Responding to the Inflation Tax*

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`Very Preliminary: Please Do Not Circulate`

Abstract

This paper adopts a mechanism design approach to study the effects of anticipated inflation on individual spending patterns, trade, and welfare. We consider various channels through which individuals respond to the inflation tax: market participation (extensive margin), search intensity (intensive margin), and substitution between money and risk-free capital. Instead of assuming a particular pricing protocol, we adopt a trading mechanism designed to maximize society's welfare, taking as given the frictions in the economy. We find that increasing search intensity in response to higher inflation is constrained-optimal behavior. There are non-monotonic effects between search intensity and inflation, and we find that output and search efforts are substitutes for intermediate levels of the inflation while they are complements for sufficiently high levels of inflation. We also discuss our results in relation to the evidence and historical anecdotes on the effect of inflation on consumer spending patterns.

Keywords: inflation, capital, hot potato effect of inflation, mechanism design

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"The public discover that it is the holders of notes who suffer taxation [from inflation] ... and they begin to *change their habits and to economize in their holding of notes.* They can do this in various ways ... [T]hey can reduce the amount of till-money and pocket-money that they keep and the average length of time for which they keep it, even at the cost of great personal inconvenience ... By these means they can get along and do their business with an amount of notes having an aggregate real value substantially less than before. In Moscow the unwillingness to hold money except for the shortest possible time reached at one period a fantastic intensity. If a grocer sold a pound of cheese, he ran off with the roubles as fast as his legs could carry him to the Central Market to replenish his stocks by changing them into cheese again, lest they lost their value before he got there; thus justifying the provision of economists in naming the phenomenon velocity of circulation!"

John Maynard Keynes (1924), *A Tract for Monetary Reform*

1 Introduction

How do individuals respond to anticipated inflation? The conventional wisdom is that as inflation rises, individuals find ways to reduce their exposure to the inflation tax. For instance, one of the most salient characteristics of high inflations is that consumers try to minimize the time they carry money for transactions by speeding up their purchases in order to get rid of depreciating cash. As Irving Fisher put it, "when depreciation is anticipated, there is a tendency among owners of money to spend it speedily." Similarly, Nassau Senior (1830) reflected in his *Three Lectures on the Cost of Obtaining Money* that: "Everybody taxed his ingenuity to find employment for a currency of which the value evaporated from hour to hour. It was passed on as it was received, as if it burned everyone's hands who touched it." These actions are costly and are part of the social cost of inflation.[1]

In this paper, we adopt a mechanism design approach to a simple monetary model to study the effects of anticipated inflation on individual consumption and trading behaviors. As in Lagos and Wright (2005) and Rocheteau and Wright (2005), our model features alternating rounds of pairwise random trades where anonymity and information frictions generate a role for fiat money, and centralized trades where agents can rebalance their portfolios. However instead of assuming a particular pricing protocol, we adopt a trading mechanism similar to Hu, Kennan, and Wallace (2009) that is designed to maximize society's welfare, taking as given the frictions in the economy.

[1] As articulated by Lucas (2000), "In a monetary economy, it is in everyone's private interest to try to get someone else to hold non-interest-bearing cash and reserves. But someone has to hold it all, so all of these efforts must simply cancel out. All of us spend several hours per year in this effort, and we employ thousands of talented and highly-trained people to help us. These person-hours are simply thrown away, wasted on a task that should not have to be performed at all."

In contrast with previous studies, mechanism design permits us to identify the socially desirable features in monetary economies from ones that arise from inefficient trading mechanisms, including many commonly used pricing protocols in the literature.

We focus on three channels through which individuals can respond to the inflation tax: market participation, search intensity, and substitution between money and risk-free capital. Along the extensive margin, agents can choose whether to participate in the decentralized market by incurring a fixed entry fee, which pins down the equilibrium composition of the market. This entry decision imparts a congestion externality so that a decrease in the measure of buyers who enter also increases the trading probability of an individual buyer. Along the intensive margin, we allow agents to choose individual search efforts by varying the intensity in which they search. We then ask how changes in anticipated inflation affects optimal decisions along the extensive margin and the intensive margin, and the extent to which this is affected by the presence of an alternative means of payment such as capital.

Our main finding is that inflation can increase the intensity with which buyers search (the intensive margin) as well as total measure buyers that choose to trade in the decentralized market (the extensive margin). Since we have employed the optimal trading mechanism, we argue that these behavior pattern are constrained efficient. Indeed these qualitative results remain even when we introduce other channels to avoid the inflation tax such as substitution towards capital.

Whereas Lucas (2000) argues the hot potato effect is socially wasteful, we prove that increasing search efforts in response to rising inflation is second-best constrained-optimal behavior compared to a world where the planner is not allowed to adjust search intensity. When we investigate a special case of our model where only buyers can choose their search intensity, we find that quantities traded would fall by more than if search intensity were simply fixed. Thus in one sense, Lucas is correct: if inflation were sufficiently low, then the first-best is feasible. At the same time however, his intuition is inconsistent with a key aspect of our findings: agents would actually be worse off if they could not work harder to find a match. Lucas argues that the search efforts must “cancel out”– but that is not true according to our analysis, at least not in a world of search frictions. There are some agents who are not matched and so searching harder can be constrained-optimal behavior.

This paper proceeds as follows. Section 1.1 reviews the related literature. Section 2 describes the environment, and Section 3 describes implementation of the optimal mechanism. In Section 4, we analyze the effects of inflation along the extensive margin and compare allocations with and without capital. Section 5 considers the effect of inflation along the intensive. margin. We conclude in Section 6.

1.1 Literature Review

The effect of inflation on individual search efforts has been addressed previously in a similar class of models, namely by Li (1994, 1995, 1997), Lagos and Rocheteau (2005), Nosal (2008), Ennis (2009), Liu, Wang, and Wright (2011), and Dong and Jiang (2011). However, the consensus from this analysis is that the extent to which the hot potato effect is present varies depending on assumptions regarding the (in)divisibility of assets and the economy's pricing mechanism.[2]

- Shopping time models
  - McCallum and Goodfriend (1987)
  - Lucas (1994) and the welfare cost of inflation
- Search models and the hot potato effect of inflation
  - Li (1994, 1995, 1997)
  - Lagos and Rocheteau (2005)
  - Nosal (2008)
  - Ennis (2009)
  - Liu, Wang, and Wright (2011)
  - Dong and Jiang (2011)
- History and anecdote
  - Sargent's "Four Big Inflations" (Europe 1920s)
    * Austria 1922
    * Hungary 1923-24
    * Poland 1923-24
    * the Weimar Republic 1923
  - Latin American inflations
    * Brazil 1990s - O'Dougherty, Maureen. 2002. *Consumption Intensified: The Politics of Middle-Class Daily Life in Brazil.* Durham & London: Duke University Press.
    * Argentina 1975-1991
    * Ecuador
  - Zimbabwe 2008

[2]For instance, Lagos and Rocheteau (2005) show that inflation induces search efforts to fall in a model with divisible money and endogenous prices determined through bargaining. However, search efforts can rise with inflation under competitive search under certain parameterizations.

2 Environment

Time is discrete and continues forever. The economy is populated by a continuum of infinitely-lived agents, divided into a set of *buyers*, denoted by $\mathbb{B}$, and a set of *sellers*, denoted by $\mathbb{S}$. Each date has two stages: the first has pairwise meetings in an OTC market and the second has centralized meetings. The first stage will be referred to as the *DM* (decentralized market) while the second stage will be referred to as *CM* (centralized market). There is a single perishable good at each stage, with the CM good taken as the numéraire. In the CM, all agents have the ability to produce and wish to consume. Agents' labels as buyers and sellers depend on their roles in the DM: only sellers can produce and only buyers wish to consume in the DM.

The numéraire good can be transformed into a capital good one for one. Capital goods accumulated at the end of period t are used by sellers at the beginning of the CM of $t+1$ to produce the numéraire good according to the linear technology Ak. Capital goods depreciate fully after one period. The rental (or purchase) price of capital in terms of the numéraire good is R_t.

There is also an intrinsically useless, perfectly divisible and storable asset called money. Let M_t denote the quantity of money in the CM of period t. All agents in the CM are price-takers, and the relative price of money in terms of the numéraire, ϕ_t, adjusts to clear the market. The gross growth rate of the money supply is constant over time and equal to γ; that is, $M_{t+1} = \gamma M_t$. New money is injected, or withdrawn if $\gamma < 1$, by lump-sum transfers, or taxes.[3] Transfers take place at the beginning of the CM and we specify that they go to buyers only. Lack of record-keeping and private information about individual trading histories rule out unsecured credit, giving a role for money and capital to serve as means of payment. In addition, individual money holdings are common knowledge in a match.[4]

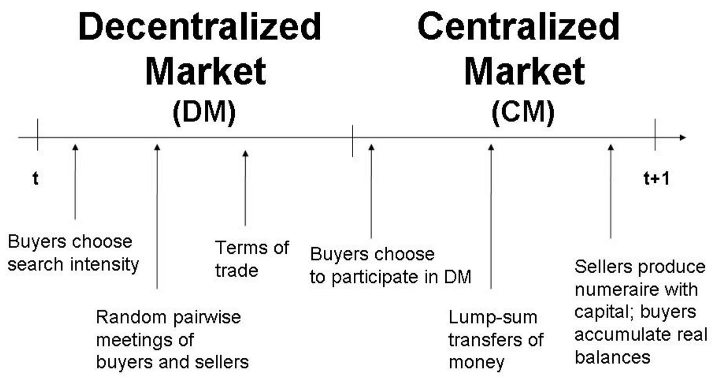
Agents are matched pairwise and at random in the DM. We normalize the measure of sellers to one and assume their search intensity is exogenously given. However buyers can choose whether to participate in the DM, and upon entering, their search intensity at the beginning of each period. The entry decision for the DM occurs at the beginning of the previous period's CM, while the choice of search intensity occurs at the beginning of the DM. Figure 1 summarizes the timing of a representative period.

Let n denote the measure of buyers who choose to enter the DM. Upon entering, each buyer chooses search intensity, $e \in \mathbb{R}_+$, at the beginning of the DM. The average search intensity of buyers

[3]The government is assumed to have enough coercive power to collect taxes in the CM, but has no coercive power in the DM. The government also cannot observe trading histories or asset holdings in neither the DM nor the CM. Hu, Kennan, and Wallace (2009) and Andolfatto (2010) consider alternative approaches to model deflation where the buyers can choose not to participate the CM in order to avoid paying taxes.

[4]We assume with no loss in generality that sellers do not carry real balances or capital across periods. See e.g. Hu and Rocheteau (2013).

Figure 1: **Timing of Representative Period**



is $\bar{e}$, defined as

$$\bar{e} = \int_{b\in[0,1]} e_b db.$$

A buyer exerting effort e to search in the DM incurs a cost in utility terms of $\psi(e)$. We assume that $\psi(1) = v > 0$.

