Geometric Mean Maximization: *An Overlooked Portfolio Approach?*

JAVIER ESTRADA

JAVIER ESTRADA
is professor and head of the Department of Financial Management at the IESE Business School in Barcelona, Spain.
jestrada@iese.edu

Academics have long advocated optimizing portfolios on the basis of mean and variance. Practitioners followed this recommendation and adopted the maximization of risk-adjusted returns, measured by the Sharpe ratio, as their basic criterion for portfolio selection. Obviously, there is an inherent plausibility in selecting the portfolio that provides the highest (excess) return per unit of volatility risk.

However, there is an alternative criterion that seems to be less popular with academics and practitioners that consists of maximizing the growth of the capital invested, thus maximizing terminal wealth. This criterion, which amounts to maximizing a portfolio's geometric mean return (or mean compound return) in principle appears to be at least just as plausible as the maximization of risk-adjusted returns. The ultimate goal of this article is to compare both criteria from an empirical perspective.

Markowitz [1952, 1959] was the first to advocate the focus on mean and variance and the selection of portfolios with the lowest risk (volatility) for a target level of return or the highest return for a target level of risk. Investors, however, find risk-adjusted returns difficult to digest. Few of them focus on the Sharpe ratio of their investments; rather, they tend to focus on whether or not their invested capital grows and the rate at which it does. Furthermore, fund management companies tend to summarize performance with the mean compound return of their funds. For both reasons, then, a potential plausible goal for portfolio managers to adopt would be to grow the capital entrusted to them at the fastest possible rate; that is, to maximize the geometric mean return of their portfolios.

At least two questions arise from this discussion. First, is the portfolio that grows at the fastest rate the one that yields the highest risk-adjusted returns? In general, that is not the case. Second, given that the portfolio that maximizes the geometric mean return and the one that maximizes the Sharpe ratio are in general different, which of the two is more attractive? Providing an empirical perspective on this question is one of the main goals of this article.

The main results can be summarized as follows. The analysis of in-sample optimizations shows that portfolios built with the goal of maximizing the growth of the capital invested are less diversified, have a higher expected return, and higher volatility than those built with the goal of maximizing risk-adjusted returns. The analysis of out-of-sample performance largely confirms the in-sample results but also provides some additional interesting insights; in particular, although portfolios built to maximize the growth of the capital invested tend to achieve their goal, those built to maximize risk-adjusted returns often do not. All in all, the results reported here suggest that academics and practitioners may be overlooking a compelling portfolio optimization criterion.

The criterion at the heart of this article has been variously referred to in the literature as the Kelley criterion, the growth optimal portfolio, the capital growth theory of investment, the geometric mean strategy, investment for the long run, maximum expected log, and herein as *geometric mean maximization* (GMM). The standard criterion accepted by academics and practitioners will be referred to in this article as Sharpe ratio maximization (SRM). Furthermore, the optimal portfolios that result from the GMM and the SRM criteria will be respectively referred to as the *G* portfolio and the *S* portfolio.

THE ISSUE

The GMM criterion is inevitably intertwined with the gambling strategy developed by Kelly [1956], who considered a gambler having noisy private information and making a large number of bets (each being a constant proportion of his capital) with cumulative effects (reinvesting gains and losses). In this setting, Kelly derived the optimal proportion of capital a gambler should bet on each round if he aimed to maximize his expected terminal wealth.

There is a vast gambling literature on the Kelly criterion that highlights some properties of Kelly's betting scheme that are shared by the GMM criterion. These properties include that the gambler never risks ruin; terminal wealth is very likely to be higher than with any other strategy; the bets may be very aggressive; the ride may be very bumpy; and betting more (less) than *K* increases (decreases) risk and decreases the growth of capital.

At the same time that Kelly was developing a gambler's optimal betting strategy, Latane [1959] was independently considering the optimal decision of an individual facing a large number of uncertain and cumulative choices. In this setting, he argued that the optimal choice of an individual aiming to maximize his expected terminal wealth was the strategy with the highest probability of leading to more terminal wealth than any other strategy. He also argued that such strategy would be the one with the highest geometric mean return.

Importantly, note that the SRM criterion is a *one*-period framework; in contrast, the GMM criterion is a *multiperiod* framework with *cumulative* results, which is consistent with the way most investors think about their portfolios. This distinction is critical because optimal decisions for a single period may be suboptimal in a multiperiod framework;[1] and the target variable in a cumulative framework (the geometric mean) is different from the target variable when gains and losses are not reinvested (the arithmetic mean).[2]

Although GMM was proposed as an alternative to mean–variance optimization, curiously, one of its strongest early supporters was Harry Markowitz. In fact, not only did he allocate the entire Chapter VI of his pioneering book (Markowitz [1959]) to "Return in the Long Run" but he also added a "Note on Chapter VI" in a later edition. Markowitz [1976] subsequently reaffirmed his support for the GMM criterion.

Samuelson [1971] admits as an obvious truth that maximizing the geometric mean return would almost certainly lead to the maximization of terminal wealth and utility if the period considered is sufficiently long, but he warns against believing in the false corollary that such a strategy would maximize expected utility (unless the underlying utility function were logarithmic), and perhaps surprisingly given his strong opposition to the criterion, he concludes that GMM "still avoids some of the even greater arbitrariness of conventional mean–variance analysis."

A more comprehensive review is beyond the scope of this article, but Christensen [2005] and Poundstone [2005] provide thorough accounts of the origins and evolution of the GMM criterion, the former from a theoretical perspective and the latter through a fascinating and entertaining story. Table 1 in MacLean, Ziemba, and Blazenko [1992] summarizes many desirable and undesirable analytical properties of the GMM criterion and provides references for each of those properties. McEnally [1986] provides a good overview of this criterion and some of the controversies surrounding it.

Empirical research on the GMM criterion has been rather scarce, which is precisely one of the voids this article attempts to fill. The earliest empirical contribution appears to be that of Roll [1973], who derives from theory some testable implications of this criterion; tests them using a sample of NYSE and AMEX stocks; and finds that the *G* portfolio is statistically indistinguishable from the market portfolio. Similarly, Fama and McBeth [1974], using a sample of NYSE stocks, cannot reject the hypothesis that over different time periods the *G* portfolio is *statistically* indistinguishable from the market portfolio. However, they find substantial *economic* differences between them, with the G portfolio having a much higher (geometric mean) return and (beta) risk than the market portfolio.

