Retirement Planning: From Z to A

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Abstract

Retirement planning is an issue that must be tackled early and solved backward. It must be tackled early because with a few working years to go there is little that can be done if an individual is not on the right path; and it must be solved backward because it makes little sense to aim for a portfolio that may not be able to sustain the desired lifestyle in retirement. This article introduces an approach that integrates the working period and the retirement period; leads the individual to consider all the relevant variables at the beginning of his journey; and enables him to start saving early to build a target portfolio specifically designed to sustain a desired retirement. The analytical framework introduced yields closed-form solutions for the target retirement portfolio and the contributions that need to be made during the working years to hit that target. The framework proposed is illustrated with an empirical base case, sensitivity analysis, and Monte Carlo simulations.

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1. Introduction

Retirement planning involves an interesting paradox. When an individual really needs to make a financial plan early in his working years, he has neither the motivation nor the information to do it; on the other hand, when the individual approaches the end of his career and has both the motivation and more information, it may already be too late. That is but one of the many reasons why most people fail to plan properly for retirement.

Compounding the previous problem is the fact that more often than not, if they plan at all, many individuals wrongly view retirement planning as two separate steps. First they focus on how much they would like to retire with, and the periodic contributions that would lead to that target portfolio; and only when approaching retirement they focus on how much they will be able to periodically withdraw from the portfolio, trying to avoid spending too little (thus reducing their standard of living unnecessarily) or too much (thus running out of money too early).

The problem with this artificial separation is that if the target portfolio has not been built with the specific goal of sustaining the desired lifestyle in retirement, it will most likely be the standard of living that would eventually have to adjust to the funds available. A much better strategy is to work backward, first determining the size of the portfolio necessary to sustain the desired lifestyle, and then determining how to build that portfolio during the working years.

* I would like to thank Jack Rader for his comments. David Vila provided research assistance. The views expressed below and any errors that may remain are entirely my own.
This artificial separation is also reflected in the literature, with a strand of articles that focuses on strategies for the accumulation (or working) period and another strand that focuses on strategies for the withdrawal (or retirement) period. However, it should be obvious that these two periods are not independent from each other at all. In fact, the ultimate goal of retirement planning is to build during the working period a portfolio specifically designed to sustain the desired lifestyle during the retirement period.

Put differently, retirement planning should be done in a sequence not often implemented in practice and hardly ever addressed properly in the literature. It should start with the desired lifestyle in retirement and an estimate of its periodic costs; it should then map that information into a target portfolio; and it should finally map the latter into a set of periodic contributions to be made during the working years.

More precisely, the approach introduced here starts with the withdrawals to be made during the retirement period and the bequest to be left at the end of it, which are considered exogenous variables. Those withdrawals together with an expected return, given by the asset allocation chosen for the retirement period, jointly determine the target retirement portfolio. And the target portfolio together with an expected return, given by the asset allocation chosen for the working period, jointly determine the contributions to be made during the working years.

Following such approach this article provides closed-form solutions for the target portfolio and the contributions to be made during the working period, given the withdrawals to be made during the retirement period, the bequest, and the returns expected for the portfolio over both the accumulation and the withdrawal periods. These expressions can be used to evaluate most of the relevant trade-offs individuals face, such as the sensitivity of the target portfolio to changes in contributions during the working years or withdrawals during retirement; or the impact of changes in expected returns during the lifecycle.

The expressions derived here are discussed in the context of an example based on the performance of U.S. stocks and bonds, first using a base case and subsequent sensitivity analysis, and then using Monte Carlo simulations. Both approaches provide valuable insights that may help individuals and financial planners to plan properly for retirement.

The rest of the article is organized as follows. Section 2 introduces the general framework proposed and discusses the analytical problems faced by an individual during both the withdrawal and the accumulation period. Section 3 applies that framework to the specific situation of an individual planning for his retirement as soon as he starts working, using an empirical base case and a Monte Carlo simulation, as well as sensitivity analysis on both. Finally, section 4 concludes with an assessment. An appendix with important technical details concludes the article.
2. The Issue

This section starts with an overview of the approach introduced and continues with a brief review of some relevant literature. Then it discusses the problems to be solved in their proper order, beginning with the withdrawal period and continuing with the accumulation period. Finally it brings both periods together by discussing the individual’s lifecycle problem.