Given n and $\bar{e}$, the number of trade matches in the DM is determined by a constant-returns-to-scale matching function that depends on *market tightness*, defined as $\theta \equiv [n\bar{e}]^{-1}$, or the ratio of sellers to the *effective* buyers searching. A high θ implies a thick market for buyers and a thin one for sellers. Given θ, the meeting probability for an individual buyer with search intensity e is $e\alpha(\theta)$ while the meeting probability of a seller is $\alpha(\theta)/\theta$. The matching function has standard properties: $\alpha(0) = 0$, $\alpha'(0) \geq 0$, $\alpha'(\theta) > 0$ for $\theta \in (0, \infty)$, $\alpha'(\infty) = 0$, and $\alpha'' < 0$. In addition, we assume that $\alpha \in [0, 1]$ for any $\theta \geq 0$, $\lim_{\theta\to\infty} \alpha = 1$, and $\lim_{\theta\to 0} \alpha(\theta)/\theta = 1$.

The instantaneous utility function of a buyer is

$$U^b(x, q, e) = u(q) - \psi(e) + x, \tag{1}$$

where q is consumption in the DM, x is the quantity consumed in the CM, and e is the buyer's search effort. A buyer's lifetime expected utility is $\mathbb{E}_0 \left\{\sum_{t=0}^{\infty} \beta^t U^b(x_t, q_t, e_t)\right\}$, where $\mathbb{E}_0$ is the expectation operator conditional time-0 information. The discount factor $\beta \in (0, 1)$ is the same for all agents and assumed to be smaller than γ throughout the analysis. Similarly, the instantaneous utility function of a seller is

$$U^s(x,q) = -c(q) + x, \tag{2}$$

where q is production in the DM and x is the quantity consumed in the CM. Lifetime utility for a seller is given by $\mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t U^s(x_t, q_t) \right\}$.

Finally, we make the following standard assumptions on primitives:

(a) $u(0) = 0$, $u'(0) = \infty$, $u'(q) > 0$, and $u''(q) < 0$ for $q > 0$;

(b) $\psi(e) \in [0, \infty)$ for all $e \in \mathbb{R}_+$, $\psi' > 0$, $\psi'' > 0$, $\psi(0) = \psi'(0) = 0$, and $\lim_{e \to 1} \psi'(e) = \infty$;

(c) $c(0) = c'(0) = 0$, $c'(q) > 0$, and $c''(q) \geq 0$;

(d) $c(q) = u(q)$ for some $q > 0$.

Let q^* be the solution to $u'(q^*) = c'(q^*)$.

3 Implementation

We study equilibrium outcomes that can be implemented by a mechanism designer in the DM called a *mechanism designer's proposal.* A *proposal* consists of four objects: *(i)* a sequence of functions in the bilateral matches, $o_t : \mathbb{R}^2_+ \to \mathbb{R}^3_+$, each of which maps the buyer's portfolio, (z_t, k_t), into a proposed allocation, $(q_t, d_{z,t}, d_{k,t}) \in \mathbb{R}_+ \times [0, z_t] \times [0, k_t]$, where q_t is the DM output produced by the seller and consumed by the buyer, $d_{z,t}$ is the transfer of real balances, and $d_{k,t}$ is the transfer of capital from the buyer to the seller; *(ii)* an initial distribution of money, μ; *(iii)* a sequence of prices for money, $\{\phi_t\}_{t=0}^{\infty}$, and a sequence of rental prices for capital, $\{R_t\}_{t=0}^{\infty}$, both in terms of the numéraire good; *(iv)* a sequence of measures of buyers entering the DM and the search intensity of those buyers, $\{n_t, e_t\}_{t=0}^{\infty}$.

The terms of trade in the DM are determined through the following game. Given agents' portfolio holdings and the associated proposed trade, both the buyer and the seller simultaneously respond with *yes* or *no*: if both respond with *yes*, then the planner proposed trade is carried out; otherwise, there is no trade. Since both agents can turn down the proposed trade, this ensures that trades are individually rational. We also require any proposed trades to be in the pairwise core so that it is also renegotiation-proof.[5] Agents in the CM trade competitively against the proposed prices, which is consistent with the pairwise core requirement in the DM due to the equivalence between the core and competitive equilibria.

[5]The pairwise core requirement can be implemented directly with a trading mechanism that adds a renegotiation stage as in Hu, Kennan, and Wallace (2009). The renegotiation stage will work as follows. An agent will be chosen at random to make an alternative offer to the one made by the mechanism. The other agent will then have the opportunity to choose between the two offers.

We denote s_b as the strategy of buyer $b \in \mathbb{B}$, which consists of four components for any given trading history h^t at the beginning of period t: *(i)* $s_b^{h^t,0}(z,k) \in \times[0,1]$ that maps the buyer's portfolio (z,k) into his search intensity, $e \in [0,1]$, at the beginning of the DM, conditional on entering the DM; *(ii)* $s_b^{h^t,1}(z,k) \in \{yes, no\}$ that, contingent on being matched in the DM, maps the buyer's portfolio (z,k) to his *yes* or *no* response in the DM, conditional on being matched with a seller; *(iii)* $s_b^{h^t,2}(z,k,a_b,a_s) \in [0,1] \times \mathbb{R}_+^2$ that maps the buyer's portfolio, (z,k), and the buyer's and seller's choices whether to accept the trade, $a_b, a_s \in \{yes, no\}$, to to his probability of entering the DM next period, and his final real money and capital holdings after the CM. The strategy of a seller $s \in \mathbb{S}$ at the beginning of period t consists of a function, $s_s^{h^t,1}(z,k) \in \{yes, no\}$, that represent the seller's response contingent on the buyer's portfolio.

Definition 1. *An equilibrium is a list,* $\langle (s_b : b \in \mathbb{B}), (s_s : s \in \mathbb{S}), \mu, \{o_t, \phi_t, R_t, n_t, e_t\}_{t=0}^{\infty} \rangle$*, composed of one strategy for each agent and the proposal* $(\mu, \{o_t, \phi_t, R_t, n_t, e_t\}_{t=0}^{\infty})$ *such that: (*i*) each strategy is sequentially rational given other players' strategies; and (*ii*) the centralized meeting clears at every date.*

In what follows, we focus on equilibria with *(i)* stationary planner proposals that use symmetric and stationary strategies in which *(ii)* agents always respond with *yes* in all DM meetings, *(iii)* the initial distribution of money is uniform across buyers who enter the DM, *(iv)* real balances are constant over time, *(v)* equilibrium measure of buyers entering the DM, n, is pinned down by a free-entry condition given the equilibrium DM behavior, and (vi) market tightness, $\theta = [n\bar{e}]^{-1}$ is pinned down by $\bar{e} = e$. Following Hu, Kennan, and Wallace (2009), we call such equilibria *simple equilibria*. In particular, in a simple equilibrium, $\phi^t = \gamma\phi_{t+1}$ for all t, and the proposal, $o_t(z_t, k_t)$, is the same across al time periods and we may write it as $o(z,k) = [q(z,k), d_z(z,k), d_k(z,k)]$.

The outcome of a simple equilibrium is summarized by a list, $(q^p, d_z^p, d_k^p, z^p, k^p, n^p, e^p)$, where (q^p, d_z^p, d_k^p) is the terms of trade in the DM, n^p is the measure of buyers entering the DM with search intensity e^p, and (z^p, k^p) are portfolios of those buyers. An equilibrium outcome, $(q^p, d_z^p, d_k^p, z^p, k^p, n^p, e^p)$, is said to be *implementable* if it is the equilibrium outcome of a simple equilibrium associated with a proposal $\{o, \phi, R, n, e\}$, and if for any portfolio (z,k), $o(z,k)$ selects a trade in the pairwise core. Given the proposals, for each (z,k), we use $\mathcal{CO}(z,k)$ to denote the set of allocations in the pairwise core.

For a given proposal, o, a given market thickness, θ, and rental price, R, let V_o^b, $V^b(z,k)$, and $W^b(z,k)$ denote the continuation values for a buyer who, after the DM, stay outside the DM, for a buyer who hold (z,k) upon entering the DM, and for a buyer who hold (z,k) upon entering the CM, respectively. Similarly, let $W^s(z,k)$ denote the continuation values for a seller holding (z,k)

upon entering the CM. The Bellman equation for a buyer in the CM solves

$$W^b(z,k) = z + Rk + \max\left\{\beta V_o^b, \max_{\hat{z}\geq 0, \hat{k}\geq 0}\left\{-\gamma\hat{z} - \hat{k} + T + \beta V^b(\hat{z},\hat{k})\right\}\right\}, \tag{3}$$

where $\hat{z}$ and $\hat{k}$ denote the real balances and capital taken into the next DM, $T = (M_{t+1} - M_t)\phi_t$ is the lump-sum transfer. In order to hold $\hat{z}$ real balances in the next period, the buyer must accumulate $\gamma\hat{z}$ units of current real balances (since the rate of return of fiat money is γ^{-1}). Notice that, due to the linear preferences in the CM, the buyer's value function is linear in wealth.

The Bellman equation for $V^b(z,k)$ is given by

$$V^b(z,k) = \max_{e\in[0,1]}\left\{-\psi(e) + e\alpha(\theta)\left\{u\left[q(z,k)\right] + W^b\left[z - d_z(z,k), k - d_k(z,k)\right]\right\} + [1 - e\alpha(\theta)]W^b(z,k)\right\}. \tag{4}$$

Using the linearity of W^b, (4) simplifies to

$$V^b(z,k) = \max_{e\in[0,1]}\left\{-\psi(e) + e\alpha(\theta)\left\{u\left[q(z,k)\right] - d_z\left(z,k\right) - Rd_k\left(z,k\right)\right\} + W^b(z,k)\right\}. \tag{5}$$

Now, the value V_o^b is then simply

$$V_o^b = W^b(0,0). \tag{6}$$

Substituting $V^b(z,k)$ and V_o^b with its expression given by (5) and (6) into (3), using the linearity of $W^b(z,k)$, and omitting constant terms, the buyer's problem in the CM can be reformulated as

$$\max\left\{0, \max_{(z,k,e)}\left\{-iz - (1+r-R)k - \psi(e) + e\alpha(\theta)\left\{u\left[q(z,k)\right] - d_z\left(z,k\right) - Rd_k(z,k)\right\}\right\}\right\}. \tag{7}$$

The free-entry condition then requires that, for the solution (z,k,e) to the above problem,

$$-iz - (1+r-R)k + e\alpha(\theta)\left\{u\left[q(z,k)\right] - d_z\left(z,k\right) - Rd_k(z,k)\right\} - \psi(e) = 0. \tag{8}$$

Using the same logic as above, the Bellman equation for a seller in the CM solves

$$W^s(z,k) = z + Rk + \max_{\widehat{k}\geq 0}\left\{A\widehat{k} - R\widehat{k}\right\}. \tag{9}$$

Sellers therefore choose k such that $F'(\widehat{k}) = R$. With $F(k) = Ak$, we therefore have $R = A$.