Grauer [1981] compares the portfolios selected by the GMM and SRM criteria. Using a sample of 20 Dow stocks and 20 NYSE portfolios, he runs over 200 optimizations and finds that these two criteria rarely select the same asset mix; that neither criterion produces highly diversified portfolios; that G portfolios are less diversified than S portfolios; and that G portfolios have higher expected return and volatility than S portfolios.

Finally, Hunt [2005a, 2005b], using a sample of 25 Australian stocks in the first case and the 30 stocks in the Dow in the second case, finds that unconstrained G portfolios are largely unattainable (implying short positions in excess of 1000% in some cases), and that long-only G portfolios are less diversified, have over twice the geometric mean return, and almost twice the volatility than do his three benchmark portfolios (the equally weighted portfolio, the minimum variance portfolio, and the portfolio with minimum risk for a target return of 15%).

METHODOLOGY

Standard modern portfolio theory establishes that the expected return (μ_p) and variance (σ_p^2) of a portfolio are given by

$$\mu_p = \sum_{i=1}^{n} x_i \mu_i \tag{1}$$

$$\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \sigma_{ij} \tag{2}$$

where x_i denotes the proportion of the portfolio invested in asset i; μ_i the expected return of asset i; σ_{ij} the covariance between assets i and j; and n the number of assets in the portfolio.

Maximizing risk-adjusted returns when risk is measured by volatility amounts to maximizing a portfolio's Sharpe ratio (SR_p). This problem is formally given by

$$\text{Max}_{x_1, x_2, \ldots, x_n} \quad SR_p = \frac{\mu_p - R_f}{\sigma_p} = \frac{\Sigma_{i=1}^{n} x_i \mu_i - R_f}{\sqrt{\Sigma_{i=1}^{n} \Sigma_{j=1}^{n} x_i x_j \sigma_{ij}}} \tag{3}$$

$$\text{Subject to} \sum_{i=1}^{n} x_i = 1 \quad \text{and} \quad x_i \geq 0 \ \text{for all } i \tag{4}$$

where R_f denotes the risk-free rate and $x_i \geq 0$ the no short-selling constraint. This is the formal expression of the criterion referred to in this article as SRM. The solution of this problem is well known and available from a wide variety of optimization packages.

The maximization of a portfolio's geometric mean return can be implemented in more than one way. Ziemba [1972], Elton and Gruber [1974], Weide, Peterson, and Maier [1977], and Bernstein and Wilkinson [1997] all propose different algorithms to solve this problem. The method proposed here is easy to implement numerically and requires the same inputs as those needed to maximize a portfolio's Sharpe ratio. Maximizing a portfolio's geometric mean return (GM_p) amounts to solving the problem formally given by

$$\text{Max}_{x_1, x_2, \ldots, x_n} \quad GM_p \approx \exp\left\{\ln(1+\mu_p) - \frac{\sigma_p^2}{2(1+\mu_p)^2}\right\} - 1 \approx \exp\left\{\ln\left(1+\sum_{i=1}^{n} x_i \mu_i\right) - \frac{\Sigma_{i=1}^{n} \Sigma_{j=1}^{n} x_i x_j \sigma_{ij}}{2(1+\Sigma_{i=1}^{n} x_i \mu_i)^2}\right\} - 1 \tag{5}$$

$$\text{Subject to} \sum_{i=1}^{n} x_i = 1 \quad \text{and} \quad x_i \geq 0 \ \text{for all } i \tag{6}$$

This is the formal expression of the criterion referred to in this article as GMM. Note that maximizing Expression (5) is obviously the same as maximizing the expression inside the brackets. In fact, Markowitz [1959] suggests approximating the geometric mean of an asset precisely with the expression $\{\ln(1 + \mu) - \sigma^2/[2(1 + \mu)^2]\}$.

Furthermore, note that Expression (5) highlights an important fact about the role that volatility plays in the GMM framework. In the SRM framework, volatility is undesirable because it is synonymous with risk; in the GMM framework, volatility is also undesirable but for a different reason, namely, because *it lowers the geometric mean return*. In other words, in the GMM framework volatility is detrimental because it lowers the rate of growth of the capital invested, thus ultimately lowering the expected terminal wealth.

EVIDENCE

This section compares the two optimization criteria considered in this article, GMM and SRM, from an empirical perspective. The sample consists of monthly returns for the entire MSCI database of 22 developed markets and 26 emerging markets, as well as monthly returns for five asset classes, namely, U.S. stocks, EAFE stocks, emerging market stocks, U.S. bonds, and U.S. real estate. All returns are in dollars and account for both capital gains and dividends. The sample period varies across assets and goes from the inception of each asset into the MSCI database through June 2008 in all cases. Exhibit A1 in Appendix A describes the data.

In-Sample Optimal Portfolios

The first step of the analysis consists of comparing the characteristics of the portfolios generated by the GMM and the SRM criteria. In order to gain a broad perspective, *G* and *S* portfolios are obtained for developed markets, emerging markets, and asset classes. Furthermore, in order to avoid drawing conclusions possibly biased by the conditions at a single point in time, *G* and *S* portfolios are obtained at three points in time, June 2008, June 2003, and June 1998. Exhibits 1 through 3 report the relevant results. In all cases *S* portfolios follow from Expressions (3) and (4) and *G* portfolios from Expressions (5) and (6); also, in all cases, the inputs of the optimization problems (expected returns, variances, and covariances) are calculated using all the data available at the time of each estimation.

Exhibit 1 focuses on developed markets; Panel A shows the composition of all portfolios and Panel B summarizes their characteristics. As the exhibit shows, the portfolios generated by the GMM criterion are substantially less diversified than those generated by the SRM criterion; at all three points in time the *S* portfolios contain at least twice as many assets (developed markets) as the *G* portfolios. And, as the exhibit also shows, the lower diversification of the *G* portfolios makes them much more volatile than the *S* portfolios.

By design, *G* portfolios have a higher geometric mean return and lower Sharpe ratios than *S* portfolios, and both characteristics are reflected in the figures displayed in Exhibit 1. The higher expected growth of *G* portfolios translates into substantially higher levels of expected terminal wealth. The last two lines of the exhibit show the expected terminal value of $100 invested at the geometric mean return of each portfolio after 10 and 20 years. As the figures clearly show, the differences are substantial; the expected terminal values of *G* portfolios (relative to *S* portfolios) are at least 20% higher for a 10-year holding period and at least 40% higher for a 20-year holding period.

The results for developed markets in Exhibit 1 are confirmed and strengthened by those for emerging markets in Exhibit 2. As this exhibit shows, *G* portfolios (relative to *S* portfolios) are less diversified, more volatile, and have lower Sharpe ratios; at the same time, they have higher expected (arithmetic and geometric) return and expected terminal values.