2.1. Overview

It is both understandable and problematic that most people postpone retirement planning when it is necessary and address it when it is probably too late. On the one hand, a young individual at the beginning of his working years can hardly be blamed for not wanting to think about retirement; on the other hand, retirement planning must start from the end. Without an estimate of the withdrawals needed to sustain the desired lifestyle in retirement, there is no way to determine a target portfolio; and without the latter, there is no way to determine the contributions needed to hit a non-existent target.

The same reasoning applies to a bequest; it is something that most people tend to consider well into their retirement, not early in their careers. And yet if an individual wants to make sure that he will leave something to the people or institutions of his choice, a rough estimate is also needed to factor into the withdrawals to be made from the retirement portfolio. The framework introduced here takes both the withdrawals and the bequest as given (exogenous variables).1

An individual also needs to consider his portfolio’s asset allocation during both the accumulation and the withdrawal periods, beginning with the latter. The reason for beginning with the latter is because it will determine the return of the portfolio during retirement, which in turn will determine how large the retirement portfolio needs to be. In other words, the less (more) aggressive the asset allocation is, the larger (smaller) the portfolio would have to be to sustain a given level of withdrawals. These three variables, the withdrawals, the bequest, and the expected return of the portfolio during retirement, determine the target portfolio, a relationship that is formalized below. That is step 1 in the approach proposed.

Having determined the target retirement portfolio, the next step is to determine the periodic contributions during the working years necessary to hit that target. These contributions are related to the rate at which the portfolio grows during the accumulation period, which again is determined by the portfolio’s asset allocation; for any given target, the less (more) aggressive the asset allocation is, the higher (lower) the periodic contributions would have to be. Thus, these two variables, the target retirement portfolio and the expected return of the portfolio during the

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1 Both Eisenberg (2006) and Benz (2020) discuss this issue.
working years determine the contributions to be made during the accumulation period. That is step 2 in the approach proposed.

2.2. Related Research

There is a vast literature on both the accumulation and the withdrawal periods, far too extensive to summarize here. Regarding the accumulation period, one of the issues that has received most attention is how to maximize the size of the retirement portfolio, which in turn involves at least two debates, namely, the choice between static or dynamic strategies; and related to the latter, the choice between declining and rising equity glidepaths.

The most popular dynamic strategy is a declining-equity glidepath, typically used by target-date funds, which features an increasingly conservative asset allocation as retirement approaches; see, for example, Donaldson et al (2019). However, this strategy has been criticized for being suboptimal in terms of capital accumulation; Basu and Drew (2009), Arnott et al (2013), and Estrada (2014), among others, show that a rising-equity glidepath leads to a larger retirement portfolio. Others argue in favor of static strategies, typically with a high allocation to stocks; see, for example, Ayres and Nalebuff (2010).

Most of the previous literature focuses on maximizing the size of the retirement portfolio, although that is not the only possible goal. Some investors may be interested in policies designed to hit a target portfolio, an issue considered by Basu et al (2011) and Estrada (2019). Other investors may be interested in obtaining increasing downside protection as retirement approaches, which is what target-date funds offer; see Estrada (2020b).

Regarding the withdrawal period, Bengen (1994) is widely considered the seminal article. Part of the subsequent literature focuses on withdrawal strategies, such as Stout (2008), Blanchett and Frank (2009), Jaconetti et al (2013), and Zolt (2014), among others. Another part of this literature focuses on asset allocation strategies, such as Blanchett (2007), Spitzer and Singh (2007), Pfau and Kitces (2014), Kitces and Pfau (2015), Estrada (2016), and Estrada and Kritzman (2019), among others. Estrada (2020a) considers both withdrawal and asset allocation strategies.

One of the characteristics of all the previous literature is that it focuses on either the accumulation or the withdrawal period. That is a valid approach if the focus is either on the target portfolio or on the sustainability of a given retirement portfolio. However, as already argued, proper retirement planning necessarily implies integrating both periods, as is done here. Pfau (2011) also considers both periods when introducing the concept of a safe savings rate.
2.3. The Withdrawal or Retirement Period

Consider an individual that at the very beginning of his working years wants to put a proper retirement plan in place. He thinks about his desired retirement, which he expects to last \( N \) years, and believes that it can be sustained with \( N \) annual withdrawals \( W_t \) through \( W_N \) from his retirement portfolio \( P \), made at the beginning of each year in retirement. He plans to keep his purchasing power constant over time, which implies that in real (inflation-adjusted) terms \( W_1 = W_2 = \ldots = W_N = W \), where \( W \) is the constant annual real withdrawal.