We are now in a position to characterize implementable outcomes.

Proposition 1. *An outcome, $(q^p, d_z^p, d_k^p, z^p, k^p, n^p, e^p)$, is implementable if and only if*

$$-iz^p - (1 + r - A)k^p + e^p\alpha(1/e^p n^p)[u(q^p) - d_z^p - Ad_k^p] - \psi(e^p) = 0, \tag{10}$$

$$d_z^p \leq z^p,\ d_k^p \leq k^p, \tag{11}$$

$$\psi'(e) = \alpha(1/n^p e^p)[u(q^p) - d_z^p - Ad_k^p], \tag{12}$$

$$-c(q^p) + d_z^p + Ad_k^p \geq 0, \tag{13}$$

and $(q^p, d_z^p, d_k^p) \in \mathcal{CO}(z^p, k^p)$.

Our goal is to find implementable outcomes that are socially optimal. Formally, given an outcome, $(q^p, d_z^p, d_k^p, z^p, k^p, n^p, e^p)$, its social welfare is given by

$$\mathcal{W}(q^p, d_z^p, d_k^p, z^p, k^p, n^p) = \frac{1}{r}\{n^p e^p \alpha(1/e^p n^p)[u(q^p) - c(q^p)] - n^p\psi(e^p) - n^p(1 + r - A)k^p\}. \tag{14}$$

We say that an outcome is *constrained efficient* if it maximizes $\mathcal{W}$ among the implementable outcomes.

4 Extensive Margin

Here we study the effect of inflation along the extensive margin. Specifically, we exogenously set $e^p = 1$ and denote $\psi(1)$ by v. An outcome then consists of $(q^p, d_z^p, d_k^p, z^p, k^p, n^p)$, and it is implementable if and only it satisfies (10), (11), (13), and the pairwise core requirement with $e^p = 1$.

In this case, the first-best level of output, capital, and measure of buyers entering that maximize (14) is given by (q^*, k^*, n^*) and is such that

$$u'(q^*) = c'(q^*), \tag{15}$$

$$k^* = 0,$$

$$\left[\alpha\left(1/n^*\right) - \alpha'\left(1/n^*\right)/n^*\right][u(q^*) - c(q^*)] = v. \tag{16}$$

Throughout this section we assume that

$$\alpha(1/n^*)[u(q^*) - c(q^*)] > v.$$

Obviously, without this assumption the buyer is not willing to participate the DM even at the first-best arrangement.

4.1 Optimal Allocation and Entry Decision with Money Alone

Here we study implementable outcomes that are socially optimal, i.e., that maximizes the social welfare defined as the discounted sum of buyers' and sellers' expected utilities, weighted by their respective measures in the DM.

We first consider the economy without production of capital. That is, we consider constrained-efficient outcomes with the additional constraint that $k^p = 0$, in which case an outcome consists of (q^p, d_z^p, z^p, n^p). The following lemma helps to find the constrained efficient outcome.

Lemma 1. *There exists (d_z^p, z^p) such that (q^p, d_z^p, z^p, n^p) is a constrained-efficient outcome if the pair (q^p, n^p) solves*

$$\max_{(q,n)} n\alpha(1/n)[u(q) - c(q)] - nv \tag{17}$$

subject to

$$\alpha(1/n)[u(q) - c(q)] \geq ic(q) + v. \tag{18}$$

Because of Lemma 1, we also call the pair (q^p, n^p) a constrained-efficient outcome if it solves (17)–(18).

Proposition 2. *For any $i \geq 0$, a constrained efficient outcome, $(q^p(i), n^p(i))$, exists, and satisfies the following.*

1. *Let*
$$i^* = \frac{\alpha(1/n^*)[u(q^*) - c(q^*)] - v}{c(q^*)} > 0.$$
Then, for all $i \in [0, i^*]$, the constrained efficient outcome, $(q^p(i), n^p(i))$, is unique, and satisfies $q^p(i) = q^*$ and $n^p(i) = n^*$.

2. *There exists $\bar{i} > i^*$ such that for all $i \in (i^*, \bar{i}]$, any constrained efficient outcome, $(q^p(i), n^p(i))$, is unique and satisfies $q^p(i) < q^*$ and $n^p(i) < n^*$. Moreover, $n^p(i)$ is strictly decreasing in $(i^*, \bar{i}]$.*

3. *For any $n \in (0, n^*]$ and $q \in (0, q^*]$, there exists $i_{n,q}$ such that for if $i > i_{n,q}$, then any constrained efficient outcome, $(q^p(i), n^p(i))$, satisfies $n^p(i) < n$ and $q^p(i) < q$.*

4.2 Optimal Allocation and Entry Decision with Money and Capital

Here we consider the economy with potential production of capital with the linear technology. Let $\bar{z}_i = \{[u(q^*) - c(q^*)] - v\}/i$ and let $\bar{k} = \{[u(q^*) - c(q^*)] - v\}/(1 + r - A)$. Then, by (10), in any implementable outcome, we have $z^p \leq \bar{z}_i$ and $k^p \leq \bar{k}$.

Lemma 2. *There exists* $(d_z^p, d_k^p) \leq (z^p, k^p)$ *such that* $(q^p, d_z^p, d_k^p, z^p, k^p, n^p)$ *is a constrained-efficient outcome if the tuple* (q^p, z^p, k^p, n^p) *solves*

$$\max_{(q,z,k,n)\in[0,q^*]\times[0,\bar{z}_i]\times[0,\bar{k}]\times[0,n^*]} n\alpha(1/n)[u(q)-c(q)] - nv - n(1+r-A)k \tag{19}$$

subject to

$$-iz - (1+r-A)k + \alpha(1/n)[u(q) - z - Ak] - v = 0, \tag{20}$$

$$-c(q) + z + Ak \geq 0. \tag{21}$$

Moreover, if $i \geq i^*$*, then for any constrained-efficient outcome,* $(q^p, d_z^p, d_k^p, z^p, k^p, n^p)$*,* (q^p, k^p, n^p) *solves (19)–(21).*

Because of Lemma 2, we also refer to a solution of (19)–(21) as a constrained-efficient outcome.

Before we present our characterization result, we first consider the case without fiat money, that is, we impose the constraint that $z^p = 0$, as a benchmark case. Under this additional constraint, the maximization problem becomes

$$\max_{(q,k,n)\in[0,q^*]\times[0,\bar{k}]\times[0,n^*]} n\alpha(1/n)[u(q)-c(q)] - nv - n(1+r-A)k \tag{22}$$

subject to

$$-(1+r-A)k + \alpha(1/n)[u(q) - Ak] - v = 0, \tag{23}$$

$$-c(q) + Ak \geq 0. \tag{24}$$

By the Extreme Value Theorem, a solution to the above problem exists and the optimal value is unique, denoted by $\mathcal{W}^k$. Moreover, by the Theorem of Maximum, the set of solutions, denoted by Ω^k, is a compact set. Let $\hat{q} = \min\{q : (q, n, k) \in \Omega_k\}$, $\hat{n} = \min\{n : (q, n, k) \in \Omega_k\}$, and $\hat{k} = \min\{k : (q, n, k) \in \Omega_k\}$.

Proposition 3. *For any* $i \geq 0$*, a constrained efficient outcome,* $(q^p(i), z^p(i), k^p(i), n^p(i))$*, exists, and satisfies the following.*

1. *Let* $i^* = \frac{\alpha(1/n^*)[u(q^*)-c(q^*)]-v}{c(q^*)} > 0$*. Then, for all* $i \in [0, i^*]$*, the constrained-efficient outcome,* $(q^p(i), z^p(i), k^p(i), n^p(i))$*, is unique, and satisfies* $q^p(i) = q^*$*,* $z^p(i) \leq c(q^*)$*,* $k^p(i) = 0$*, and* $n^p(i) = n^*$*.*

2. *There exists* $\bar{i} > i^*$ *such that for all* $i \in (i^*, \bar{i}]$*, the unique constrained-efficient outcome,* $(q^p(i), z^p(i), k^p(i), n^p(i))$*, satisfies* $q^p(i) < q^*$*,* $z^p(i) = c(q^p)$*,* $k^p = 0$*, and* $n^p(i) < n^*$*. Moreover,* $n^p(i)$ *is strictly decreasing in* $i \in (i^*, \bar{i}]$*.*

3. *There exists $\tilde{i} \geq \bar{i}$ such that, if $i > \tilde{i}$, any constrained-efficient outcome, $(q^p(i), z^p(i), k^p(i), n^p(i))$, satisfies $k^p(i) > 0$. Moreover, $z^p(i)$ converges to 0 as i goes to infinity but the maximum welfare converges to Ω_k.*

5 Intensive Margin

Here we study the effect of inflation along the intensive margin, where buyers can choose the intensity in which they search. There is a exogenously given measure of buyers entering the DM, normalized to 1, and each buyer $b \in \mathbb{B} = [0, 1]$ chooses the search intensity, $e \in [0, 1]$, at the beginning of the DM. In this case, an outcome may be denoted by $(q^p, d_z^p, d_k^p, z^p, k^p, e^p)$, and it is implementable if and only it satisfies (10) with weak inequality, (11), (12), (13), and the pairwise core requirement with $n^p = 1$. In this case, the unconstrained first-best allocation is given by the following: $q^p = q^*$, $k^p = 0$, and $e^p = e^*$ given by

$$\left[\alpha(1/e^*) - \alpha'(1/e^*)/e^*\right][u(q^*) - c(q^*)] = \psi'(e^*).$$

5.1 Optimal Allocation and Search Intensity with Money Alone

Here we consider the economy without production of capital, that is, we consider the constrained-efficient outcomes with the additional constraint that $k^p = 0$. Then, an outcome may be written as (q^p, d_z^p, z^p, e^p). The following lemma helps to find the constrained-efficient outcome.

Lemma 3. *There exists z^p such that (q^p, d_z^p, z^p, e^p) is a constrained-efficient outcome if and only if the triple (q^p, d_z^p, e^p) solves*

$$\max_{(q,d,e)} e\alpha(1/e)[u(q) - c(q)] - \psi(e) \tag{25}$$

subject to

$$-id_z + e\alpha(1/e)[u(q) - d_z] - \psi(e) \geq 0, \tag{26}$$

$$\psi'(e) = \alpha(1/e)[u(q) - d_z], \tag{27}$$

$$e\alpha(1/e)[d_z - c(q)] \geq 0. \tag{28}$$

Because of Lemma 3, we also call the triple (q^p, d_z^p, e^p) a constrained-efficient outcome if it solves (25)–(28).

