x \) is non-increasing. A single curvature parameter \( \gamma \leq 1 \) controls the extent to which people are able to discriminate between probability ratios and differences in time delays, inducing subproportionality and substationarity at the same time. If \( t = 0 \), then the weighting function in (4) becomes

\[ w(p) = e^{-\left(\ln 1/p\right)^\gamma}, \quad (6) \]

which is the probability weighting function axiomatized by Prelec (1998). And if \( p = 1 \), then the discount function in (2) becomes

\[ f(r_xt) = e^{-\left(r_xt\right)^\gamma}. \quad (7) \]

The form \( f(rt) = e^{-\left(rt\right)^\gamma} \) has been axiomatized by Ebert and Prelec (2007).\(^2\) Both (6) and

\(^2\)Compound invariance is the key condition on risk preferences that yields \( w(p) = e^{-\left[\ln 1/p\right]^\gamma} \); and scale invariance is the key condition on time preferences that yields \( f(rt) = e^{-\left[rt\right]^\gamma} \). Not surprisingly, scale invariance is formally equivalent to compound invariance once we replace \( t \) by \( \ln 1/p\).
fit the experimental data quite well (Wu et al., 2004; Ebert and Prelec, 2007; Booij et al., 2010) and we expect the combined form to be a good candidate for parametric modeling.

Another possibility for \( d \) is a piecewise linear function with two segments of slopes \( \delta_h \) and \( \delta_l \), respectively, with \( \delta_h > \delta_l \). This function reproduces the beta-delta model of discounting (Laibson, 1997). This model could also incorporate subendurance if \( r_x \) were decreasing.

### 4.3 Constant Sensitivity to Risk and Time Distance

The PTT model under proportionality and subendurance can be called the \textit{expected magnitude-dependent discounted utility} (EMDU) model, and it is given by

\[
V(x, p, t) = pe^{-r_xt}v(x),
\]

for some non-increasing \( r_x \). Under EMDU, time is exponentially discounted, but the rate may be outcome dependent. In this case, subendurance and substationarity become equivalent.

**Proposition 6** Under A1-A4 and proportionality,

\textit{Subendurance} \iff \textit{Substationarity} \iff \textit{EMDU}.

To see that subendurance induces substationarity, suppose we are indifferent to a smaller sooner reward, \((x, 1, 0)\), and a larger later reward, \((y, 1, t), y > x > 0, t > 0\). If \( r_x > r_y \) and we add a common delay \( \Delta \), then \( e^{-r_x\Delta} < e^{-r_y\Delta} \) so that \( x \) is discounted more than \( y \). Therefore, the prospect \((y, 1, t + \Delta)\) becomes more attractive than \((x, 1, \Delta)\), producing substationarity.

It is easy to show that isoendurance and proportionality necessarily imply stationarity, yielding the \textit{expected discounted utility} (EDU) model:

\[
V(x, p, t) = pe^{-r_0t}v(x).
\]

In fact, stationarity is a strong property that cannot coexist with strict subendurance or with strict subproportionality.
Proposition 7 Under A1-A4,

Stationarity ↔ Isoendurance and Proportionality ↔ EDU.

5 Time Discounting and Attitudes Towards Risk

Time and risk preferences are often described using relationships between the observable values of $x$, $p$ and $t$. For example, this is how one defines the discount rate and the risk premium. In this section, we assume that $v$, $d$ and $r$ can be differentiated, and we explore how the shape of $v$, $d$, and $r$ affect the properties of the discount rate and the risk premium.

For future reference, let $\alpha(x) = xv'(x)/v(x) > 0$ be the elasticity of $v(x)$, and constant elasticity corresponds to $v(x) = x^\alpha$, $\alpha > 0$.

5.1 Decreasing Impatience and the Magnitude Effect

Given any delayed prospect $(y, p, t) \in \mathcal{M}^+, t > 0$, let $x \in (0, y)$ be the smaller immediate reward with the same probability such that $(x, p, 0) \sim (y, p, t)$. The (continuous time) discount rate, $\rho$, is defined as $x = ye^{-\rho t}$, or

$$\rho = \frac{1}{t} \ln \frac{y}{x}. \quad (10)$$

The discount rate is a measure of impatience. In the EDU model, the discount rate is constant and independent of $t$ and $y$. Empirically, discount rates decrease as $t$ increases, and decrease as $y$ increases. These properties are known as decreasing impatience and the magnitude effect, respectively. The evidence for decreasing impatience and the magnitude effect is strong and abundant (Thaler, 1981; Frederick et al., 2002). Formally, we define decreasing impatience as $\partial \rho / \partial t \leq 0$, and the magnitude effect as $\partial \rho / \partial y \leq 0$. Increasing impatience and the reverse magnitude effect are defined as $\partial \rho / \partial t \geq 0$ and $\partial \rho / \partial y \leq 0$, respectively.