The main difference between Exhibits 1 and 2 is simply one of degree; that is, the differences between *G* portfolios and *S* portfolios are amplified in the case of emerging markets relative to those discussed before for developed markets. To illustrate, in the case of emerging markets, *G* portfolios have (relative to *S* portfolios) at least twice the volatility, at least twice the expected terminal wealth over 10 years, and at least four times the expected terminal wealth over 20 years. As shown in Exhibit 1, these differences are substantially lower for developed markets.

Finally, Exhibit 3 shows the results for portfolios of asset classes, and again the results confirm and strengthen those of the previous two exhibits. One of the interesting results of Exhibit 3 is the possible extreme concentration of *G* portfolios, which contain only one asset class in June 2008 and June 1998 and two in June 2003; *S* portfolios, in contrast, contain four asset classes in all cases. The rest of the relative characteristics of *G* and *S* portfolios in this exhibit are quantitatively different but qualitatively the same, as with those discussed before for developed and emerging markets.

The results in Exhibits 1–3 are in general consistent with those previously reported in the few articles that explore the empirical characteristics of *G* portfolios. In particular, they are consistent with the results in Grauer [1981], who reports that *G* and *S* portfolios usually have a substantially different asset mix. They are also consistent with the results in Hunt [2005a, 2005b], who reports that *G* portfolios are less diversified, have a much higher return, and much higher volatility than *S* portfolios.

Leverage

The results in the previous section show that, as expected by design, *G* portfolios exhibit higher expected growth and terminal wealth than *S* portfolios. Still, however desirable these two characteristics may be, at least two arguments may be raised against adopting the GMM

EXHIBIT 1
In-Sample Optimal Portfolios: Developed Markets

	June 2008		June 2003		June 1998	
	S	G	S	G	S	G
Panel A: Portfolio Weights						
Australia	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Austria	13.8%	0.0%	7.7%	0.0%	0.0%	0.0%
Belgium	10.6%	0.0%	16.7%	2.2%	17.9%	0.0%
Canada	3.6%	0.0%	0.0%	0.0%	0.0%	0.0%
Denmark	33.0%	12.3%	22.3%	0.0%	11.4%	0.0%
Finland	0.6%	9.2%	0.0%	12.4%	2.2%	0.0%
France	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Germany	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Hong Kong	11.8%	56.1%	11.9%	60.7%	7.1%	49.8%
Ireland	0.0%	0.0%	0.0%	0.0%	18.4%	10.8%
Italy	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Japan	3.7%	0.0%	5.3%	0.0%	0.0%	0.0%
Netherlands	3.2%	0.0%	3.9%	0.0%	20.2%	8.3%
New Zealand	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Norway	1.0%	5.5%	0.0%	0.0%	0.0%	0.0%
Portugal	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Singapore	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Spain	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Sweden	12.4%	16.9%	12.8%	24.7%	10.7%	31.1%
Switzerland	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
U.K.	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
U.S.	6.2%	0.0%	19.4%	0.0%	12.0%	0.0%
Panel B: Portfolio Characteristics						
n	11	5	8	4	8	4
μ_p	1.3%	1.6%	1.2%	1.6%	1.4%	1.8%
σ_p	4.4%	7.1%	4.3%	8.0%	4.1%	7.0%
SR_p	0.220	0.179	0.208	0.166	0.242	0.187
GM_p	1.2%	1.4%	1.1%	1.3%	1.4%	1.5%
Annualized σ_p	15.1%	24.8%	15.0%	27.7%	14.3%	24.2%
Annualized GM_p	15.2%	17.5%	14.0%	16.8%	17.6%	19.7%
TV10	$413	$502	$370	$473	$505	$604
TV20	$1,709	$2,516	$1,372	$2,240	$2,549	$3,653

Notes: Panel A shows the weight of each country in the optimal portfolios, and Panel B shows some of their characteristics, including the number of markets in each portfolio (n), arithmetic mean return (μ_p), volatility (σ_p), Sharpe ratio (SR_p), geometric mean return (GM_p), and the terminal value of $100 invested at GM_p after 10 years (TV10) and 20 years (TV20). Mean returns, volatility, and Sharpe ratios in Panel B are monthly magnitudes. The monthly risk-free rates used are 0.33% (June 2008), 0.29% (June 2003), and 0.44% (June 1998). The data are described in Exhibit A1.

criterion. First, it may be argued that it is possible to invest in a levered *S* portfolio with the same level of risk and higher expected return than the *G* portfolio. Second, it may be argued that it is possible to invest in a levered *S* portfolio with lower risk than and the same expected growth as the *G* portfolio. Both arguments are considered in this section.

To illustrate both issues, consider Exhibit 4, which focuses on asset classes in June 2008. The securities market line depicted has an intercept of 0.33% (the risk-free rate) and a slope of 0.267 (the Sharpe ratio). As shown in

EXHIBIT 2
In-Sample Optimal Portfolios: Emerging Markets

	June 2008		June 2003		June 1998	
	S	**G**	**S**	**G**	**S**	**G**
Panel A: Portfolio Weights						
Argentina	1.9%	18.4%	3.6%	23.0%	2.1%	24.5%
Brazil	3.4%	30.4%	5.1%	19.6%	2.8%	14.3%
Chile	1.7%	0.0%	14.8%	0.0%	4.9%	0.0%
China	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Colombia	2.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Czech Republic	11.1%	0.0%	0.0%	0.0%	0.0%	0.0%
Egypt	11.7%	21.5%	0.0%	0.0%	0.0%	0.0%
Hungary	0.0%	0.0%	0.0%	0.0%	11.4%	34.8%
India	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Indonesia	0.0%	0.0%	1.0%	0.0%	0.0%	0.0%
Israel	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Jordan	8.2%	0.0%	0.0%	0.0%	0.0%	0.0%
Korea	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Malaysia	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Mexico	12.7%	0.0%	21.1%	2.4%	8.4%	0.0%
Morocco	42.3%	0.0%	47.8%	0.0%	69.7%	0.0%
Pakistan	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Peru	2.1%	0.0%	0.0%	0.0%	0.0%	0.0%
Philippines	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Poland	0.1%	0.0%	2.3%	15.1%	0.8%	26.4%
Russia	2.8%	29.7%	1.1%	39.9%	0.0%	0.0%
South Africa	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Sri Lanka	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Taiwan	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Thailand	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Turkey	0.0%	0.0%	3.1%	0.0%	0.0%	0.0%
Panel B: Portfolio Characteristics						
n	12	4	9	5	7	4
μ_p	1.9%	2.9%	1.5%	2.9%	2.7%	3.8%
σ_p	4.0%	9.9%	4.3%	12.6%	3.3%	10.9%
SR_p	0.388	0.263	0.274	0.211	0.681	0.311
GM_p	1.8%	2.5%	1.4%	2.2%	2.7%	3.3%
Annualized σ_p	13.7%	34.3%	15.0%	43.7%	11.5%	37.7%
Annualized GM_p	23.7%	33.8%	17.9%	29.5%	37.0%	46.9%
TV10	\$840	\$1,837	\$520	\$1,331	\$2,333	\$4,673
TV20	\$7,062	\$33,733	\$2,705	\$17,705	\$54,428	\$218,397