Proposition 4. *For any $i \geq 0$, a constrained efficient outcome, $(q^p(i), d_z^p(i), e^p(i))$, exists, and satisfies the following.*

1. *Let*
$$i^* \equiv \frac{e^*\psi'(e^*) - \psi(e^*)}{u(q^*) - \psi'(e^*)/\alpha(1/e^*)}.$$
Then, $i^* > 0$ *and for all* $i \in [0, i^*]$*, the unique constrained-efficient outcome,* $(q^p(i), d_z^p(i), e^p(i))$*, satisfies* $q^p(i) = q^*$*,* $e^p(i) = e^*$*, and* $d_z^p(i) = d_z^* \equiv u(q^*) - \psi'(e^*)/\alpha(1/e^*)$*.*

2. *There exists* $\bar{i} > i^*$ *such that for all* $i \in (i^*, \bar{i}]$*, the unique constrained-efficient outcome,* $(q^p(i), d_z^p(i), e^p(i))$*, satisfies* $q^p(i) < q^*$*,* $d^p(i) < d_z^*$*,* $e^p(i) > e^*$*,* $\frac{d}{di}e^p(i) > 0$*.*

3. *For any* $e \in (0, 1]$*, there exists* $i_e > i^*$ *such that if* $i > i_e$*, then any constrained-efficient outcome,* $(q^p(i), d_z^p(i), e^p(i))$*, satisfies* $q^p(i) < q^*$*, and* $d^p(i) < d_z^*$*,* $e^p(i) < e$*.*

4. *Suppose that* $\psi''(0) \in (0, \infty)$ *and that for some* $\delta > 0$*,* $\lim_{q\to 0}(c^{-1} \circ u)'(q)q^{0.5+\delta} > 0$*. Then, for any* i*, equilibrium is monetary, that is,* $d_z^p(i) > 0$*.*

5.2 Optimal Allocation and Search Intensity with Money and Capital

Figures

Extensive Margin with Money Only

Figure 2: Output (Money Only)

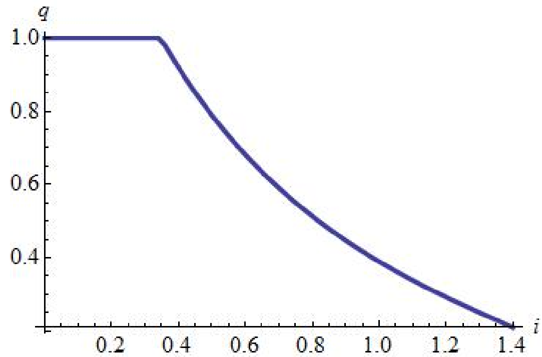


Figure 3: Monetary Payment (Money Only)

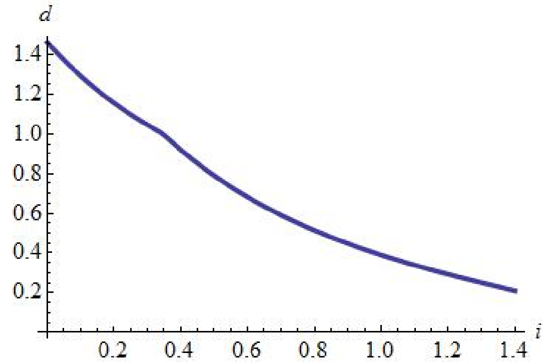


Figure 4: Measure of Buyers (Money Only)

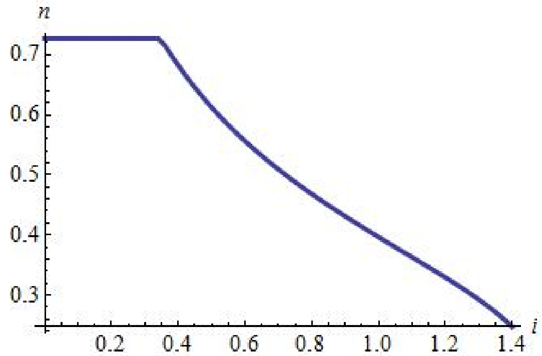
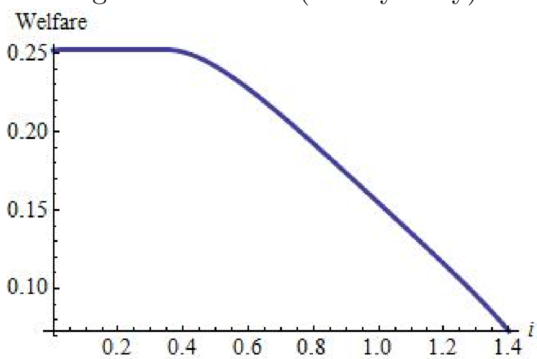


Figure 5: Welfare (Money Only)



Extensive Margin with Money and Capital

Figure 6: Output

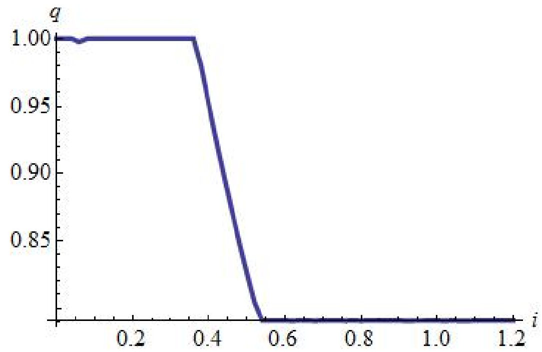


Figure 7: Measure of Buyers

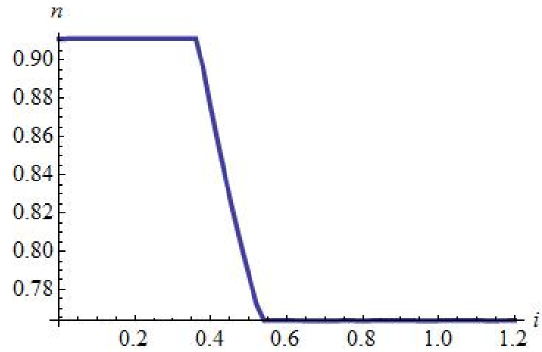


Figure 8: Real Balances

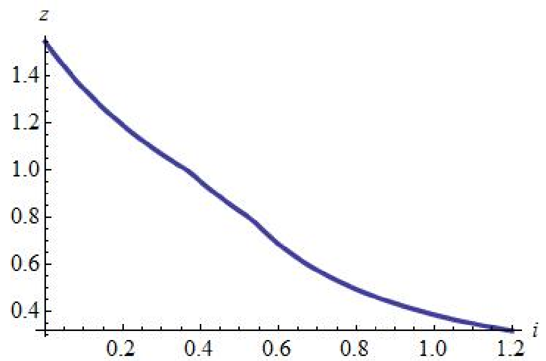
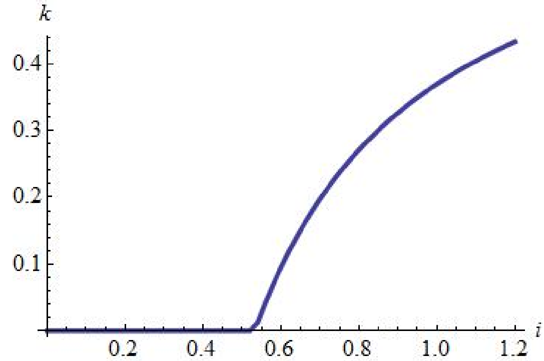


Figure 9: Capital



Intensive Margin With Money Only

Figure 10: Output (Money Only)

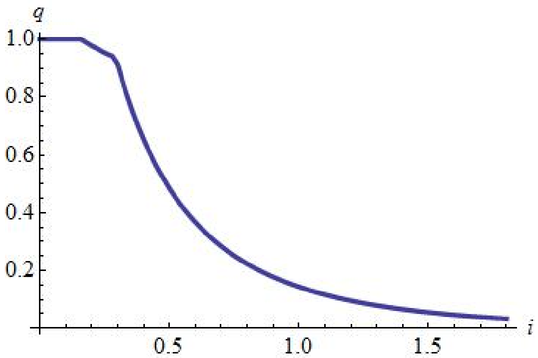


Figure 11: Buyer's Search Intensity (Money Only)

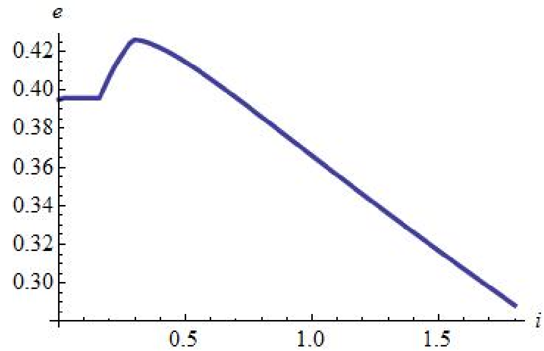


Figure 12: Real Balances (Money Only)

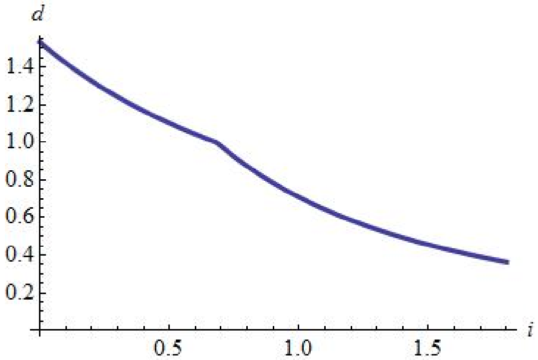
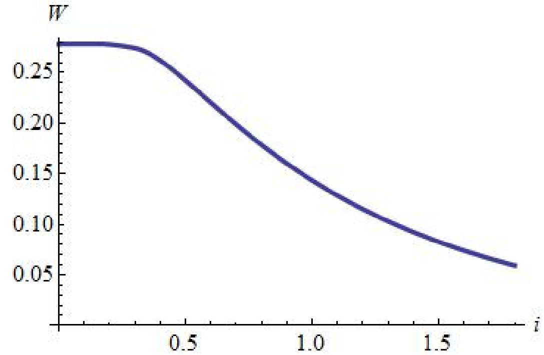


Figure 13: Welfare (Money Only)