Our individual plans to leave a bequest \( B \), which can be thought of as a final withdrawal that fully exhausts the portfolio, made one year after \( W_N \). Finally, he needs to select an asset allocation for his portfolio, which in turn will determine \( R \), the annualized return he expects to obtain during the retirement period. As shown in the appendix, our individual’s problem can be expressed as

\[
P \cdot (1 + R)^N - W \cdot \sum_{t=1}^{N} (1 + R)^t - B = 0
\]

where \( t \) indexes periods, measured in years. Given \( B, W, \) and \( R \) this expression can be solved for the target portfolio \( P^* \); that is,

\[
P^* = \frac{B + W \sum_{t=1}^{N} (1+R)^t}{(1+R)^N}
\]

Finally, simplifying the sum by using the expression for the sum of a geometric series (see the appendix), we obtain

\[
P^* = \frac{B - W \cdot (1+R) \left( \frac{1-(1+R)^N}{R} \right)}{(1+R)^N}
\]

which formalizes the relationship between the target retirement portfolio, the constant annual real withdrawal, the bequest, and the return of the portfolio during retirement.

2.4. The Accumulation or Working Period

Having determined his target portfolio \( P^* \), and expecting to work for \( M \) years, our individual then needs to select an asset allocation for his portfolio during his working years. The selected asset allocation will in turn determine \( S \), the annualized return he expects to obtain from his portfolio during the accumulation period.

Our individual aims to make \( M-1 \) annual contributions to his portfolio, the first at the end of his first working year, and the last one year before retirement. Thus, at the end of the accumulation period he aims to liquidate his portfolio, implement his new (more conservative) asset allocation, and immediately take the first withdrawal to fund his first year in retirement.
Finally, our individual expects to keep his contributions constant in real terms, which implies that
$C_1 = C_2 = ... = C_{M-1} = C$, where $C$ is the constant annual real (inflation-adjusted) contribution. As shown in the appendix, our individual’s problem can be expressed as

$$C \cdot \sum_{t=1}^{M-1} (1 + S)^t = P^*$$

(4)

where $t$ again indexes periods, measured in years. Given $S$ and $P^*$, this expression can be solved for the constant annual real contribution ($C$); that is,

$$C^* = \frac{P^*}{\sum_{t=1}^{M-1} (1 + S)^t}$$

(5)

Finally, simplifying the sum by using the expression for the sum of a geometric series (see the appendix), we obtain

$$C^* = -\frac{P^*}{\frac{(1+S)^M}{S}}$$

(6)

which formalizes the relationship between the constant annual real contribution, the target retirement portfolio, and the return of the portfolio during the working years.

2.5. The Lifecycle

The previous two sections describe the problem of our representative individual, considered and solved in proper order; that is, from the end (the retirement period) to the beginning (the working period). Expression (3) maps the withdrawals to be made in retirement, the bequest to be left, and the expected return of the portfolio in retirement to the target portfolio. Expression (6), in turn, maps the target portfolio determined in the previous step and the expected return of the portfolio during the working years to the contributions to be made during the accumulation period.

Those two expressions, (3) and (6), summarize the framework introduced, which in fact can be summarized in just one expression by inserting (2) into (5) or (3) into (6); that is,

$$C^* = \frac{W \cdot \sum_{t=1}^{N}(1+R)^t + B}{\sum_{t=1}^{M-1} (1+S)^t} = -\frac{B - W \cdot (1+R) \left( \frac{1-(1+R)^N}{R} \right)}{\frac{(1+S)^M}{S}}$$

(7)

where the only endogenous variable is $C$, depending on the exogenous values of $N$, $M$, $R$, $S$, $W$, and $B$. In words, how much our individual needs to periodically contribute to his retirement portfolio during the accumulation period depends on how many years he expects to work ($M$), the return of his portfolio during those years ($S$), how many years he expects to be in retirement ($N$), the
return of his portfolio during those years \((R)\), how much he aims to periodically withdraw from his portfolio \((W)\), and the size of the bequest he aims to leave \((B)\).