If $v$ takes the power form, then decreasing impatience and the magnitude effect are driven
independently by a concave $d$ and a decreasing $r$, respectively.\footnote{In this case, the discount rate can be explicitly calculated as
\[ \rho = \frac{d(\ln 1/p + r\alpha t) - d(\ln 1/p)}{\alpha t}. \] (11)
Hence, if $d$ is linear and $r$ is decreasing, then we have constant impatience and the magnitude effect. By contrast, if $d$ is concave and $r$ is constant, then we have decreasing impatience and no magnitude effect.}

**Proposition 8** Under A1-A4, if $v$ exhibits constant elasticity, then

a) **Subproportionality** $\leftrightarrow$ Decreasing impatience.

b) **Subendurance** $\leftrightarrow$ The magnitude effect.

There is recent empirical evidence supporting the correlation between subproportionality and decreasing impatience. Epper et al. (2009) elicited the weighting function and the discount function, and find that an increase in the subproportionality index by 0.1, the $\gamma$ parameter in (6), is associated with the discount factor decreasing by 2\% each year.

If the elasticity of $v$ is not constant, then decreasing impatience and the magnitude effect become *negatively correlated*. If the elasticity of $v$ is decreasing, then decreasing impatience is reinforced and the magnitude effect is weakened; and if the elasticity of $v$ is increasing, then decreasing impatience is weakened and the magnitude effect reinforced.

For instance, consider the expo-power form (Holt and Laury, 2002),

$$ v(x) = \frac{1 - e^{-cx^b}}{c}, b > 0, c \neq 0. $$

The elasticity, $\alpha(x) = bcx^b/(e^{cx^b} - 1)$, is decreasing. If $c = 0$, the form reduces to $v(x) = x^b$. Decreasing elasticity reinforces decreasing impatience, to the point that decreasing impatience holds even if $d$ is linear. In contrast, decreasing elasticity weakens the magnitude effect, to the point that the reverse magnitude effect holds if $r$ is constant. It is only when $r$ is sufficiently decreasing and compensates for the effect of decreasing elasticity that the magnitude effect shows. Formally,
**Proposition 9** Under A1-A4, if \(v\) exhibits non-increasing elasticity, then

\[a)\] Subproportionality \(\rightarrow\) Decreasing impatience.

\[b)\] Isoendurance \(\rightarrow\) The reversed magnitude effect.

And if \(v\) exhibits non-decreasing elasticity, then

\[c)\] Proportionality \(\rightarrow\) Increasing impatience.

\[d)\] Subendurance \(\rightarrow\) The magnitude effect.

If \(d\) is linear and \(r\) is constant (EDU model), then the magnitude effect and its reversed form are equivalent to non-decreasing and non-increasing elasticity of \(v\), respectively. This fact, coupled with Proposition 9, produces the following result.

**Proposition 10** Under A1-A4 and stationarity,

\[a)\] The magnitude effect \(\rightarrow\) Increasing impatience.

\[b)\] The reverse magnitude effect \(\rightarrow\) Decreasing impatience.

Hence, stationarity, decreasing impatience and the magnitude effect are incompatible (Noor, 2009). In the past, the magnitude effect has presented difficulties to bring risk and time preferences together in a unified framework, especially if one wants to include decreasing impatience. Noor (2010) uses a calibration argument rejecting a curvature based explanation for the magnitude effect. According to Green and Myerson (2004, p.788), the magnitude effect constitutes the “most striking evidence to discard that the same decision making process underlies risk and time decision”. Many propose that the value function for risk and time are different.

Our proposal is that the value function for risk and for time can be the same. A decreasing probabilistic discount rate is the innovation that elegantly accommodates the magnitude effect and decreasing impatience together. Our theoretical argument is consistent with the evidence provided by Abdellaoui et al. (2010), who estimated the value function using delayed simple prospects, finding that \(v\) is stationary whereas \(w\) depends on the delay.
5.2 Risk Premium

A simple way to observe attitudes towards risk is to compare the expected value of a prospect to its certainty equivalent, the difference representing the risk premium. Formally, for any risky prospect \((y, p, t) \in \mathcal{M}^+\), let \(x \in (0, y)\) be the sure smaller reward with the same delay such that \((x, 1, t) \sim (y, p, t)\). The risk premium, \(\pi\), is defined as

\[
\pi = yp - x. \tag{12}
\]

How does the risk premium vary with \(p\)? For the comparison between prospects of different probabilities to be fair, we fix \(py\) to some constant \(\mu\), and we then explore how the risk premium, \(\pi\), changes as a function of \(a\), the risk distance. We increase \(a\) from zero to infinity, while \(p = e^{-a}\) decreases and \(y = \mu e^a\) increases. Formally, \(\pi(a)\) solves

\[
(\mu - \pi(a), 1, t) \sim (\mu e^a, e^{-a}, t).
\]

Clearly, \(\pi(0) = 0\), and the sign of \(\pi'(0)\) indicates the risk attitude for \(p\) close to one.

**Proposition 11** Under A1-A4, if

\[
-r' \mu t < (>) 1 - \frac{\alpha(\mu)}{d'(r, t)},
\]

then for probabilities close to one the risk premium will be positive (negative).

If \(t\) is not too large, \(r\) does not decrease too fast, or \(d'(0)\) is not too small, then we expect to observe risk aversion for probabilities close to one. This is what is observed in experiments with no delay. Risk aversion is reasonable, because if \(a' > a\), then \((\mu e^{a'}, e^{-a'}, t)\) is a mean preserving spread of \((\mu e^a, e^{-a}, t)\).