Intensive Margin with Money and Capital

Figure 14: Output

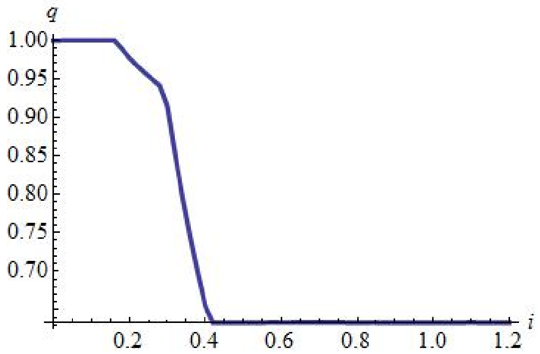


Figure 15: Buyer's Search Intensity

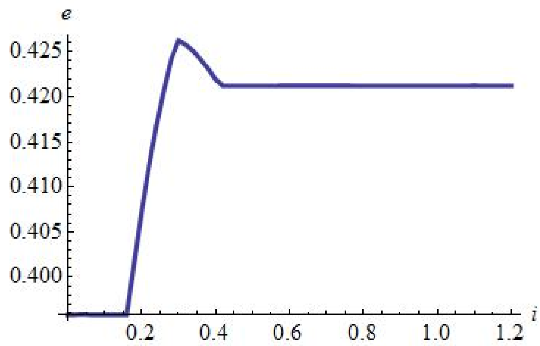


Figure 16: Real Balances

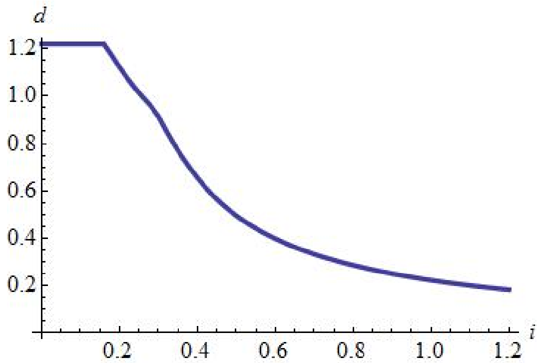
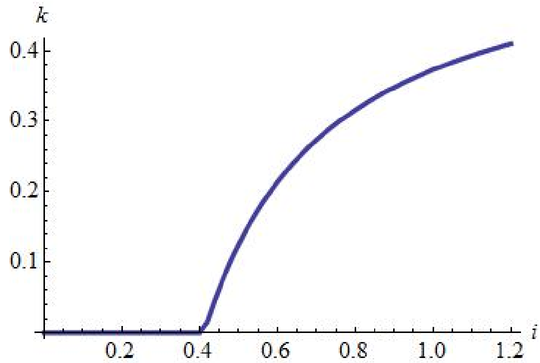


Figure 17: Capital



Appendix

Proof of Proposition 2

(1) To determine the first-best allocation, we examine the unconstrained problem (17) without (18). The first-best allocation (q^*, n^*) solves (15) and (16) and are uniquely determined. When $i \in [0, i^*]$, (q^*, n^*) satisfies (10), (11), (13), and the pairwise core requirement with $e^p = 1$, and hence is implementable.

(2) When $i > i^*$, we consider the maximization problem (17) subject to (18) at equality.

Claim 1. (i) (18) binds for all $i > i^*$; (ii) $q < q^*$ for $i > i^*$.

Proof. **(i)** Consider the Lagrangian associated with the maximization problem (17) subject to (18):

$$\begin{aligned}\mathcal{L}(q,n;\lambda) &= n\alpha\,(1/n)\,[u(q)-c(q)] - nv \\ &+ \lambda\{-ic(q) + \alpha(1/n)[u(q)-c(q)] - v\}\end{aligned}$$

where $\lambda \geq 0$ is the Lagrange multiplier associated with (18). The first-order necessary conditions with respect to q and n are

$$\begin{aligned}[\alpha(1/n^p)(n^p+\lambda)][u'(q^p)-c'(q^p)] - \lambda i c'(q^p) &= 0 \\ \left[\alpha(1/n^p) - \alpha'(1/n^p)\left(\frac{1}{n^p} - \frac{\lambda}{(n^p)^2}\right)\right][u(q^p)-c(q^p)] &= v.\end{aligned}$$

To show that (18) binds for $i > i^*$; i.e. $\lambda > 0$, suppose by contradiction that $\lambda = 0$. Then $q^p = q^*$ and $n^p = n^*$, which implies that

$$i \leq \frac{\alpha(1/n^*)[u(q^*)-c(q^*)] - v}{c(q^*)} = i^*$$

Hence, $i \leq i^*$, a contradiction. Consequently for $i > i^*$, $\lambda > 0$, which implies that

$$\frac{u'(q^p)}{c'(q^p)} = 1 + \frac{\lambda i}{\alpha(1/n^p)(n^p+\lambda)} \tag{29}$$

(ii) To verify that $q^p < q^*$ for $i > i^*$, consider (29). As $\lambda > 0$, we therefore have $q^p < q^*$. □

Given $i > i^*$, the planner problem with (18) at equality simplifies to a choice of n. That is, the optimal trading mechanism solves (17) subject to

$$\alpha\,(1/n)\,[u(q)-c(q)] = ic(q) + v.$$

Let $q = g(n,i) > 0$ solve

$$\alpha\left(\frac{1}{n}\right)[u(g(n,i)) - c(g(n,i))] = ic(g(n,i)) + v.$$

Then, the maximization problem is

$$\max_{n\in[0,n^*]} nic(g(n,i)).$$

Given i, the first order condition with respect to n is $f(n,i) = 0$ where

$$f(n,i) = i[c[g(n,i)] + nc'[g(n,i)]g_n(n,i)].$$

To show that $n^p(i) < n^*$ and $n^p(i)$ is strictly decreasing in i, we apply the Implicit Function Theorem (IFT) in the neighborhood of the first-best. To do so, we first determine the signs of the second derivatives, $f_n(n^*,i^*)$ and $f_i(n^*,i^*)$. In this exercise, n is taken parametrically and is not treated as a function of i.

Let $h(q) = u(q) - c(q)$. Differentiating $\alpha(1/n)[h(g(n,i))] = ic(g(n,i)) + v$, we have

$$\begin{aligned}
g_n(n,i) &= \frac{\alpha'(1/n)h(q)}{n^2[\alpha(1/n)h'(q) - ic'(q)]};\\
g_i(n,i) &= \frac{c(q)}{\alpha(1/n)h'(q) - ic'(q)};\\
g_{ni}(n,i) &= \frac{c'(q)g_n(n,i)[\alpha(1/n)h'(q) - ic'(q)]}{[\alpha(1/n)h'(q) - ic'(q)]^2}\\
&- \frac{c(q)[\alpha(1/n)h''(q)g_n(n,i) - (1/n^2)\alpha'(1/n)h'(q) - ic''(q)g_n(n,i)]}{[\alpha(1/n)h'(q) - ic'(q)]^2};\\
g_{nn}(n,i) &= \frac{[\alpha(1/n)h'(q) - ic'(q)]\{n^2h'(q)g_n(n,i)\alpha'(1/n) - \alpha''(1/n)h(q)\}}{[\alpha(1/n)h'(q) - ic'(q)]^2n^4}\\
&- \frac{[\alpha'(1/n)h(q)]\left\{n^2\alpha(1/n)h''(q)g_n - h'(q)\alpha'(1/n) + 2n[\alpha(1/n)h'(q) - ic'(q)] - n^2ic''(q)g_n\right\}}{[\alpha(1/n)h'(q) - ic'(q)]^2n^4}.
\end{aligned}$$

Since $h'(q^*) = 0$ and noting i^* can be rewritten as

$$i^* = \frac{\alpha'(1/n^*)h(q^*)}{c(q^*)n^*},$$

we have

$$\begin{aligned}
g_n(n^*,i^*) &= -\frac{c(q^*)}{n^*c'(q^*)} < 0;\\
g_i(n^*,i^*) &= -\frac{c(q^*)}{i^*c'(q^*)} < 0;\\
g_{ni}(n^*,i^*) &= \frac{c(q^*)}{i^*n^*c'(q^*)} + \frac{c(q^*)^2[\alpha(1/n^*)h''(q^*) - i^*c''(q^*)]}{(i^*)^2n^*[c'(q^*)]^3};
\end{aligned}$$

$$\begin{aligned}
g_{nn}(n^*,i^*) &= \frac{\alpha''(1/n^*)h(q^*)}{i^*(n^*)^4c'(q^*)}\\
&+ \frac{\alpha'(1/n^*)h(q^*)\left[\frac{n^*c(q^*)}{c'(q^*)}[\alpha(1/n^*)h''(q^*) - i^*c''(q^*)] + 2n^*i^*c'(q^*)\right]}{(i^*)^2[c'(q^*)]^2(n^*)^4}\\
&= \frac{i^*\alpha''(1/n^*)h(q^*)c'(q^*) + \alpha'(1/n^*)h(q^*)\left[\frac{n^*c(q^*)}{c'(q^*)}[\alpha(1/n^*)h''(q^*) - i^*c''(q^*)] + 2n^*i^*c'(q^*)\right]}{(i^*)^2[c'(q^*)]^2(n^*)^4}
\end{aligned}$$

Thus,

$$f(n^*,i^*) = i^*[c[g(n^*,i^*)] + n^*c'[g(n^*,i^*)]g_n(n^*,i^*)] = 0.$$

Moreover for $i^* > 0$, the second partial derivatives are

$$\begin{aligned}
f_n(n^*,i^*) &= i^*\left\{n^*c''[g(n^*,i^*)]g_n^2(n^*,i^*) + n^*c'[g(n^*,i^*))]g_{nn}(n^*,i^*) + 2c'[g(n^*,i^*)]g_n(n^*,i^*)\right\}\\
&= \frac{i^*c''(q^*)c(q^*)^2}{n^*[c'(q^*)]^2} + \frac{\alpha''(1/n^*)h(q^*)c'(q^*)}{(n^*)^3c'(q^*)}\\
&+ \frac{\alpha'(1/n^*)h(q^*)\left\{\frac{n^*c(q^*)}{c'(q^*)}[\alpha(1/n^*)h''(q^*) - i^*c''(q^*)]\right\}}{i^*c'(q^*)(n^*)^3} + 2\frac{i^*c(q^*)}{n^*} - 2\frac{i^*c(q^*)}{n^*}\\
&= \frac{i^*c''(q^*)c(q^*)^2}{n^*[c'(q^*)]^2} + \frac{\alpha''(1/n^*)h(q^*)}{(n^*)^3} + \frac{\alpha'(1/n^*)h(q^*)\left\{\frac{n^*c(q^*)}{c'(q^*)}[\alpha(1/n^*)h''(q^*) - i^*c''(q^*)]\right\}}{i^*c'(q^*)(n^*)^3}\\
&= \frac{[\alpha'(1/n^*)h(q^*)]^2c''(q) - [\alpha'(1/n^*)h(q^*)]^2c''(q)}{i[c'(q)]^2(n^*)^3}\\
&+ \frac{\alpha''(1/n^*)h(q^*)}{(n^*)^3} + \frac{\alpha'(1/n^*)h(q^*)\left\{\frac{n^*c(q^*)}{c'(q^*)}[\alpha(1/n^*)h''(q^*)]\right\}}{i^*c'(q^*)(n^*)^3}\\
&= \underbrace{\frac{\alpha''(1/n^*)h(q^*)}{(n^*)^3}}_{(-)} + \underbrace{\frac{\alpha'(1/n^*)h(q^*)\left\{\frac{n^*c(q^*)}{c'(q^*)}[\alpha(1/n^*)h''(q^*)]\right\}}{i^*c'(q^*)(n^*)^3}}_{(-)} < 0.
\end{aligned}$$