3. Evidence

This section discusses an implementation of the plan, first with a base case and sensitivity analysis, and then with Monte Carlo simulations. Data on stocks, bonds, and inflation over the 1928-2019 period are from Damodaran’s web page.\(^2\) Stocks are represented by the S&P 500, bonds by 10-year U.S. Treasury Notes, and inflation by the Consumer Price Index (CPI). All returns are annual, real (adjusted by inflation), and account for capital gains/losses and cash flows paid. Over this 92-year period, stocks and bonds delivered annualized returns of 6.5% and 1.9%, with volatility of 19.5% and 8.1%; their correlation over the sample period was 0.05.

3.1. Base Case

At the beginning of his career, our representative individual expects to live 30 years in retirement. He believes the cost of his lifestyle can be sustained by 30 constant annual real withdrawals of $60,000 from his portfolio, made at the beginning of each of his years in retirement. He aims to leave as a bequest the equivalent of five years of annual withdrawals ($300,000). During retirement, he thinks he will feel comfortable with a 40/60 stock/bond allocation, from which he expects an annualized real return of 4.2%, based on the performance of this allocation over the 1928-2019 period. The target retirement portfolio consistent with these conditions follows from expression (3) and is equal to

\[
P^* = \frac{300,000 - 60,000 \cdot (1 + 0.042) \cdot (1 - (1 + 0.042)^{-10})}{(1 + 0.042)^{10}} = $1,142,637
\]

At the beginning of his career, our representative individual aims to work for 40 years. During this period he thinks he will feel comfortable with a more aggressive 60/40 stock/bond allocation, from which he expects an annualized real return of 5.2%, based on the performance of this allocation over the 1928-2019 period. Given that he targets a $1,142,637 retirement portfolio, his constant annual real contribution follows from (6) and is equal to

\[
C^* = - \frac{1,142,637}{(1 + 0.052)^{40}} = $9,079
\]

This annual contribution to the retirement portfolio could also be found directly from (7), without specifying the target portfolio; that is,

\[^2\text{http://people.stern.nyu.edu/adamodar/New_Home_Page/datafile/histretSP.html}\]
Having all this information, our representative individual is now able to determine the expected path of his retirement portfolio over his 70-year lifecycle, shown in Exhibit 1. Note that at the end of the first working year the portfolio only contains the first contribution; from that point on and until the end of the accumulation period, the portfolio receives annual (real) contributions of $9,079 and compounds at 5.2%. Note, also, that after the last withdrawal of $60,000 at the beginning of year 70 (the end of year 69 in Exhibit 1), the portfolio’s value is $287,908, which compounds over one more period at 4.2% and ends with the intended bequest of $300,000.

Exhibit 1: Retirement Portfolio – Expected Path

This exhibit shows the expected path of a portfolio \( P \) over 70 years \( Y \), 40 in the working period and 30 in the retirement period. \( P \) receives real contributions of $9,079 at the end of years 1 through 39 and earns a 5.2% real return during the first 40 years. \( P \) is also subject to 30 real withdrawals of $60,000 at the beginning of years 40 through 70, leaves a bequest of $300,000 at year-end 70, and earns a 4.2% real return during the last 30 years. All figures in dollars.

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Importantly, note that the retirement date (year 40, highlighted) is the breakpoint; until then the portfolio compounds at 5.2%, and from then on it compounds at 4.2%. Note, also, that on retirement date the value of the portfolio ($1,082,637) is not equal to the target portfolio ($1,142,637); the difference between those two values is equal to the first withdrawal, made to fund the expenses of the first year in retirement.

Specifying the expected path of a retirement portfolio is important because it provides markers to determine whether the individual’s plan is on track. Some companies aim to guide investors during their working years by specifying how many times their salary they should have saved at different points in time;\(^3\) Exhibit 1 shows directly the value the retirement portfolio should have at the end of each year over the lifecycle. Should the individual find himself away

\(^3\) Fidelity, for example, suggests that individuals should have 3 (6) \([8]\) times their annual salary saved by the time they reach 40 (50) \([60]\) and refers to these figures as saving factors; see Fidelity (2018).
from this path, he can consider policies to get back on track; Estrada (2019, 2020) shows that adjusting contributions or withdrawals is a more effective way to do so than adjusting a portfolio’s asset allocation.