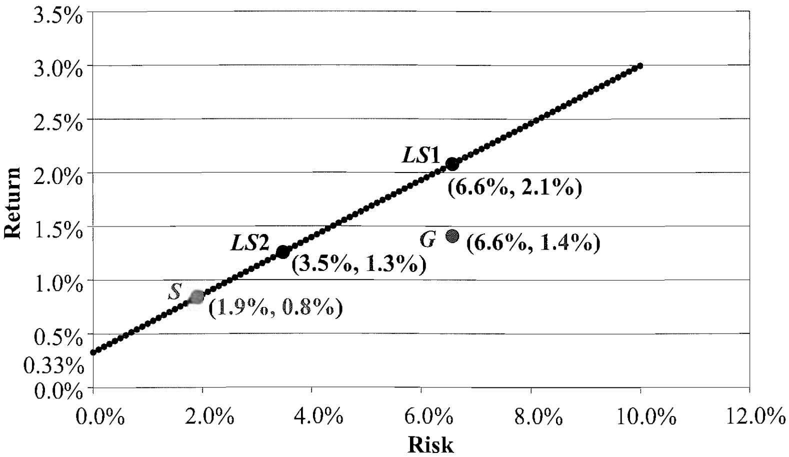
Notes: This exhibit shows optimal portfolios of emerging markets and some of their characteristics. The optimizations are performed in June 2008, June 2003, and June 1998 based on all the data available at each point in time. S portfolios are obtained from expressions (3)-(4) and G portfolios from expressions (5)-(6). Panel A shows the weight of each country in the optimal portfolios, and panel B shows some of their characteristics, including the number of markets in each portfolio (n), arithmetic mean return (μ_p), volatility (σ_p), Sharpe ratio (SR_p), geometric mean return (GM_p), and the terminal value of \$100 invested at GM_p after 10 years (TV10) and 20 years (TV20). Mean returns, volatility, and Sharpe ratios in panel B are monthly magnitudes. The monthly risk-free rates used are 0.33% (June 2008), 0.29% (June 2003), and 0.44% (June 1998). The data are described on Exhibit A1 in Appendix A.

EXHIBIT 3
In-Sample Optimal Portfolios: Asset Classes

	June 2008		June 2003		June 1998	
	S	G	S	G	S	G
Panel A: Portfolio Weights						
U.S. Stocks	4.8%	0.0%	9.3%	77.4%	75.7%	100.0%
EAFE Stocks	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
EM Stocks	16.9%	100.0%	7.7%	22.6%	4.0%	0.0%
U.S. Bonds	64.8%	0.0%	68.7%	0.0%	16.6%	0.0%
U.S. Real Estate	13.5%	0.0%	14.3%	0.0%	3.7%	0.0%
Panel B: Portfolio Characteristics						
n	4	1	4	2	4	1
μ_p	0.8%	1.4%	0.8%	1.1%	1.4%	1.5%
σ_p	1.9%	6.6%	1.8%	4.4%	3.0%	3.5%
SR_p	0.267	0.165	0.299	0.174	0.321	0.318
GM_p	0.8%	1.2%	0.8%	1.0%	1.3%	1.5%
Annualized σ_p	6.6%	22.8%	6.3%	15.2%	10.3%	12.0%
Annualized GM_p	10.3%	15.3%	10.3%	12.1%	17.5%	19.3%
TV10	$266	$416	$266	$314	$500	$586
TV20	$707	$1,732	$705	$985	$2,496	$3,437

Notes: This exhibit shows optimal portfolios of asset classes and some of their characteristics. The optimizations are performed in June 2008, June 2003, and June 1998 based on all the data available at each point in time. S portfolios are obtained from expressions (3)-(4) and G portfolios from expressions (5)-(6). Panel A shows the weight of each asset class in the optimal portfolios, and panel B shows some of their characteristics, including the number of asset classes in each portfolio (n), arithmetic mean return (μ_p), volatility (σ_p), Sharpe ratio (SR_p), geometric mean return (GM_p), and the terminal value of $100 invested at GM_p after 10 years (TV10) and 20 years (TV20). Mean returns, volatility, and Sharpe ratios in panel B are monthly magnitudes. The monthly risk-free rates used are 0.33% (June 2008), 0.29% (June 2003), and 0.44% (June 1998). The data are described on Exhibit A1 in Appendix A.

EXHIBIT 4
Leverage: Asset Classes, June 2008



Notes: This exhibit shows the securities market line for asset classes in June 2008; its intercept is given by the risk-free rate (0.33%) and its slope by the Sharpe ratio (0.267). It also shows the risk and return of the portfolios that result from maximizing the Sharpe ratio (S) and the geometric mean return (G) as well as those of a levered portfolio that results from levering S to obtain the same level of risk than that of the G portfolio (LS1), and another levered portfolio that results from levering S to obtain the same growth (geometric mean) as that of the G portfolio (LS2).

Exhibit 3, the *S* portfolio has an expected return of 0.8% and a volatility of 1.9%; the *G* portfolio, in turn, has an expected return of 1.4% and a volatility of 6.6%. These two portfolios are emphasized in Exhibit 4 and their (arithmetic and geometric) mean return, volatility, and Sharpe ratio are reported again in Exhibit 5.

Consider first the *LS*1 portfolio, which is a levered *S* portfolio designed to match the volatility of the *G* portfolio. It could be argued that *LS*1 dominates *G* because, at the same level of risk, it has a higher expected return (which in turn implies, as the figures in Exhibit 5 confirm, that it also has a higher Sharpe ratio and higher expected growth). However, as the last line of Exhibit 5 shows, *LS*1 would require going long 344% the *S* portfolio (and short 244% the risk-free rate), a level of leverage nearly impossible to obtain for many investors. In other words, investors that can bear the volatility of the *G* portfolio will find that *LS*1 is a better choice; but they may also find that this better choice is not attainable.