$$
\begin{aligned}
f_i(n^*,i^*) &= i^*\left\{c'[g(n^*,i^*)]g_i(n^*,i^*)+n^*c''[g(n^*,i^*)]g_n(n^*,i^*)g_i(n^*,i^*)+n^*c'[g(n^*,i^*)]g_{ni}(n^*,i^*)\right\} \\
&+ \underbrace{c[g(n^*,i^*)]+n^*c'[g(n^*,i^*)]g_n(n^*,i^*)}_{=0} \\
&= -c(q^*)+\frac{c''(q^*)c(q^*)^2}{c'(q^*)^2}+i^*n^*c'(q^*)g_{ni}(n^*,i^*)) \\
&= -c(q^*)+c(q^*)+\frac{c''(q^*)c(q^*)^2}{c'(q^*)^2}+\frac{c(q^*)^2}{i^*[c'(q^*)]^2}[\alpha(1/n^*)h''(q^*)-i^*c''(q^*)] \\
&= \frac{c''(q^*)c(q^*)^2}{c'(q^*)^2}+\frac{c(q^*)^2}{i^*[c'(q^*)]^2}[\alpha(1/n^*)h''(q^*)-i^*c''(q^*)] \\
&= \frac{i^*c''(q^*)c(q^*)^2-i^*c''(q^*)c(q^*)^2}{i^*[c'(q^*)]^2}+\frac{c(q^*)^2\alpha(1/n^*)h''(q^*)}{i^*[c'(q^*)]^2} \\
&= \underbrace{\frac{c(q^*)^2\alpha(1/n^*)h''(q^*)}{i^*[c'(q^*)]^2}}_{(-)}<0.
\end{aligned}
$$

Since $f(n^*,i^*)=0$ and $f_n(n^*,i^*)<0$, by the Implicit Function Theorem, there exists an open neighborhood $(n_0,n_1)\times(i_0,i_1)$ around the first-best (n^*,i^*) and a continuously differentiable function, $\xi:(i_0,i_1)\to(n_0,n_1)$ such that for $i\in(i_0,i_1)$, the function $\xi(i)$ gives the unique value of $n\in(n_0,n_1)$ such that

$$f(\xi(i),i)=0.$$

As $f(n^*,i^*)=0$ and $f_n(n^*,i^*)<0$, $\xi(i)$ is also the local maximizer in the open neighborhood (n_0,n_1). Moreover, since $\xi(i)$ is continuously differentiable by the IFT, there exists a threshold $\bar{i}>i^*$ such that for $i\in(i^*,\bar{i}]$,

$$\xi'(i)=-\frac{f_i(\xi(i),i)}{f_n(\xi(i),i)}.$$

Then by the IFT, the first-order effect of i on n at (i^*,n^*) is given by

$$\xi'(i^*)=-\frac{f_i(n^*,i^*)}{f_n(n^*,i^*)}.$$

Since $f_n(n^*,i^*)<0$ and $f_i(n^*,i^*)<0$, we therefore have

$$\xi'(i^*)<0.$$

Consequently for $i\in(i^*,\bar{i}]$, $n^p(i)$ is strictly decreasing in i. In addition, $n^p(i)<n^*$ by the continuity

of $\xi(i)$.

Proof of Lemma 2

If $i \leq i^*$, then, by Proposition 2, the first-best allocation is implementable with money alone. Moreover, as is shown in the proof of that proposition, we may choose $d_z^p = z^p$ in the constrained efficient outcome and hence it solves the problem (19)-(21). The pairwise core requirement is obviously satisfied.

Clearly, for any solution, (q^p, k^p, n^p), to (19)-(21), the outcome

$$(q^p, d_z^p, d_k^p, z^p, k^p, n^p) = (q^p, z^p, k^p, z^p, k^p, n^p)$$

also satisfies (10)-(13). Here we show that, if $i > i^*$, then for any constrained efficient outcome, $(q^p, d_z^p, d_k^p, z^p, k^p, n^p)$, $d_z^p = z^p$, $k^p = d_k^p$, and (q^p, z^p, k^p, n^p) satisfies (19)-(21) and belongs to the set $[0, q^*] \times [0, \bar{z}] \times [0, \bar{k}] \times [0, n^*]$. Notice that if $q^p > q^*$, then we may decrease q^p and increase z^p so that (10) is unchanged but the welfare is increased. So $q^p \leq q^*$. Moreover, if $n^p > n^*$, then, because $q^p \leq q^*$, we may decrease n^p and increase z^p to keep (10) unchanged but the welfare is increased. This also implies the pairwise core requirement is satisfied.

Now suppose that $i > i^*$. Suppose that $(q^p, d_z^p, d_k^p, z^p, k^p, n^p)$ is an implementable outcome w.r.t. i. We have two claims.

(a) $k^p = d_k^p$. Suppose that $k^p > d_p^k$. Then we may decrease k^p (and increase z^p proportionally to keep (10) in tact) and increase $\mathcal{W}$, a contradiction.

(b) $z^p = d_z^p$.

(b.1) Suppose that $q^p < q^*$ and $z^p > d_z^p$. Let $d_z' = d_z^p + \epsilon < z^p$ be such that $u(q^p) + \epsilon \leq u(q^*)$. Let q' be such that $u(q') = u(q^p) + \epsilon$. Then,

$$-iz^p - (1+r-A)k^p + \alpha(1/n^p)[u(q') - d_z' - Ad_k^p] = -iz^p - (1+r-A)k^p + \alpha(1/n^p)[u(q^p) - d_z^p - Ad_k^p] = v,$$

and

$$\begin{aligned} -c(q') + d_z' + Ad_k^p &= -[c(q') - c(q^p) - \epsilon] + [-c(q^p) + d_z^p + Ad_k^p] \\ &\geq \epsilon - c'(q')(q' - q^p) \geq \epsilon - u'(q')(q' - q^p) \geq \epsilon - [u(q') - u(q^p)] = 0. \end{aligned}$$

Thus, $(q^p, d_z', d_k^p, z^p, k^p, n^p)$ is implementable but has higher welfare as $q' > q^p$, a contradiction.

(b.2) Suppose that $q^p = q^*$ and $z^p > d_z^p$. If $n^p < n^*$, then we can decrease z^p and increase n^p to keep (10) unchanged but increase the welfare, a contradiction. Suppose that $n^p = n^*$. By Proposition 2, it must be the case that $k^p = d_k^p > 0$. Then we may increase d_z^p and decrease k^p to make

$d_z' + Ak' = d_z^p + Ak^p$ while changing z^p so that (10) is unchanged, but the welfare is increased, a contradiction.

Proof of Proposition 3

Proof of Lemma 3

Call (14) subject to (10), (11), (12), and (13) program $\mathcal{A}$ and (25) subject to (26), (27), and (28) program $\mathcal{B}$.

Assume (q^p, e^p, d^p, z^p) is a solution to $\mathcal{A}$. As $z^p \geq d_z^p$ the following is true:

$$-iz^p + e^p\alpha(1/e^p)[u(q^p) - d_z^p] - \psi(e^p) \leq -id_z^p + e^p\alpha(1/e^p)[u(q^p) - d_z^p] - \psi(e). \tag{30}$$

As (q^p, d^p, z^p, e^p) is a solution to $\mathcal{A}$, (10) holds. This, with (30) implies (26). Note that the maximands in $\mathcal{A}$ and $\mathcal{B}$ are equivalent, that (12) implies (27), and that (13) implies (28). Therefore, (q^p, d_z^p, e^p) is a solution to $\mathcal{B}$.

Assume (q^p, d_z^p, e^p) is a solution to $\mathcal{B}$. When $z^p = d_z^p$, (26) implies (10). Note that the maximands in $\mathcal{A}$ and $\mathcal{B}$ are equivalent, that (27) implies (12), and that (28) implies (13). Therefore, (q^p, d_z^p, z^p, e^p) is a solution to $\mathcal{A}$.

Proof of Proposition 4

(1) To determine the first-best allocation, we examine the unconstrained problem (25) without constraints (26), (27), and (28). The first-best allocation, (q^*, e^*), solves

$$u'(q^*) = c'(q^*), \tag{31}$$

$$\left[\alpha(1/e^*) - \alpha'(1/e^*)/e^*\right][u(q^*) - c(q^*)] = \psi'(e^*). \tag{32}$$

Then from (26), we have

$$d_z^p = d_z^* \equiv u(q^*) - \psi'(e^*)/\alpha(1/e^*).$$

When $i \in [0, i^*]$, the first-best solution (q^*, e^*, d_z^*) satisfies constraints (26), (27), and (28). To verify that (26) holds, we need $i \leq \frac{e^*\psi'(e^*) - \psi(e^*)}{u(q^*) - \psi'(e^*)/\alpha(1/e^*)}$, which is satisfied if and only if $i \leq i^*$, where i^* is defined as $i^* \equiv \frac{e^*\psi'(e^*) - \psi(e^*)}{u(q^*) - \psi'(e^*)/\alpha(1/e^*)}$. Condition (27) holds by construction of $d_z^p = d^*$. For (28) to hold, we need $d_z^* \geq c(q^*)$, which is satisfied by the pairwise core requirement.

In addition, we verify that (q^*, e^*) is the unique solution to (**??**). First, given e^*, the objective

(25) is concave in q since $[u(q) - c(q)]$ is a concave function. Second, given q^*, (25) is strictly concave in e if $e\alpha(1/e)$ is strictly concave. To verify, for any $e_1 \neq e_2$ and $\lambda \in (0,1)$, we have

$$\alpha(1/\lambda e) > \alpha(1/e)$$
$$\lambda e_1 \alpha[1/(\lambda e_1)] + (1-\lambda)e_2\alpha[1/((1-\lambda)e_2)] > \lambda[e_1\alpha(1/e_1)] + (1-\lambda)[e_2\alpha(1/e_2)].$$

By definition of strict concavity, $e\alpha(1/e)$ is strictly concave, and therefore $e\alpha(1/e)[u(q^*) - c(q^*)] - \psi(e)$ is strictly concave in e. Since $e^*\alpha(1/e^*)[u(q) - c(q)] - \psi(e^*)$ is concave in q and $e\alpha(1/e)[u(q^*) - c(q^*)] - \psi(e)$ is strictly concave in e, by e.g. Theorem 21.7 in Simon and Blume (1994), (q^*, e^*) is the unique global maximizer of (25). In addition, d_z^* is uniquely determined by (27).