3.2. Sensitivity Analysis

Let’s recap. At the beginning of his 40-year career our individual considers his 30-year retirement, makes an estimate of its annual costs ($60,000), decides the bequest he wants to leave ($300,000), and selects the asset allocation he will feel comfortable with (40/60), from which he expects a 4.2% annualized return. This enables him to determine a target retirement portfolio ($1,142,637), and after selecting the asset allocation he will feel comfortable with during his working years (60/40), from which he expects a 5.2% annualized return, he finds the annual real contributions he needs to make to his retirement portfolio ($9,079). The path the portfolio is expected to follow over our individual’s 70-year lifecycle is depicted in Exhibit 1.

Expressions (3) and (6), or just expression (7), summarize the framework introduced here for proper retirement planning. This framework can also be used to explore the impact of adjusting some variables the individual controls on the size of the retirement portfolio, and ultimately on the contributions to be made during the working period, thus enabling the evaluation of many relevant trade-offs.

To explore this issue consider the three panels of Exhibit 2, each with the base case highlighted. Panel A, based on expression (3), shows target retirement portfolios; in the top row it shows stock/bond allocations during retirement, and in the first column it shows levels of the constant annual withdrawal. Note that for any given level of withdrawals, the less (more) aggressive the asset allocation is, the larger (smaller) the target portfolio would need to be; thus, if a target portfolio looks too large and therefore unreachable, the individual could aim for a smaller portfolio and compensate with a more aggressive asset allocation in retirement. Also, given the asset allocation, the higher (lower) the standard of living desired by the individual, reflected in higher (lower) withdrawals, then the larger (smaller) is the retirement portfolio he would need. Expression (3) can be used to explore such trade-offs.

Panel B, based on expression (6), shows levels of the constant annual contribution during the accumulation period; in the top row it shows stock/bond allocations during the working years, and in the first column it shows target retirement portfolios. Note that given the target portfolio, the less (more) aggressive the asset allocation is, the larger (smaller) the annual contributions would need to be; thus, if an asset allocation is too aggressive for the individual, he could select a more conservative one and compensate by increasing his annual contributions.

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4 Recall that the individual aims to leave as a bequest five years of annual withdrawals; hence different levels of withdrawals also imply different levels of bequest.
Also, given the asset allocation, if the individual desires a higher (lower) standard of living, reflected in a larger (smaller) retirement portfolio, then he would need to make larger (smaller) contributions during his working years. Expression (4) can be used to explore such trade-offs.

**Exhibit 2: Sensitivity Analysis**

This exhibit shows three panels. Panel A, based on expression (3), shows target retirement portfolios, with asset allocations during retirement in the top row and annual withdrawals in the first column. Panel B, based on expression (6), shows annual contributions to the retirement portfolio, with asset allocations during the working years in the top row and target retirement portfolios in the first column. Panel C, based on expression (7), also shows annual contributions to the retirement portfolio, with asset allocations during the retirement period in the top row, and asset allocations during the accumulation period in the first column. All figures in dollars.