In the specifications given in (5), \(r'\) can take large negative values for \(x\) small. Thus, if \(t\) is sufficiently large and/or \(r\) decreases sufficiently, then subjects will in fact become risk seeking for probabilities close to one. In fact, the preference reversal 1-3 in Table 1 provides evidence for this. If \(-r'_\mu\) becomes large for small \(\mu\), then a small delay, perhaps the minutes it
takes between making a choice and receiving the reward, may already be sufficient to induce risk seeking preferences. In fact, this behavior has been observed and coined the *peanuts effect* (Hershey and Schoemaker, 1980). Subendurance provides a parsimonious explanation for the *peanuts effect*, avoiding the the artificial modification of the value function near the origin (Weber and Chapman, 2005, p. 45). A symmetric effect would explain risk aversion for tiny losses.

What is the behavior of the risk premium as $a$ increases / $p$ decreases? In experiments with no delay, most subjects take risks for prospects of small probability. Risk aversion for moderate to high probabilities, together with risk seeking preferences for small probabilities is known as the four-fold pattern, as applied to gains (Kahneman and Tversky, 1979; Bruhin et al., 2010). Thus, $\pi(a)$ increases first but eventually starts decreasing, even to the point where it takes negative values. The concavity of $d$ is necessary, but not sufficient, to explain this pattern. Indeed, for the risk premium to be eventually decreasing, we need $d'$ to be smaller than $\alpha$.

**Proposition 12** Under A1-A4 and subendurance, if

$$\liminf_{a \to \infty} \frac{d'(a + r\mu e^a t)}{\alpha(\mu e^a)} > 1 \quad (\limsup_{a \to \infty} \frac{d'(a + r\mu e^a t)}{\alpha(\mu e^a)} < 1),$$

then the risk premium is eventually increasing (decreasing).

The four-fold pattern is one of the many patterns that can occur under the PTT model. To demonstrate this, observe the lines A, B, C and D in Figure 1. The examples are calculated using experimentally plausible curvature parameters, and illustrate four general patterns for the risk premium. For instance, the elasticity of $v$ is assumed to be constant or decreasing, and less than one in all cases.

**A.** Any elasticity, proportionality and isoendurance (EDU model). The risk premium is not affected by $t$, and it increases, if and only if, $\alpha$ is less than one. Increasing risk premium is the rational pattern when facing a mean preserving spread in risk.
Figure 1: Risk premium as a function of \( a \). \( \nu \) is expo-power, \( d \) is power, and \( r(y) = r_0 + r_1/y \). We set \( \mu = 10 \) and \( t = 2 \) months. Line A: \( b = 0.8, c = 0; \gamma = 1; r_0 = 0.02, \) and \( r_1 = 0 \). Line B: same as A but with \( r_1 = 2 \). Line C: \( b = 0.8, c = 0.05; \gamma = 0.7; r_0 = 0.02, \) and \( r_1 = 0 \). Line D: same as C, but with \( c = 0 \).

B. Any elasticity, proportionality and subendurance (EMDU model). Because of subendurance, the risk premium decreases for small \( a \) (Proposition 11) but because \( d' = 1 > \alpha \), the risk premium eventually increases (Proposition 12). Thus, the opposite of the four-fold pattern holds.

C. Decreasing elasticity, subproportionality and isoendurance. Any pattern is possible because the ratio \( d'(a + r_{\mu e t})/\alpha(\mu e^a) \) could cross 1 as many times as is desired. In the example, the ratio begins above 1, before dropping below it and finally, it tends to \( \infty \). Accordingly, the risk premium first increases, it then decreases and eventually, it increases.

D. Constant elasticity, subproportionality and isoendurance. Because \( \alpha \) is constant, if \( d'(0) \) is sufficiently large, then the risk premium will be positive for \( a \) close to zero. As \( a \) increases, if \( d' \) tends to zero, then \( d'(a)/\alpha(\mu e^a) \to 0 \). The risk premium eventually decreases, yielding the four-fold pattern for gains.
5.3 Cross Effects

The discount rate may depend on the probability. Reversal 4-6 in Table 1 provides evidence that adding risk to both the sooner smaller reward and the larger later rewards produces a shift of preference towards the larger later reward.

Proposition 13 Under A1-A4 and subproportionality, $\partial \rho / \partial p \geq 0$. Moreover, if $d$ is strictly concave, then the discount rate strictly increases with $p$.

The risk premium may depend on the delay. Noussair and Wu (2006) adopts the EU framework and concludes that “a substantial fraction of subjects exhibit a greater level of risk aversion for lotteries resolved and paid in the present rather than in the future.” Allowing for probability weighting, Abdellaoui et al. (2010) found that when lotteries are played immediately, 77% of the subjects exhibit risk aversion. This percentage declines to 75% and 67% when the delay of lottery receipts is set to 6 and 12 months, respectively. They use both simple and two-outcome delayed prospects and Table 1, reversal 1-3 also shows that a common delay makes subjects shift preference towards the more risky prospect.