(2) When $i \in (i^*, \bar{i}]$ for some $\bar{i} > i^*$, we consider the maximization problem (25) subject to (26) and (27) at equality, but without (28).

Claim 1. (i) The seller's participation constraint, (28), holds with strict inequality at the optimum; (ii) the buyer's participation constraint, (26), binds for all $i > i^*$; (iii) $q^p < q^*$ for $i > i^*$.

Proof. **(i)** To show that (28) is slack at (q^*, e^*, d^*), notice that at the first-best, we have $d_z^* = u(q^*) - \psi'(e^*)/\alpha(1/e^*)$, and hence, (28) is slack if and only if

$$u(q^*) - \psi'(e^*)/\alpha(1/e^*) > c(q^*),$$

or equivalently,

$$u(q^*) - c(q^*) > \psi'(e^*)/\alpha(1/e^*).$$

Now, by the definition of e^*, we have

$$\left[\alpha(1/e^*) - \alpha'(1/e^*)/e^*\right][u(q^*) - c(q^*)] = \psi'(e^*),$$

and hence the above inequality is reduced to

$$\alpha(1/e^*) > \left[\alpha(1/e^*) - \alpha'(1/e^*)/e^*\right],$$

which is satisfied since $e^* > 0$ and $\alpha' \cdot > 0$.

(ii) To show that (26) binds for all $i > i^*$, consider the Lagrangian associated with (25), (26), (28), and (27) :

$$\begin{aligned}\mathcal{L}(q, d_z, e; \lambda, \mu, \eta) &= e\alpha(1/e)[u(q) - c(q)] - \psi(e) \\ &+ \lambda\{-id + e\alpha(\theta)[u(q) - d_z] - \psi(e)\} \\ &+ \mu\{e\alpha(1/e)[d_z - c(q)]\} \\ &+ \eta\{\psi'(e) - \alpha(\theta)[u(q) - d_z]\},\end{aligned}$$

where $\lambda \geq 0$, $\mu \geq 0$, and $\eta \geq 0$ are the Lagrange multipliers associated with (26), (28), and (27) respectively. By Theorem M.K.2 in e.g. Mas-Collel, Whinston, and Green (1995), the necessary conditions for an optimum are

$$e^p[(1+\lambda)u'(q^p) - (1+\mu)c'(q^p)] = \eta u'(q^p), \tag{33}$$

$$(1+\lambda)\psi'(e^p) - \eta\left[\psi''(e^p) + \frac{\alpha'(1/e^p)}{e^2}[u(q^p) - d_z^p]\right] \tag{34}$$

$$= \left[\alpha(1/e^p) - \frac{\alpha'(1/e^p)}{e^p}\right][(1+\lambda)u(q^p) - (1+\mu)c(q^p) + (\mu - \lambda)d^p],$$

$$i\lambda = \alpha(1/e^p)[e^p(\mu - \lambda) + \eta]. \tag{35}$$

Combining (33) and (35) yield

$$\frac{u'(q^p)}{c'(q^p)} = \frac{1}{1 - i\lambda[e^p\alpha(1/e^p)(1+\mu)]^{-1}}. \tag{36}$$

(iii) To verify that (26) binds for all $i > i^*$, i.e. $\lambda > 0$, suppose by contradiction that $\lambda = 0$. Then from (36), $q^p = q^*$, which from (33) and $\lambda = 0$ implies $-e^p\mu = \eta$. Since $e^p \geq 0$, $\mu \geq 0$, and $\eta \geq 0$, this implies $\eta = \mu = 0$, and hence $e^p = e^*$. But this implies that $i < \frac{e^*\psi'(e^*) - \psi(e^*)}{u(q^*) - \psi'(e^*)/\alpha(1/e^*)}$, or $i < i^*$, a contradiction. Hence $\lambda > 0$.

Next, we prove that $q^p < q^*$ for $i > i^*$. From (36), $q^p \neq q^*$ unless $\lambda = 0$, which is violated when $i > i^*$. Now suppose that $q^p > q^*$. Consider the following deviation: the planner could decrease q^p to q^* to increase trade surplus while changing $d_z^p \geq c(q^*)$ such that this deviation is incentive compatible. This deviation would raise welfare and is incentive feasible; a contradiction. Hence $q^p < q^*$. □

Observe that with (26) at equality and (27), the maximization problem simplifies to a choice of e for a given i. Given i and a choice of e, (26) at equality and (27) implies a unique solution for

q and d_z as functions of e and i:

$$d_z(e,i) = \frac{e\psi'(e)-\psi(e)}{i}; \quad (37)$$

$$q(e,i) = u^{-1}\left(\frac{e\psi'(e)-\psi(e)}{i}+\frac{\psi'(e)}{\alpha(1/e)}\right). \quad (38)$$

Let

$$g(e,i) \equiv \frac{e\psi'(e)-\psi(e)}{i}+\frac{\psi'(e)}{\alpha(1/e)}$$

and $f(x)=u^{-1}(x)$. Then $f'(g(e,i))=\frac{1}{u'[f(g,e))]}>0$, in which case $1-c'[f(g(e,i))]f'(g(e,i))>0$ for $q<q^*$ and $1-c'[f(g(e,i))]f'(g(e,i))=0$ for $q=q^*$. The objective function can therefore be written

$$F(e,i)=e\alpha(1/e)\left[g(e,i)-c\left[f(g(e,i))\right]\right]-\psi(e).$$

Claim 2. There exists an $\bar{i}$ such that for all $i\in[i^*,\bar{i}]$, there is a unique maximizer $e^p(i)$ for $\max_e F(e,i)$. Moreover, $\frac{d}{de}e^p(i)>0$ for all $i\in[i^*,\bar{i}]$.

Proof. The proof for $\frac{d}{di}e^p(i)>0$ applies the Implicit Function Theorem (IFT) in e.g. Theorem M.E.1 in Mas-Collel, Whinston, and Green (1995). First we show that

$$\frac{\partial^2}{\partial e^2}F(e^*,i^*)<0. \quad (39)$$

The first partial derivative of $F(e,i)$ with respect to e is

$$\begin{aligned}
\frac{\partial F(e^*,i^*)}{\partial e} &= \alpha(1/e^*)\left[g(e^*,i^*)-c\left[f(g(e^*,i^*))\right]\right]-\frac{e^*\alpha'(1/e^*)}{e^{*2}}\left[g(e^*,i^*)-c\left[f(g(e^*,i^*))\right]\right] \\
&+ e\alpha(1/e^*)\left[\frac{\partial g(e^*,i^*)}{\partial e}-c'[f(g(e^*,i^*))]f'(g(e^*,i^*))\frac{\partial g(e^*,i^*)}{\partial e}\right]-\psi'(e^*) \\
&= \left[\alpha(1/e^*)-\frac{\alpha'(1/e^*)}{e^*}\right]\left[g(e^*,i^*)-c\left[f(g(e^*,i^*))\right]\right] \\
&+ e^*\alpha(1/e^*)\frac{\partial g(e^*,i^*)}{\partial e}\underbrace{\left[1-c'[f(g(e^*,i^*))]f'(g(e^*,i^*))\right]}_{=0\text{ at }q=q^*}-\psi'(e^*) \\
&= 0.
\end{aligned}$$

The second partial derivative is

$$
\begin{aligned}
\frac{\partial^2 F(e^*,i^*)}{\partial e^2} &= \left[\alpha(1/e^*) - \frac{\alpha'(1/e^*)}{e^*}\right]\frac{\partial g(e^*,i^*)}{\partial e}\underbrace{[1-c'[f(g(e^*,i^*))]f'(g(e^*,i^*))]}_{=0 \text{ if } q=q^*} \\
&+ \underbrace{[g(e^*,i^*) - c\left[f(g(e^*,i^*))\right]]}_{(+)\text{ since } u(q^*)-c(q^*)>0} \underbrace{\frac{\alpha''(1/e^*)}{e^{*3}}}_{(-)\text{ since } \alpha''<0} - \underbrace{\psi''(e^*)}_{(+)\text{ since } \psi''>0} \\
&+ \underbrace{e\alpha(1/e^*)\frac{\partial g(e^*,i^*)}{\partial e}^2}_{(+)} \underbrace{[-c''[f(g(e^*,i^*))] - c'[f(g(e^*,i^*))f'']}_{(-)} \underbrace{f'(g(e^*,i^*))^2}_{(+)} \\
&< 0.
\end{aligned}
$$

This verifies that $\frac{\partial^2}{\partial e^2}F(e^*,i^*) < 0$.

Then, because $\frac{\partial}{\partial e}F(e^*,i^*) = 0$, by (39) and by the IFT, there is an open neighborhood $(e_0,e_1)\times(i_0,i_1)$ around (e^*,i^*) and a continuously differentiable implicit function $\omega : (i_0,i_1) \to (e_0,e_1)$ such that for all $i \in (i_0,i_1)$, $\omega(i)$ is the unique $e \in (e_0,e_1)$ such that

$$\frac{\partial}{\partial e}F(\omega(i),i) = 0.$$

Since $\frac{\partial}{\partial e}F(e^*,i^*) = 0$ and (39), we know that $\omega(i)$ is the local maximizer in the neighborhood (e_0,e_1).

To show that it is also the global maximizer, first consider $M(i) = \max_{e\notin(e_0,e_1)} F(e,i)$. By the Theorem of the Maximum in e.g. Stokey and Lucas (1989), $M(i)$ is continuous and $M(i^*) < F(e^*,i^*)$. Let $\delta = F(e^*,i^*) - M(i^*) > 0$. Then by continuity, there exists am $i_2 \in (i^*,i_1]$ such that if $i \in [i^*,i_2]$, then $M(i) \le M(i^*) + \delta/3 < F(e^*,i^*) - \delta/3 \le F(\omega(i),i)$. Hence, for all $i \in [i^*,i_2]$, $\omega(i)$ maximizes $F(\cdot,i)$. Moreover, because the function $-c(q(e,i)) + d(e,i)$ is continuous and because $-c(q^*)+d^* \ge 0$, it follows from continuity that there exists an $i_3 \in (i^*,i_2]$ such that for all $i \in (i^*,i_3]$, $-c(q(\omega(i),i)) + d(\omega(i),i) \ge 0$. Thus, for each $i \in (i^*,i_3]$, $(q^p,e^p,d^p) = (q(\omega(i),i),\omega(i),d(\omega(i),i))$ is the constrained efficient outcome.