EXHIBIT 5
Leverage

	June 2008				June 2003				June 1998			
	S	G	LS1	LS2	S	G	LS1	LS2	S	G	LS1	LS2
DMs												
μ_p	1.3%	1.6%	1.9%	1.5%	1.2%	1.6%	2.0%	1.5%	1.4%	1.8%	2.1%	1.6%
σ_p	4.4%	7.1%	7.1%	5.3%	4.3%	8.0%	8.0%	5.6%	4.1%	7.0%	7.0%	4.9%
SR_p	0.220	0.179	0.220	0.220	0.208	0.166	0.208	0.208	0.242	0.187	0.242	0.242
GM_p	1.2%	1.4%	1.6%	1.4%	1.1%	1.3%	1.6%	1.3%	1.4%	1.5%	1.9%	1.5%
x_S			164.0%	121.7%			185.5%	130.0%			168.8%	118.6%
EMs												
μ_p	1.9%	2.9%	4.2%	2.6%	1.5%	2.9%	3.7%	2.5%	2.7%	3.8%	7.9%	3.3%
σ_p	4.0%	9.9%	9.9%	5.9%	4.3%	12.6%	12.6%	8.0%	3.3%	10.9%	10.9%	4.3%
SR_p	0.388	0.263	0.388	0.388	0.274	0.211	0.274	0.274	0.681	0.311	0.681	0.681
GM_p	1.8%	2.5%	3.7%	2.5%	1.4%	2.2%	3.0%	2.2%	2.7%	3.3%	7.3%	3.3%
x_S			249.3%	149.3%			291.6%	186.1%			326.8%	127.7%
ACs												
μ_p	0.8%	1.4%	2.1%	1.3%	0.8%	1.1%	1.6%	1.0%	1.4%	1.5%	1.6%	1.5%
σ_p	1.9%	6.6%	6.6%	3.5%	1.8%	4.4%	4.4%	2.3%	3.0%	3.5%	3.5%	3.4%
SR_p	0.267	0.165	0.267	0.267	0.299	0.174	0.299	0.299	0.321	0.318	0.321	0.321
GM_p	0.8%	1.2%	1.9%	1.2%	0.8%	1.0%	1.5%	1.0%	1.3%	1.5%	1.5%	1.5%
x_S			344.0%	182.3%			242.0%	127.8%			116.7%	115.8%

Notes: This exhibit shows the arithmetic mean return (μ_p), geometric mean return (GM_p), volatility (σ_p), and Sharpe ratio (SR_p) of S and G portfolios, all taken from Exhibit 1 for developed markets (DMs); from Exhibit 2 for emerging markets (EMs); and from Exhibit 3 for asset classes (ACs). It also shows the arithmetic mean return, geometric mean return, volatility, and Sharpe ratio of two levered portfolios that result from a short position in the risk-free rate and a long position (x_S) in the S portfolio; one levered portfolio (LS1) is designed to match the volatility of the G portfolio, and the other (LS2) to match the growth (geometric mean) of the G portfolio. Mean returns, volatility, and Sharpe ratios are monthly magnitudes. The monthly risk-free rates used are 0.33% (June 2008), 0.29% (June 2003), and 0.44% (June 1998). The data are described in Exhibit A1.

Consider now the *LS*2 portfolio, which is a levered *S* portfolio designed to match the growth (geometric mean return) of the *G* portfolio. On the positive side, *LS*2 has lower volatility than *G*; on the negative side, *LS*2 has a lower expected return than *G* and, as Exhibit 5 shows, also requires a substantial amount of leverage. More precisely, it requires going long 182.3% the *S* portfolio (and short 82.3% the risk-free rate), a level of leverage much lower than that required to implement *LS*1 but still very high for many investors.

Exhibit 5 shows for developed markets, emerging markets, and asset classes, as well as for June 2008, June 2003, and June 1998, the (arithmetic and geometric) mean return, volatility, and Sharpe ratio of all *S*, *G*, *LS*1, and *LS*2 portfolios. Importantly, it also shows the size of the position that must be taken in *S* (x_S) to implement the levered portfolios. In some cases the size of this position is moderate (lower than 120-130%) and in some cases very high (over 300%). On average across all assets (developed markets, emerging markets, and asset classes) and all dates (June 2008, June 2003, and June 1998) considered, *LS*1 requires going long 232.1% the *S* portfolio (and short 132.1% the risk-free rate); the corresponding number for *LS*2 is 139.9% (short 39.9% the risk-free rate).

These results can be interpreted in a variety of ways. It can be argued that the possibility of leverage renders GMM irrelevant because *G* portfolios can be dominated in one or more dimensions by levered versions of the *S* portfolio. Investors that can tolerate the volatility of a *G* portfolio would prefer to invest in a levered *S* portfolio with the same volatility but a higher expected return and growth than for the *G* portfolio. However, the results discussed show that the leverage required may be unattainable (or undesirable) to many investors.

On the other hand, investors that desire to attain the growth and terminal wealth of a G portfolio may prefer to invest in a levered S portfolio with the same expected growth but lower volatility than the G portfolio. Unlike the previous case, this would require a more moderate amount of leverage. However, this would not render GMM irrelevant; it may still be appealing to investors that cannot or do not want to use leverage and to long-only mutual funds.

Out-of-Sample Performance

The (arithmetic and geometric) mean return, volatility, Sharpe ratio, and terminal wealth of the portfolios discussed in the In-Sample Optimal Portfolios section are all *expected* magnitudes. In other words, they are the characteristics expected from each portfolio given the historical performance of the assets included in them. However, it would be useful to explore the *actual* behavior of the portfolios selected by the GMM and SRM criteria; that is the issue addressed in this section. Exhibit 6 summarizes the relevant results of the analysis.

Panel A of Exhibit 6 summarizes the performance of a $100 investment in the optimal portfolios determined in June 1998, passively held through the end of June 2008. Panel B, in turn, summarizes the performance of a $100 investment in the optimal portfolios determined in June 1998; passively held through June 2003; rebalanced to the optimal portfolios determined at that point in time; and passively held through the end of June 2008. This rebalancing halfway into the holding period, as the exhibit shows, does not affect the qualitative results substantially.

As expected given the results of the previous in-sample analysis, G portfolios have higher risk than S portfolios regardless of whether risk is measured by standard deviation, semideviation, beta, or the minimum monthly return. This result applies to all the assets considered, namely, developed markets, emerging markets, and asset classes.