Proposition 14 Under A1-A6, $\partial \pi / \partial t \leq 0$. Moreover, if $d$ is strictly concave or $r$ strictly decreasing, then the risk premium strictly decreases with $t$.

Propositions 13 and 14 have dynamic implications. For a prospect having a risky reward on a fixed calendar date, the passage of time makes $t$ decrease. Thus, as the prospects come closer in time, the decision maker will become more risk averse. Thus, a manager may approve of a risky project to be executed in the far future, but cancel the project as it gets closer in time. Similarly, due to the partial resolution of uncertainty, if the probabilities of success increase, then the manager may apply a higher discount rate.

6 Parameter-free Elicitation of $r$, $v$ and $d$

Suppose we wish to elicit $r_x$, $x > 0$ ($r(0)$ can be derived using continuity), then choose some arbitrary $p$, $t$ such that $(x, p, t) \in M^+$ and $\theta \in (0, 1)$. Matching or bisection should be used
to find $\Delta$ such that $(x, p\theta, t) \sim (x, p, t + \Delta)$. The PTT evaluation yields

$$r_x = \frac{1}{\Delta} \ln \frac{1}{\theta}.$$  

To better estimate $r_x$, one may want to use several values of $p$ and $t$, and average the results.

The method to elicit $v$ and $d$ resembles the trade-off method used in rank dependent models (Abdellaoui, 2000). Choose a high outcome $x_1 > 0$ and $\theta \in (0, 1)$, and because of the uniqueness property of the representation, $d(\ln 1/\theta)$ and $v(x_1)$ can be set at any desirable positive number. Matching or bisection are used to find $x_2 < x_1$ such that $(x_2, 1, 0) \sim (x_1, \theta, 0)$. Iteratively, a decreasing standard sequence should be found $x_k < x_{k-1} < \ldots < x_2 < x_1$ such that $(x_k, 1, 0) \sim (x_{k-1}, \theta, 0), k = 2, ..., K$. The PTT evaluation yields

$$v(x_k) = \beta v(x_{k-1}), \text{ or}$$

$$v(x_k) = \beta^{k-1} v(x_1), k = 2, ..., K,$$

where $\beta = e^{-d(\ln 1/\theta)}$. Finally, we use the standard sequence to elicit $d$. For $k = 2, ..., K$, use matching or bisection to find $p_k$ such that $(x_k, 1, 0) \sim (x_1, p_k, 0), k = 2, ..., K$. The PTT evaluation yields

$$d(\ln 1/p_k) = (k - 1)d(\ln 1/\theta), \ k = 2, ..., K.$$

By setting $\theta$ closer to one, we have a smaller step size, values of $\beta$ closer to one, and we obtained a finer grid for both $v$ and $d$. This comes at the expense of having to use more steps $K$ to approximately cover the same range of outcomes and risk distances.

7 Conclusions

We have proposed a preference model that is parsimonious, and that has prospect theory and hyperbolic discounting as special cases. The model is more than a natural extension of prospect theory to temporal lotteries. Besides exhibiting the properties of rank dependent utility and decreasing impatience, the model is compatible with the magnitude effect and other experimental evidence in the domain of risk and time. The magnitude effect,
although strongly documented, has not been readily accepted as part of standard models. We attribute this to the inability of existing models to square the magnitude effect with decreasing impatience. In the PTT model, the common ratio effect, decreasing impatience and the magnitude effect can consistently coexist.

Andersen et al. (2008) show that to measure time preferences accurately one needs to account for the utility/value function $v$. We go one step further, and propose that risk and time preferences follow from the combined effect of the functions $v$, $d$ and $r$. Because the discount rate is a function of $v$, $d$ and $r$, one would expect the direct measurement of discount rates to be very unstable, as is indeed the case (Frederick et al., 2002). We propose a method to elicit $v$, $d$ and $r$ directly. Results from Baucells et al. (2009) suggest that $r$ decreases with age ($r$ is multiplied by a factor of 0.95 each year that passes) and wealth (doubling your wealth/income reduces $r$ by a factor of 0.9). Thus, it would be interesting to replicate these results and to discover other factors that may affect the trade-off between probability and time (Killeen, 2009).

The PTT model predicts that risk and time preferences are correlated via the psychological distance function $d$. Some recent evidence supports this prediction (Epper et al., 2009). Additional evidence is provided by comparing risk and time preferences in separated domains. The weighting function for losses seems to be more elevated and closer to linear than the weighting function for gains. In the time domain, subject discount losses less than gains, and discounting for losses is more exponential than for gains (Tversky and Kahneman, 1986).

In prescriptive applications, one may be interested on how risk preferences can be manipulated using time, or time preferences modified by introducing risk (e.g., in the energy and climate change debate, in marketing design, or in the design of financial products). The PTT model clearly implies that by manipulating time and risk one can influence the discount rate and the risk premium. In particular, adding delay unambiguously reduces risk aversion, as documented by Abdellaoui et al. (2010).
The extension of the PTT model to domains with multiple risky outcomes and/or multiple rewards over time is an open and non-trivial problem. A more approachable problem is to distinguish between the time at which uncertainty is resolved and the time one receives the outcome (Wu, 1999), or to consider that \( x \) is a multi-dimensional object, such as a consumption bundle or a multi-attribute consequence.