Finally, by IFT again, $\omega(i)$ is continuously differentiable and there exists an $\bar{i} \in (i^*,i_3]$ such that for all $i \in (i^*,\bar{i}]$,

$$\omega'(i) = -\frac{\partial^2}{\partial e\partial i}F(\omega(i),i)/\frac{\partial^2}{\partial e^2}F(\omega(i),i) > 0.$$

This result follows directly from the continuity of the derivative and by the IFT that the first-order effect of i on e at i^* and e^* is given by

$$\omega'(i^*) = -\frac{\partial^2}{\partial e \partial i} F(e^*, i^*) / \frac{\partial^2}{\partial e^2} F(e^*, i^*) > 0.$$

Because we have shown that $\frac{\partial^2}{\partial e \partial i} F(e^*, i^*) < 0$, it suffices to show that

$$\frac{\partial^2}{\partial e \partial i} F(e^*, i^*) > 0. \tag{40}$$

To compute (40), we differentiate to obtain

$$\begin{aligned}
\frac{\partial^2}{\partial e \partial i} F(e^*, i^*) = {} & \alpha(1/e^*) \frac{\partial}{\partial i} g(e^*, i^*) \underbrace{\{1 - c'[f(g(e^*, i^*))] f'(g(e^*, i^*))\}}_{= 0 \text{ if } q = q^*} \\
& + e^* \alpha'(1/e^*)(-1/e^{*2}) \frac{\partial}{\partial i} g(e^*, i^*) \underbrace{\{1 - c'[f(g(e^*, i^*))] f'(g(e^*, i^*))\}}_{= 0 \text{ if } q = q^*} \\
& + e^* \alpha(1/e^*) \frac{\partial^2}{\partial e \partial i} g(e^*, i^*) \underbrace{\{1 - c'[f(g(e^*, i^*))] f'(g(e^*, i^*))\}}_{= 0 \text{ if } q = q^*} \\
& + e^* \alpha(1/e^*) \frac{\partial}{\partial i} g(e^*, i^*) \{-c''[f(g(e^*, i^*))][f'(g(e^*, i^*))]^2 - c'[f(g(e^*, i^*))] f''\} \frac{\partial}{\partial e} g(e^*, i^*) \\
= {} & e^* \alpha(1/e^*) \underbrace{\frac{\partial}{\partial i} g(e^*, i^*)}_{(-)} \{- \underbrace{c''[f(g(e^*, i^*))][f'(g(e^*, i^*))]^2}_{(+)} - \underbrace{c'[f(g(e^*, i^*))] f''}_{(+)}\} \underbrace{\frac{\partial}{\partial e} g(e^*, i^*)}_{(+)} \\
> {} & 0.
\end{aligned}$$

This verifies that $\frac{\partial^2}{\partial e \partial i} F(e^*, i^*) > 0$. Hence, for $i \in (i^*, \bar{i}]$, $\frac{d}{di} e^p(i) > 0$ and by continuity of $\omega(i)$, $e^p > e^*$.

Finally, to show that $d^p < d^*$ for $i \in (i^*, \bar{i}]$, we have from (27),

$$d^p = u(q^p) - \psi'(e^p)/\alpha(1/e^p) < u(q^*) - \psi'(e^*)/\alpha(1/e^*) = d^*,$$

since $q^p < q^*$ and $e^p > e^*$. □

Claim 3. For all $i \in [i^*, \bar{i}]$, $e^p(i)$ solves $\max_e F(e, i)$ if and only if $(q(e^p(i)), e^p(i), d(e^p(i), i))$ solves (25) to (28).

(4) By (27), $d^p = u(q^p) - \psi'(e^p)/\alpha(1/e^p)$. Thus, we may rewrite (26) and (28) as

$$\frac{\psi'(e^p)}{\alpha(1/e^p)} + \frac{e^p\psi'(e^p) - \psi(e^p)}{i} \geq u(q^p), \tag{41}$$

$$u(q^p) - c(q^p) \geq \frac{\psi'(e^p)}{\alpha(1/e^p)}. \tag{42}$$

Let $\bar{\bar{e}}$ be defined such that

$$\frac{\psi'(\bar{\bar{e}}))}{\alpha(1/\bar{\bar{e}})} = u(q^*) - c(q^*).$$

Then,

$$\frac{\psi'(e^p)}{\alpha(1/e^p)} \leq u(q^p) - c(q^p) \leq u(q^*) - c(q^*) = \frac{\psi'(\bar{\bar{e}}^p)}{\alpha(1/\bar{\bar{e}}^p)}.$$

Hence for any i and for any constrained-efficient outcome, $e^p(i) \leq \bar{\bar{e}}$ since $\frac{\psi'(e^p)}{\alpha(1/e^p)}$ is increasing in e^p.

Fix some $e \in (0, \bar{\bar{e}}]$. Define q_e by

$$u(q_e) - c(q_e) = \psi'(e)/\alpha(1/e).$$

Since $\psi'(e)/\alpha(1/e)$ is increasing in e and continuous for $e \in (0, 1]$, it follows that $q_e \in (0, q^*]$ is uniquely determined and varies continuously in e. Let

$$i(e) = \frac{e\psi'(e) - \psi(e)}{u(q_e) - \psi'(e)/\alpha(1/e)}.$$

Then, $i(e) \in (0, \infty)$ and is continuous in e. Now we show that if $i > i(e)$, then (e, q) does not satisfy (41) and (42) w.r.t. i for any q. Suppose, by contradiction, that (e, q) satisfies (41) and (42) w.r.t. i. Then, by (42),

$$u(q) - c(q) \geq \psi'(e)/\alpha(1/e) = u(q_e) - c(q_e),$$

and hence, $q \geq q_e$. But by (41),

$$u(q) \leq \frac{\psi'(e)}{\alpha(1/e)} + \frac{e\psi'(e) - \psi(e)}{i} < \frac{\psi'(e)}{\alpha(1/e)} + \frac{e\psi'(e) - \psi(e)}{i(e)} = u(q_e),$$

which implies that $q < q_e$, a contradiction.

Finally, for each $e \in (0, \bar{\bar{e}}]$, let

$$i_e = \max\{i(e') : e' \in [e, \bar{\bar{e}}]\}.$$

Notice that i_e is well-defined because $i(e)$ is continuous and $[e, \bar{\bar{e}}]$ is a compact set. Now, if $i > i_e$, then for any $e' \in [e, \bar{\bar{e}}]$, $i > i_{e'}$ and hence (e', q) does not satisfy (41) and (42) w.r.t. i for any q. Thus, $e^p(i) < e$.

(4) (*Minor comment: need to redefine f and g since these variables already used*) Let $f(e) = \frac{\psi'(e)}{\alpha(1/e)}$ and let $g(e,i) = \frac{e\psi'(e)-\psi(e)}{i}$. Then, $f' > 0$, $f(0) = 0$, $f(1) = \infty$, $g_e > 0$, $g(0,i) = 0$, and $g(1,i) = \infty$.

Let $\tilde{q}(e,i) = u^{-1}(f(e) + g(e,i))$. Then, (q,e) satisfies (41) and (42) if $q = \tilde{q}(e,i)$ and if

$$u(q) - c(q) \geq f(e),$$

that is, if

$$f(e) + g(e,i) - c \circ u^{-1}(f(e) + g(e,i)) \geq f(e),$$

which is equivalent to

$$c^{-1} \circ u(g(e,i)) \geq f(e) + g(e,i).$$

First we show that for e close to 1,

$$c^{-1} \circ u(g(e,i)) < f(e) + g(e,i). \tag{43}$$

To see this, notice that because $c^{-1} \circ u$ is concave and because of the Inada conditions,

$$\lim_{e \to 1} \frac{c^{-1} \circ u(g(e,i))}{g(e,i)} = 0$$

and hence, for e sufficiently close to 1,

$$\frac{c^{-1} \circ u(g(e,i))}{g(e,i)} < 1 < 1 + \frac{f(e)}{g(e,i)}.$$

Now we show that for e sufficiently small,

$$c^{-1} \circ u(g(e,i)) > f(e) + g(e,i). \tag{44}$$

First notice that $g_e(e,i) = e\psi''(e)/i$ and

$$f'(e) = \frac{\psi''(e)\alpha(1/e) + \psi'(e)\alpha'(1/e)/e^2}{\alpha(1/e)^2}.$$

Let $\psi''(0) = A$. Then,

$$\lim_{e\to 0} f'(e) = \frac{A + (\psi'(e)/e)(\alpha'(1/e)/e)}{\alpha(1/e)^2} \in [A, 2A],$$

where $\lim_{e\to 0}\alpha(1/e) = 1$, $\lim_{e\to 0}\alpha'(1/e)/e \leq \lim_{e\to 0}\alpha(1/e) = 1$, and $\lim_{e\to 0}\psi'(e)/e = \psi''(0) = A$. Thus, for sufficiently small e, $g_e(e,i) \in (eA/2i, 2Ae/i)$ and $f'(e) \in (A/2, 4A)$. Thus, for such e's,

$$g(e,i) \in ((A/4i)e^2, (A/i)e^2) \text{ and } f(e) \in ((A/2)e, (4A)e),$$

and hence, for such e's,

$$g(e,i) + f(e) < (4A)e + (A/i)e^2.$$

However, for sufficiently small q, $(c^{-1} \circ u)'(q) > Kq^{-0.5-\delta}$ for some $K > 0$, and hence $c^{-1} \circ u(q) > Kq^{0.5-\delta}$ for all such q's. Thus, for e sufficiently small,

$$c^{-1} \circ u(g(e,i)) \geq Kg(e,i)^{0.5-\delta} > ((A/4i)e^2)^{0.5-\delta} \equiv Le^{1-\delta}.$$

Because $\lim_{e\to 0} \frac{Le^{1-\delta}}{(4A)e+(A/i)e^2} = \infty$, it follows that, for e sufficiently small,

$$c^{-1} \circ u(g(e,i)) > Le^{1-\delta} > (4A)e + (A/i)e^2 \geq f(e) + g(e,i).$$

This proves (44).

Now, by (43) and (44), and by the Intermediate Value Theorem, there exists $\tilde{e}_i > 0$ such that

$$g(\tilde{e}_i, i) = c \circ u^{-1}(f(\tilde{e}_i) + g(\tilde{e}_i, i)).$$

Then, $(\tilde{q}(\tilde{e}_i, i), \tilde{e}_i)$ satisfies (41) and (42). Moreover, it has positive welfare:

$$\alpha(1/\tilde{e}_i)[u(\tilde{q}(\tilde{e}_i, i)) - c(\tilde{q}(\tilde{e}_i, i))] = \psi'(\tilde{e}_i)$$

and hence

$$\tilde{e}_i\alpha(1/\tilde{e}_i)[u(\tilde{q}(\tilde{e}_i, i)) - c(\tilde{q}(\tilde{e}_i, i))] - \psi(\tilde{e}_i) = \tilde{e}_i\psi'(\tilde{e}_i) - \psi(\tilde{e}_i) > 0.$$