The higher volatility of G portfolios, however, is in some cases more than compensated by higher returns. Although S portfolios are designed to produce higher Sharpe ratios than G (and all other) portfolios, this is not achieved out of sample in three of the six cases considered, namely, developed markets with and without rebalancing and asset classes with rebalancing. In these three cases, G portfolios outperform S portfolios in terms of risk-adjusted returns when risk is measured by both the standard deviation and the semideviation. In other words, G portfolios outperform S portfolios out of sample on the basis of both Sharpe ratios and Sortino ratios.[3]

On the other hand, G portfolios, designed to maximize growth, do outperform S portfolios in terms of mean compound return and terminal wealth in all cases

Exhibit 6
Out-of-Sample Performance

	Developed Markets		Emerging Markets		Asset Classes	
	S	G	S	G	S	G
Panel A: No Rebalancing						
μ_p	0.6%	0.9%	1.3%	1.6%	0.4%	0.3%
GM_p	0.5%	0.8%	1.2%	1.2%	0.4%	0.2%
σ_p	4.9%	6.0%	4.4%	8.2%	3.4%	4.3%
Σ_p	3.4%	3.8%	2.5%	5.5%	2.3%	3.1%
β_p	1.1	1.2	0.6	1.4	0.8	1.0
Min	–13.8%	–15.2%	–11.0%	–35.8%	–11.4%	–13.9%
Max	12.4%	21.2%	16.8%	21.8%	8.4%	10.0%
$(\mu_p - R_f)/\sigma_p$	0.053	0.093	0.199	0.143	0.012	–0.014
μ_p/Σ_p	0.187	0.249	0.503	0.284	0.185	0.107
Annualized σ_p	16.8%	20.7%	15.2%	28.4%	11.7%	15.0%
Annualized GM_p	6.5%	9.6%	14.9%	15.5%	4.6%	2.8%
TV	$188	$250	$402	$422	$156	$132

EXHIBIT 6 (continued)

	Developed Markets		Emerging Markets		Asset Classes	
	S	G	S	G	S	G
Panel B: With Rebalancing						
μ_p	0.7%	1.0%	1.3%	1.6%	0.4%	0.6%
GM_p	0.6%	0.8%	1.2%	1.3%	0.4%	0.5%
σ_p	4.9%	6.0%	4.4%	8.3%	3.3%	4.6%
Σ_p	3.4%	3.8%	2.5%	5.5%	2.3%	3.2%
β_p	1.1	1.2	0.6	1.4	0.7	1.0
Min	−13.8%	−15.2%	−12.0%	−35.8%	−11.4%	−13.9%
Max	12.4%	21.2%	15.8%	19.8%	8.4%	10.0%
$(\mu_p-R_f)/\sigma_p$	0.072	0.095	0.198	0.149	0.009	0.044
μ_p/Σ_p	0.216	0.250	0.494	0.296	0.184	0.186
Annualized σ_p	16.8%	20.9%	15.2%	28.6%	11.3%	15.9%
Annualized GM_p	7.7%	9.8%	14.9%	16.3%	4.5%	6.0%
TV	$209	$254	$400	$452	$155	$179

Notes: This exhibit shows, for developed markets, emerging markets, and asset classes, the out-of-sample performance of ex-ante optimal portfolios defined as those that maximize the Sharpe ratio (S) according to Expressions (3) and (4) or mean compound return (G) according to Expressions (5) and (6). Panel A summarizes the performance of $100 invested in the optimal portfolios formed in June 1998, described in Exhibits 1–3, and passively held through June 2008. Panel B shows the performance of $100 invested in the optimal portfolios formed in June 1998, described in Exhibits 1–3; passively held through June 2003; rebalanced to the optimal portfolios formed in June 2003, described in Exhibits 1–3; and passively held through June 2008. Performance measures include the arithmetic mean return (μ_p), geometric mean return (GM_p), volatility (σ_p), semideviation with respect to 0 (Σ_p), beta with respect to the world market (β_p), lowest (Min) and highest (Max) return, all expressed in monthly magnitudes, as well as the terminal value of the $100 investment (TV). The risk-free rate (R_f) used for the calculation of $(\mu_p - R_f)/\sigma_p$ is the monthly average over the June 1998–June 2008 period (0.39%). The data are described in Exhibit A1.

but one (asset classes without rebalancing). Interestingly, in the only case in which an *S* portfolio outperforms a *G* portfolio in terms of growth, the *G* portfolio is extremely concentrated and fully invested in U.S. stocks. This result suggests that, when attempting to maximize future growth, the GMM criterion may make a highly concentrated and risky bet that in the end may not pay off. The underperformance of the *G* portfolio in this specific case may be a useful reminder that the GMM criterion does not *guarantee* outperformance in terms of growth; it merely maximizes the *probability* that growth and terminal wealth will be higher than those obtained from any other strategy.

More generally, note that the shorter the investment horizon is, the less certain the outperformance of the *G* portfolio will be. This is simply because in a short holding period, any run of low returns in the *G* portfolio (or of high returns in the *S* portfolio) will largely determine the terminal wealth. In other words, in the short term anything can happen because the final outcome may be dominated simply by (good or bad) luck. The longer the holding period is, however, the lower the impact of luck is, and, therefore, the more likely is the *G* portfolio to outperform all other portfolios. For this reason, GMM is usually thought of as a *long*-term investment strategy.

GMM OR SRM?

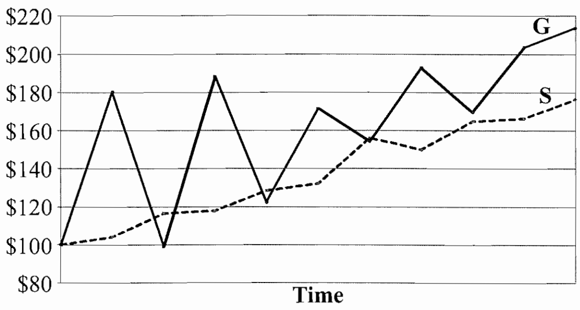
There is little doubt that both GMM and SRM have attractive properties. The question, then, is which one should investors adopt as the standard criterion. Currently, SRM seems to be the preferred choice, but should it be? This section rounds up the previous discussion by outlining some of the conditions that would make GMM the more attractive criterion.

Exhibit 7 considers two hypothetical assets, *G* and *S*. Panel A depicts their performance, and Panel B formally summarizes that performance, which can be thought of as representative of the long-term behavior of both assets. In relative terms, *S* has lower volatility and higher risk-adjusted returns; *G*, on the other hand, grows at a faster rate. Which of the two assets is more attractive?