In summary, the PTT model offers a unified view of risk and time preference, setting a promising framework to build theory and to unify a large body of empirical evidence, as well as providing experimentalists with new sets of testable propositions.

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**References**


**Proofs**

**Proof of Proposition 1.** If $V(x, p, t) = pv(x, t)$, then Choice 1 implies $v(9, 0) - 80\%v(12, 0) > 0$. Thus, $10\%[v(9, 0) - 80\%v(12, 0)] > 0$, contradicting Choice 2. If $V(x, p, t) = e^{-rt}v(x, p)$, then Choice 4 implies $v(100, 1) - e^{-r4}v(110, 1) > 0$. Thus, $e^{-r26}[v(100, 1) - e^{-r4}v(110, 1)] > 0$, contradicting Choice 5. It follows that $V(x, p, t) = pe^{-rt}v(x)$ contradicts both 1-2 and 4-5.

If $V(x, p, t) = f(t)v(x, p)$, then Choice 1 implies $f(0)[v(9, 100\%) - v(12, 80\%)] > 0$. By monotonicity, $f(t) > 0$, $t \in [0, \infty)$. It follows that $v(9, 100\%) - v(12, 80\%) > 0$ and $f(3m)[v(9, 100\%) - v(12, 80\%)] > 0$, contradicting Choice 3. If $V(x, p, t) = w(p)v(x, t)$, then Choice 4 implies $w(1)[v(100, 0) - v(110, 4w)] > 0$. By monotonicity, $w(p) > 0$, $p \in (0, 1]$. It follows that $v(100, 0) - v(110, 4w) > 0$ and $w(50\%)[v(100, 0) - v(110, 4w)] > 0$, contradicting Choice 6. It follows that $V(x, p, t) = w(p)f(t)v(x)$ contradicts both 1-3 and 4-6.
If $V(x, p, t) = w(p, t)v(x)$, then Choice 7 implies $v(5)[g(90\%, 0) - g(100\%, 1m)] > 0$. By monotonicity, $v(x) > 0$, $x > 0$. It follows that $g(90\%, 0) - g(100\%, 1m) > 0$ and $v(100)[g(90\%, 0) - g(100\%, 1m)] > 0$, contradicting Choice 8. ■

Proof of Proposition 2. Let $V(x, p, t)$ satisfy the continuity and monotonicity conditions given by A1-A2. First, we claim that A1-A3 hold if and only if

$$V(x, p, t) = V(x, pe^{-rt}, 0),$$

(13) for some continuous function $r_x > 0$ such that $V(x, pe^{-rt}, 0)$ is increasing in $x$.

(13) $\rightarrow$ A1-A3: If $V(x, p, t)$ satisfies A1-A2, then so does $V(x, pe^{-rt}, 0)$. To see that (13) implies A3, let $(x, p, t), (x, q, s) \in \mathcal{M}^+$, $\Delta > 0$, and $\theta \in (0, 1)$. If $(x, p, t+\Delta) \sim (x, p\theta, t)$, then $V(x, pe^{-r_x(t+\Delta)}, 0) = V(x, p\theta e^{-r_xt}, 0)$. By continuity and monotonicity, $pe^{-r_x(t+\Delta)} = p\theta e^{-r_xt}$ and $e^{-r_xt} = \theta$. It follows that $q e^{-r_x(s+\Delta)} = q\theta e^{-r_xs}$ and

$$V(x, q, s + \Delta) = V(x, q e^{-r_x(s+\Delta)}, 0) = V(x, q\theta e^{-r_xs}, 0) = V(x, q\theta, s).$$

A1-A3 $\rightarrow$ (13): If $t = 0$ or $(x, p, t) \in \mathcal{M}^0$, then the results holds trivially. Let $(x, p_0, t_0) \in \mathcal{M}^+, t_0 > 0$ and $\Delta_0 \in (0, t_0]$. By monotonicity, $(x, p_0, t_0 - \Delta_0) \succ (x, p_0, \Delta_0) \succ (x, 0, 0)$. By continuity and monotonicity, there is a unique $\theta_0 \in (0, 1)$ such that $(x, \theta_0 p_0, t_0 - \Delta_0) \sim (x, p_0, t_0)$. Define $r_0 = \frac{1}{\Delta_0} \ln \frac{1}{\theta_0}$. In principle, $r_0$ may depend on $x$, $p_0$ and $t_0$. We will show that $r_0$ is independent of $p_0$ and $t_0$. Consider the change of variable $z = e^{-rt}$. Noting that $e^{-r_0(t-\Delta_0)} = \theta_0^{-1}z_0$, we can write $(x, \theta_0 p_0, t_0 - \Delta_0) \sim (x, p_0, t_0)$ as $(x, \theta_0 p_0, \theta_0^{-1}z_0) \sim (x, p_0, z_0)$. By A3, this indifference is independent of $p_0$ and $z_0$, i.e.,

$$\nabla(x, p, z) = \nabla(x, \theta_0 p, \theta_0^{-1}z), \text{ for all } p \in (0, 1], z \in (0, \theta_0].$$