It depends. All investors would prefer the higher terminal value of *G*, but not all investors would be able

EXHIBIT 7
GMM vs. SRM

Panel A



Panel B

	S	G
μ_p	6.0%	15.8%
σ_p	6.1%	42.6%
SR_p	0.989	0.371
GM_p	5.8%	7.9%
TV10	$176	$214
TV20	$310	$456
TV30	$547	$973

Notes: Panel A shows the 10-year performance of two hypothetical assets, S and G. Panel B shows the arithmetic mean return (μ_p), volatility (σ_p), Sharpe ratio (SR_p) and geometric mean return (GM_p) of both assets, as well as the terminal value of $100 invested at GM_p after 10 years (TV10), 20 years (TV20), and 30 years (TV30).

to take its very high volatility. This is, precisely, Samuelson's [1971] point: Preferences do play a role and it is not certain that all investors would prefer *G* just because it grows at a faster rate; *G* is also far riskier and some investors may avoid it for that reason. This is particularly true given that although an asset (say, a technology stock) may be expected to grow at a faster rate than another (say, a utility stock), it is not certain that it will do so over a given holding period, particularly over a short one.

What conditions would make *G* the more attractive asset? Besides preferences reflecting a relatively low degree of risk aversion, two seem to stand out: The *longer* and *more certain* the investment horizon, the more attractive G becomes. As mentioned above, in a short holding period, luck plays an important role, but its impact decreases as the holding period increases; hence, the longer the holding period, the more likely is *G* to outperform S. In the short term, a utility stock may be preferred over a technology stock, but in the long term, the technology stock may be the more plausible choice. For this reason, the longer the investment horizon, the more plausible the choice of G.

Furthermore, how certain an investor is about his holding period also plays an important role. Some investors may have the intention of saving for the long term but may be forced to sell sooner than expected. If an investor's savings are not substantial and are meant to take care of all unforeseen contingencies, the likelihood of having to liquidate the holdings before the end of the expected holding period may be high. In this case, it is not clear that G, given its high volatility, is the better choice. If, on the other hand, the savings can be put away with the certainty that they will not be needed in the short or medium term, then G, given its higher expected terminal value, may be the more attractive choice.

In short, then, SRM may be a more plausible criterion than GMM for relatively more risk-averse investors, those with a short investment horizon, and those that are uncertain about the length of their holding period. GMM, on the other hand, may be a more plausible criterion than SRM for relatively less risk-averse investors, those with a long investment horizon, and those that are likely to stick to their expected (long) holding period.

AN ASSESSMENT

There is little doubt that SRM is the standard criterion of portfolio selection currently chosen by academics and practitioners. This article poses the question whether it *should* be. The in-sample analysis of the evidence reported and discussed in this article shows that *G* portfolios are less diversified, have a higher (arithmetic and geometric) mean return, and higher volatility than *S* portfolios. The evidence also shows that levered versions of *S* portfolios that aim to match some characteristics of *G* portfolios may require more leverage than many investors can or are willing to obtain. And the out-of-sample analysis shows that although GMM tends to achieve its goal of maximizing terminal wealth, SRM often does not achieve its goal of maximizing risk-adjusted returns.

Why then the general preference for SRM over GMM? It is not entirely clear. *G* portfolios are in fact less diversified and riskier than *S* portfolios; but at the same time they compound the invested capital faster, thus delivering a higher terminal wealth. Is it then the case, as Samuelson [1971] and Markowitz [1976] imply, that many investors (and therefore the practitioners that manage their portfolios) are willing to sacrifice long-term return in exchange for short-term stability? Perhaps that is part of the reason. Mauboussin [2006] suggests that portfolio managers may not be fond of GMM because, more often than not, they are forced to focus on the short term rather than on the long term. He also suggests that investors may find it difficult to deal with the high volatility of the portfolios selected by this criterion.

Nevertheless, it still remains the case that GMM has several desirable characteristics. It is by design equipped to deal with a multiperiod horizon and the reinvestment of capital; it maximizes the probability of ending with more wealth than any other strategy; it minimizes the time to reach any target level of wealth; it empirically does tend to achieve out of sample its goal of maximizing the growth of the capital invested; and it is simple to implement.

And yet GMM seems to have taken a back seat to SRM. The results reported and discussed in this article challenge the conventional wisdom and raise an important question: Are academics and practitioners largely overlooking a compelling portfolio optimization criterion?

APPENDIX A

EXHIBIT A1
Data and Summary Statistics

Developed	**AM**	**SD**	**Start**	**Emerging**	**AM**	**SD**	**Start**
Australia	1.1%	6.8%	Dec/69	Argentina	2.8%	16.1%	Dec/87
Austria	1.1%	5.9%	Dec/69	Brazil	3.1%	15.5%	Dec/87
Belgium	1.2%	5.5%	Dec/69	Chile	1.8%	7.0%	Dec/87
Canada	1.1%	5.5%	Dec/69	China	0.5%	11.0%	Dec/92
Denmark	1.3%	5.4%	Dec/69	Colombia	1.7%	9.4%	Dec/92
Finland	1.4%	9.1%	Dec/87	Czech Rep.	1.9%	8.0%	Dec/94
France	1.1%	6.4%	Dec/69	Egypt	2.3%	9.1%	Dec/94
Germany	1.1%	6.1%	Dec/69	Hungary	2.1%	9.9%	Dec/94
Hong Kong	1.8%	10.4%	Dec/69	India	1.2%	8.5%	Dec/92
Ireland	0.9%	5.6%	Dec/87	Indonesia	2.0%	14.9%	Dec/87
Italy	0.9%	7.1%	Dec/69	Israel	1.0%	7.1%	Dec/92
Japan	1.0%	6.3%	Dec/69	Jordan	0.8%	5.1%	Dec/87
Netherlands	1.2%	5.2%	Dec/69	Korea	1.2%	11.1%	Dec/87
New Zealand	0.7%	6.6%	Dec/87	Malaysia	1.0%	8.7%	Dec/87
Norway	1.4%	7.6%	Dec/69	Mexico	2.3%	9.2%	Dec/87
Portugal	0.7%	6.4%	Dec/87	Morocco	1.6%	5.4%	Dec/94
Singapore	1.3%	8.3%	Dec/69	Pakistan	1.3%	11.0%	Dec/92
Spain	1.1%	6.3%	Dec/69	Peru	2.0%	8.8%	Dec/92
Sweden	1.4%	6.8%	Dec/69	Philippines	0.9%	9.4%	Dec/87
Switzerland	1.1%	5.2%	Dec/69	Poland	2.4%	14.5%	Dec/92
U.K.	1.1%	6.4%	Dec/69	Russia	3.2%	16.9%	Dec/94
U.S.	0.9%	4.4%	Dec/69	South Africa	1.4%	7.8%	Dec/92
Asset Classes				Sri Lanka	0.9%	9.9%	Dec/92
U.S. Stocks	1.0%	4.0%	Dec/87	Taiwan	1.1%	10.9%	Dec/87
EAFE Stocks	0.7%	4.6%	Dec/87	Thailand	1.2%	11.3%	Dec/87
EM Stocks	1.4%	6.6%	Dec/87	Turkey	2.4%	17.3%	Dec/87
U.S. Bonds	0.7%	2.0%	Dec/87	**World Market**			
U.S. Real Estate	0.9%	3.8%	Dec/87	AC World	0.8%	4.0%	Dec/87