(14) Suppose that we have chosen $\Delta = \Delta_0/2$ and let $\theta$ be such that $(x, p\theta_0, t_0 - \Delta) \sim (x, p_0, t_0)$. By PTT, we have that this indifference holds for $p_1 = p\theta_0$ and $t_1 = t_0 - \Delta$ so that $(x, \theta^2 p_0, t_0 - \Delta_0) \sim (x, p_0, t_0)$. By transitivity, $(x, \theta^2 p_0, t_0 - \Delta_0) \sim (x, \theta_0 p_0, t_0 - \Delta_0)$ and $\theta = \theta_0^{1/2}$. The same argument can be repeated using $\Delta = \Delta_0/n$, to show that $\theta = \theta_0^{1/n}$. Hence, $\theta$ and $z$ can chosen as close to one as desired. Choose any such $p$ and $z$. Letting $\theta' \in (\theta z, \theta/p]$, use $p' = \theta' p/\theta$ and $z' = \theta z/\theta'$ to verify that (14) holds for all values of $\theta$ such that $(x, p, t), (x, \theta p, \theta^{-1}z) \in \mathcal{M}^+$. For fixed $x$, $\nabla$ is a generalized homogeneous functional equation with $c = 0$, $c_1 = -1$ and $c_2 = 1$ (Aczél, 1989, Proposition 2, p. 346). The solution is given by $\nabla(x, p, z) = H(x, pz)$. Observing that $H(x, pz) = \nabla(x, pz, 1)$, using $V(x, p, t) = \nabla(x, p, e^{-rt})$, and noting that $r$ may depend on $x$, (13) follows.

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This proves the claim. Using A4, Fishburn (1970, Theorem 5.4) implies that on \((x, p, 0) \in \mathcal{M}^+\), \(V(x, p, 0) = w(p)v(x)\), for some continuous functions \(v(x)\) and \(w(p)\), having \(v(0) = w(0) = 0\). Moreover, for any strictly positive numbers \(\kappa, \kappa', \eta\), the functions \(\kappa \> \eta\) and \(\kappa' \> \eta\) also represent \(\geq\) in the positive domain. The normalization \(w(1) = 1\) fixes \(\kappa'\). Finally, if \(V(x, p, 0) = w(p)v(x)\), then \(V(x, p, t) = w(pe^{x+t})v(x)\).

**Proof of Proposition 3.** a) Given \((x, p, t) \in \mathcal{M}^+, t > 0\) and \(s < t\), use A1-A2 to find \(q \in (0, p)\) such that \((x, p, t) \sim (x, q, s)\). By PTT, \(v(x)e^{-d(r_x,t+\ln 1/p)} = v(x)e^{-d(r_x,s+\ln 1/q)}\) and, because \(d\) is strictly increasing, \(\ln p/q = r_x(t-s)\). Choose any \(y > x\). Subendurance hold iff \((y, q, s) \equiv (y, p, t)\), which is equivalent to \(v(y)e^{-d(r_y,s+\ln 1/q)} \leq v(y)e^{-d(r_y,t+\ln 1/p)}\), or

\[
\text{Subendurance} \iff r_y(t-s) \leq \ln \frac{p}{q} = r_x(t-s) [r \text{ non-increasing}].
\]

b) Under isoendurance, the inequality becomes an equality \([r \text{ constant}]\).

**Proof of Proposition 4.** Given \((x, p, t) \in \mathcal{M}^+\) and \(y > x\), use A1-A2 to find \(\theta \in (0, 1)\) such that \((x, p, t) \sim (y, p\theta, t)\). Define \(a = \ln 1/p + r_x t\) and \(\Delta = \ln 1/\theta\). By PTT, \(v(x)/v(y) = e^{-d(a+\Delta)-d(a)}\). Choose any \(q \in (0, p)\) so that \(a' = \ln 1/q + r_x t > a\).

a) Subproportionality holds iff \((x, q, t) \equiv (y, q\theta, t)\), which is equivalent to \(e^{-d(a'+\Delta)-d(a')} \geq v(x)/v(y) = e^{-d(a+\Delta)-d(a)}\), or

\[
\text{Subproportionality} \iff d(a+\Delta) - d(a) \geq d(a'+\Delta) - d(a') \quad [d \text{ concave}].
\]

b) Under proportionality, then inequality becomes an equality, leading to \(d(a+\Delta)-d(a) = d(a'+\Delta) - d(a')\). Because \(d(0) = 0\), if \(a' = 0\), then \(d(a+\Delta) = d(a) + d(\Delta)\). This is the Cauchy equation whose unique solution is \(d\) linear (Aczél, 1989, p.17).

**Proof of Proposition 5.** Given \((x, p, t) \in \mathcal{M}^+\) and \(y > x\), use A1-A2 to find \(\Delta' > 0\) such that \((x, p, t) \sim (y, p, t + \Delta')\). Define \(a = \ln 1/p + r_x t\) and \(\Delta = r_y \Delta' - (r_x - r_y) t\) so that \(a + \Delta = \ln 1/p + r_y (t + \Delta')\). By PTT, \(v(x)/v(y) = e^{-d(a+\Delta)-d(a)}\), showing that \(\Delta > 0\). Choose \(s > t\) so that \(a' = \ln 1/p + r_x s > a\). Let

\[
\delta = (s-t)(r_x-r_y)
\]

so that \(d(a' + \delta) - d(\delta) = \ln 1/p + r_y (s + \Delta')\). Substationarity holds iff \((x, p, s) \equiv (y, p, s + \Delta')\), which is equivalent to \(e^{-d(a'+\Delta-\delta)-d(a')} \geq v(x)/v(y) = e^{-d(a+\Delta)-d(a)}\), or

\[
\text{Substationarity} \iff d(a+\Delta) - d(a) \geq d(a'+\Delta - \delta) - d(a').
\]

Assume subendurance \([\delta \geq 0]\). If \(d\) is concave, then \(d(a+\Delta) - d(a) \geq d(a'+\Delta) - d(a') \geq d(a'+\Delta - \delta) - d(a')\) [substationarity]. For the converse, assume that \(d\) is not concave: for
some $a' > a$, $d(a + \Delta) - d(a) = d(a' + \Delta) - d(a') - \epsilon$. If $r_x$ is non-increasing and positive, then $r_x$ has a limit as $x$ increases. We will increase $x$, keep $a$, $a'$ and $\Delta$ unchanged, and find $p$, $y$ and $s$ by solving $a = \ln 1/p + r_xt$, $v(y) = v(x)e^{[d(a+\Delta) - d(a)]}$, and $a' = \ln 1/p + r_x s$, respectively. For $x$ sufficiently large, $r_x$ can be made arbitrarily close to $r_y$ and $\delta$ arbitrarily close to zero. Because $\delta \to 0$, we have that

$$d(a' + \Delta) - d(a') - \epsilon \geq d(a' + \Delta - \delta) - d(a') \to d(a' + \Delta) - d(a'),$$

yielding a contradiction. Hence, $d$ is necessarily concave. ■

**Proof of Proposition 6.** If $d$ is linear, then substationarity as given in (15) becomes $\delta \geq 0$, which is equivalent to subendurance. ■

**Proof of Proposition 7.** In view of (15), it is immediate to see that stationarity is equivalent to

$$d(a + \Delta) - d(a) = d(a' + \Delta - \delta) - d(a').$$

(16)

Under isoendurance [$\delta = 0$] and proportionality [d linear], stationarity holds trivially.

For the converse, let $y > x > 0$. Suppose for some $x, y, r_x > r_y$. In (16), note that $\Delta > 0$ is independent of $s$, and we can adjust $s$ so that $\delta = (s - t)(r_x - r_y) = \Delta$. By (16), $\delta = \Delta$ implies $\Delta = 0$, a contradiction. Hence, $r_x \leq r_y$ and $\delta \leq 0$.

Suppose for some $x, y, r_x < r_y$. If for some $a' > a$ and $\Delta > 0$, $d(a + \Delta) - d(a) > d(a' + \Delta) - d(a')$, then (16) implies that $\delta > 0$, a contradiction. Hence, if $r_x < r_y$, then $d$ is linear of convex. By monotonicity,

$$V(y, p, t) - V(x, p, t) = e^{-d(a + (r_y - r_x)t)}v(y) - e^{-d(a)}v(x) > 0, a = \ln 1/p + r_xt.$$

By the properties of $r$ and $d$, $d(a + (r_y - r_x)t) - d(a) \to \infty$ as $t \to \infty$, contradicting monotonicity. Hence $r$ is necessarily constant [isoendurance], in which case (16) becomes equivalent to proportionality, as shown in the proof of Proposition 4b. ■

**Proof of Proposition 8.** Using $v(x) = x^a$ and solving for $V(ye^{-pt}, p, t) = V(y, p, 0)$, one obtains (11). As for a) and b), we use results from the proof of Proposition 9.

a) If $\alpha(x) = \alpha(y)$, then (18) becomes an equality, and $\partial \rho / \partial t \leq 0$ iff $d$ is concave.

b) If $\alpha(x) = \alpha(y)$, then $\partial \rho / \partial y \leq 0$ iff $r' _y \leq 0$. ■

**Proof of Proposition 9.** Using $V(y, p, t) = V(x, p, 0)$, the discount rate $\rho$ is given by the implicit function:

$$v(ye^{-pt}) - e^{-[d(ln 1/p + r_xt) - d(ln 1/p)]}v(y) = 0.$$  

(17)
a) If $\alpha$ is non-increasing, then

$$d(\ln 1/p + r_y t) - d(\ln 1/p) = \ln \frac{v(y)}{v(x)} = \int_x^y \frac{v'(t)}{v(t)} dt = \int_x^y \frac{\alpha(z)}{z} dz \leq \alpha(x) \ln \frac{y}{x}. \tag{18}$$

Apply the implicit function theorem to (17), replace $x = ye^{-\mu t}$ and $v(x) = v(y)e^{-[d(\ln 1/p + r_y t) - d(\ln 1/p)]}$, use (18), and the concavity of $d$, to obtain decreasing impatience.