Notes: This exhibit shows, for the series of monthly returns, the arithmetic mean (AM) and standard deviation (SD) of all markets and asset classes in the sample, both calculated between the beginning (Start) and the end (Jun/08) of each asset's sample period. The returns of all individual markets, as well as those of EAFE (Europe, Australasia, and the Far East) stocks and EM (Emerging Markets) stocks are summarized by MSCI indices. The returns of U.S. Bonds are summarized by the 10-year government bond total return index (from Global Financial Data) and those of U.S. Real Estate by the FTSE NAREIT (All REIT) total return index. The world market is summarized by the MSCI All Country (AC) World Index. All returns are in dollars and account for capital gains and dividends.

ENDNOTES

An expanded version of this article is available upon request. I would like to thank Tom Berglund, Mark Kritzman, Jack Rader, Rawley Thomas, and participants of the seminars at the University of Miami, University of South Florida, and Torcuato Di Tella University for their comments. Juan Nadal provided research assistance. The views expressed herein and any errors that may remain are entirely my own.

[1]As an example, consider two investments, one with a 5% certain return, and another with a 50-50 chance of a 200% gain or a 100% loss. Although this second alternative (with an expected value of 50%) may be, at least to some investors, more attractive than the first when making a one-time choice, it is a bad choice for *all* investors in a (long-term) multiperiod framework with reinvestment of gains and losses. This is the case because sooner or later the 100% loss will occur and wipe out all the capital accumulated.

[2]This had been recognized before Latane [1959]. Williams [1936] had previously argued that a speculator who repeatedly risks capital plus profits should focus on the geometric (rather than on the arithmetic) mean. Furthermore, Kelly [1956] had previously argued that a gambler restricted to bet the same absolute amount repeatedly (thus not reinvesting his proceeds) should focus on the arithmetic (rather than on the geometric) mean.

[3]Sortino ratios are generally defined as $(R_p - B)/\Sigma_{pB}$, where R_p denotes the return of the portfolio; B a benchmark return chosen by the investor; and Σ_{pB} the semideviation of the portfolio returns with respect to the benchmark B (that is, volatility below B). The benchmark considered in Exhibit 6 is $B = 0$. For a practical introduction to downside risk, see Estrada [2006]; for mean–semivariance portfolio optimization, see Estrada [2008].

REFERENCES

Bernstein, William, and David Wilkinson. "Diversification, Rebalancing, and the Geometric Mean Frontier." Unpublished manuscript, 1997.

Christensen, Morten. "On the History of the Growth Optimal Portfolio." Unpublished manuscript, 2005.

Elton, Edwin, and Martin Gruber. "On the Maximization of the Geometric Mean with Lognormal Return Distribution." *Management Science*, 21 (1974), pp. 483-488.

Estrada, Javier. "Downside Risk in Practice." *Journal of Applied Corporate Finance*, 18 (2006), 117-125.

——. "Mean-Semivariance Optimization: A Heuristic Approach." *Journal of Applied Finance*, Spring/Summer 2008, pp. 57-72.

Fama, Eugene, and James McBeth. "Long-Term Growth in a Short-Term Market." *Journal of Finance*, 29 (1974), pp. 857-885.

Grauer, Robert. "A Comparison of Growth Optimal and Mean Variance Investment Policies." *Journal of Financial and Quantitative Analysis*, 16 (1981), pp. 1-21.

Hunt, Ben. "Growth Optimal Investment Strategy Efficacy: An Application on Long Run Australian Equity Data." *Investment Management and Financial Innovations*, 1 (2005a), pp. 8-22.

——. "Feasible High Growth Investment Strategy: Growth Optimal Portfolios Applied to Dow Jones Stocks." *Journal of Asset Management*, 6 (2005b), pp. 141-157.

Kelly, John. "A New Interpretation of Information Rate." *Bell System Technical Journal*, 35 (1956), pp. 917-926.

Latane, Henry. "Criteria for Choice among Risky Ventures." *Journal of Political Economy*, 67 (1959), pp. 144-155.

MacLean, Leonard, William Ziemba, and George Blazenko. "Growth Versus Security in Dynamic Investment Analysis." *Management Science*, 38 (1992), pp. 1562-1585.

Markowitz, Harry. "Portfolio Selection." *Journal of Finance*, 7 (1952), pp. 77-91.

——. *Portfolio Selection. Efficient Diversification of Investments*. John Wiley & Sons, 1959.

——. "Investment for the Long Run: New Evidence for an Old Rule." *Journal of Finance*, 31 (1976), pp. 1273-1286.

Mauboussin, Michael. "Size Matters." Mauboussin on Strategy, Legg Mason Capital Management, February 1, 2006.

McEnally, Richard. "Latane's Bequest: The Best of Portfolio Strategies." *The Journal of Portfolio Management*, 12 (1986), pp. 21-30.

Poundstone, William. *Fortune's Formula—The Untold Story of the Scientific Betting System That Beat the Casinos and Wall Street*. New York, NY: Hill and Wang, 2005.

Roll, Richard. "Evidence on the 'Growth-Optimum' Model." *Journal of Finance*, 28 (1973), pp. 551-566.

Samuelson, Paul. "The 'Fallacy' of Maximizing the Geometric Mean in Long Sequences of Investing or Gambling." *Proceedings of the National Academy of Sciences*, 68 (1971), pp. 2493-2496.

Weide, James, David Peterson, and Steven Maier. "A Strategy Which Maximizes the Geometric Mean Return on Portfolio Investments." *Management Science*, 23 (1977), pp. 1117-1123.

Williams, John. "Speculation and the Carryover." *Quarterly Journal of Economics*, 50 (1936), pp. 436-455.

Ziemba, William. "Note on Optimal Growth Portfolios When Yields Are Serially Correlated." *Journal of Financial and Quantitative Analysis*, 7 (1972), pp. 1995-2000.