$$\frac{\partial \rho}{\partial t} = -\frac{d'(\ln 1/p + r_y t)r_y v(x) - v'(x)x \rho}{-v'(x)tx} = \frac{d'(\ln 1/p + r_y t)r_y t - \alpha(x) \ln(y/x)}{\alpha(x)t^2} \leq \frac{d'(\ln 1/p + r_y t)[d(\ln 1/p + r_y t) - d(\ln 1/p)]}{\alpha(x)t^2} \leq 0.$$

c) If $d$ is linear, then it follows from (18) that $r_y t = \int_x^y \frac{\alpha(z)}{z} dz$. If $\alpha$ is non-decreasing, then $\alpha(x) \ln(y/x) - r_y t = \int_x^y \frac{\alpha(x)-\alpha(z)}{z} dz \leq 0$, which implies that $\partial \rho/\partial t \geq 0$.

d) If $\alpha$ is non-decreasing, then

$$\frac{v'(x)y - v'(y)v(x)/v(y)}{v'(x)tx} = \frac{1}{ty} - \frac{v'(y)v(x)}{v'(x)tx} = \frac{1}{ty} - \frac{\alpha(y)}{\alpha(x)ty} = \frac{1}{ty} \left[1 - \frac{\alpha(y)}{\alpha(x)}\right] \leq 0.$$

Apply the implicit function theorem to (17) and use $r'_y \leq 0$ to obtain

$$\frac{\partial \rho}{\partial y} = \frac{-v'(x)x/y + d'(\ln 1/p + r_y t)r'_y tv(x) - v'(y)v(x)/v(y)}{-v'(x)tx} = \frac{d'(\ln 1/p + r_y t)r'_y}{\alpha(x)} + \frac{1}{ty} \left[1 - \frac{\alpha(y)}{\alpha(x)}\right] \leq 0.$$

b) If $r'_y = 0$, then $\partial \rho/\partial y \leq 0$ if $\alpha$ is non-decreasing. ■

**Proof of Proposition 10.** Noting that proportionality is a particular case of sub-proportionality, by Proposition 9a, non-increasing $\alpha$ implies decreasing impatience. If $d$ is linear, Proposition 9c shows that non-decreasing $\alpha$ implies increasing impatience. If $d$ is linear and $r'_y = 0$, then $\partial \rho/\partial y \leq 0$ holds if $\alpha(x) \leq \alpha(y)$. Thus, the magnitude effect implies non-decreasing $\alpha$, which implies increasing impatience; and the reverse magnitude effect implies non-increasing $\alpha$, which implies decreasing impatience. ■

**Proof of Proposition 11.** From the proof of Proposition 12 it follows that the sign of $\pi'(0)$ is given by the sign of $d'(r_\mu t)(1 + r'_\mu \mu t) - \alpha_\mu$. ■

**Proof of Proposition 12.** Let $p = e^{-\alpha}$ and $y = \mu e^\alpha$. $\pi(a)$ is implicitly defined as

$$v(\mu - \pi) - e^{-[d(\alpha+r_\mu e^a)-d(r_\mu e^a)]}v(\mu e^a) = 0. \tag{19}$$
Applying the implicit function theorem, using $\mu - \pi = x$ and $e^{-[d(a + r_{t\mu}t) - d(r_{t\mu}t)]}v(\mu e^a) = v(x)$, yields

$$\pi'(a) = \frac{d'(a + r_{t\mu}t)(1 + r'_{t\mu}e^a) - \alpha(\mu e^a)}{v'(x)/v(x) - d'(r_{t\mu}t)r'_{t\mu}t}. $$

Because $v$ is increasing and $r$ is non-increasing the denominator is positive. Because $r$ is positive and non-increasing, $r'$ tends to zero as $y$ increases. If $\liminf_{a \to \infty} d'(a)/\alpha(\mu e^a) > 1$, then the numerator will tend to a positive number, and $\pi$ will eventually increase. The opposite will occur if $\limsup_{a \to \infty} d'(a)/\alpha(\mu e^a) < 1$. ■

**Proof of Proposition 13.** Applying the implicit function theorem to (17) and using $d$ concave yields

$$\frac{\partial \rho}{\partial p} = \frac{d'(\ln 1/p + r_{t\mu}t) - d'(\ln 1/p + r_{t\mu}t)\alpha(x)pt}{\alpha(x)pt} \geq 0.$$  

The inequality is strict if $d$ is strictly concave. ■

**Proof of Proposition 14.** Applying the implicit function theorem to (19) yields

$$\frac{\partial \pi}{\partial t} = \frac{d'(\ln 1/p + r_{t\mu}t)r_y - d'(r_{t\mu}t)r_x}{v'(x)/v(x) - d'(r_{t\mu}t)r'_{t\mu}t}. $$

Because $v$ is increasing and $r$ is non-increasing the denominator is positive. Because $v(y) > v(x)$, it follows from (19) that $a = \ln 1/p + r_{t\mu}t > r_{t\mu}t$. If $d$ is increasing and concave, then $d'(r_{t\mu}t) \geq d'(a) > 0$; and if $r$ is non-increasing and positive, then $r_x \geq r_y > 0$. Hence, $d'(r_{t\mu}t)r_x \geq d'(a)r_y$ and $\partial \pi/\partial t \leq 0$. The inequality is strict if either $d$ is strictly concave or $r$ is strictly decreasing. ■