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<tr>
<td>1,092,637</td>
<td>12,477</td>
<td>11,078</td>
<td>9,816</td>
<td>8,681</td>
<td>7,859</td>
<td>7,106</td>
<td>6,585</td>
</tr>
<tr>
<td><strong>1,142,637</strong></td>
<td>13,048</td>
<td>11,585</td>
<td>10,265</td>
<td><strong>9,079</strong></td>
<td>8,218</td>
<td>7,431</td>
<td>6,886</td>
</tr>
<tr>
<td>1,192,637</td>
<td>13,619</td>
<td>12,092</td>
<td>10,714</td>
<td>9,476</td>
<td>8,578</td>
<td>7,756</td>
<td>7,187</td>
</tr>
<tr>
<td>1,242,637</td>
<td>14,190</td>
<td>12,599</td>
<td>11,164</td>
<td>9,873</td>
<td>8,937</td>
<td>8,082</td>
<td>7,489</td>
</tr>
<tr>
<td>1,292,637</td>
<td>14,761</td>
<td>13,106</td>
<td>11,613</td>
<td>10,270</td>
<td>9,297</td>
<td>8,407</td>
<td>7,790</td>
</tr>
<tr>
<td>Panel C</td>
<td>10/90</td>
<td>20/80</td>
<td>30/70</td>
<td>40/60</td>
<td>50/50</td>
<td>60/40</td>
<td>70/30</td>
</tr>
<tr>
<td>30/70</td>
<td>16,332</td>
<td>15,039</td>
<td>13,898</td>
<td>13,048</td>
<td>12,279</td>
<td>11,581</td>
<td>11,069</td>
</tr>
<tr>
<td>40/60</td>
<td>14,501</td>
<td>13,353</td>
<td>12,340</td>
<td>11,585</td>
<td>10,902</td>
<td>10,282</td>
<td>9,827</td>
</tr>
<tr>
<td>50/50</td>
<td>12,849</td>
<td>11,832</td>
<td>10,934</td>
<td>10,265</td>
<td>9,660</td>
<td>9,111</td>
<td>8,708</td>
</tr>
<tr>
<td><strong>60/40</strong></td>
<td>11,364</td>
<td>10,464</td>
<td>9,670</td>
<td><strong>9,079</strong></td>
<td>8,543</td>
<td>8,058</td>
<td>7,701</td>
</tr>
<tr>
<td>70/30</td>
<td>10,287</td>
<td>9,472</td>
<td>8,754</td>
<td>8,218</td>
<td>7,734</td>
<td>7,294</td>
<td>6,971</td>
</tr>
<tr>
<td>80/20</td>
<td>9,302</td>
<td>8,565</td>
<td>7,915</td>
<td>7,431</td>
<td>6,993</td>
<td>6,595</td>
<td>6,304</td>
</tr>
<tr>
<td>90/10</td>
<td>8,619</td>
<td>7,937</td>
<td>7,335</td>
<td>6,886</td>
<td>6,480</td>
<td>6,112</td>
<td>5,841</td>
</tr>
</tbody>
</table>

Finally, panel C, based on expression (7), also shows levels of the constant annual contribution; in the top row it shows asset allocations during the retirement period, and in the first column it shows asset allocations during the accumulation period. The trade-off between how aggressive an asset allocation is and how much the individual would need to contribute to the portfolio during his working years is clear; all else equal, the less (more) aggressive an asset allocation during the accumulation or the withdrawal period is, the more (less) the individual would need to contribute to his portfolio. Expression (7) can be used to explore such trade-offs.

**3.3. Monte Carlo Simulation**

Mike Tyson has famously said that everybody has a plan until they get punched in the mouth; much the same could be said about individuals planning for retirement. Plans hardly ever go as expected, even if an individual has the ability or the luck to guess correctly the annualized
return of his portfolio. As is well-known, when a portfolio is subject to contributions or withdrawals, the annualized return obviously matters but so does the sequence of returns. When an individual is regularly withdrawing funds from a portfolio, an early sequence of bad returns can ruin a retirement plan, a problem known as sequence of returns risk.

For this reason, besides the base case and sensitivity analysis already discussed, it is insightful to perform a Monte Carlo simulation to get a sense of how the relevant variables behave when things do not go exactly as planned. Exhibit 3 summarizes the results of 10,000 scenarios over an individual’s lifecycle, based on the statistical properties of the 60/40 and the 40/60 allocations over the 1928-2019 period. The parameters on which the simulations are based are those already discussed for the base case, except for the bequest, which now is whatever is left at the end of the retirement period. Panel A focuses on the retirement portfolio (the end of the accumulation period), and panel B on the bequest (the end of the retirement period).

Exhibit 3: Monte Carlo Simulation
This exhibit summarizes the results of 10,000 scenarios over an individual’s lifecycle based on the statistical properties of the 60/40 and the 40/60 stock/bond allocations over the 1928-2019 period. Panel A considers the end of the accumulation period (the retirement portfolio) and panel B considers the end of the retirement period (the bequest). ‘Failure’ is the percentage of scenarios in which the portfolio was depleted before the end of the retirement period. All figures but ‘Failure’ in dollars.

<table>
<thead>
<tr>
<th>Panel A</th>
<th>Retirement Portfolio</th>
<th>Retirement Portfolio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Median</td>
<td>1,170,203</td>
<td></td>
</tr>
<tr>
<td>1st percentile</td>
<td>365,252</td>
<td>90th percentile</td>
</tr>
<tr>
<td>5th percentile</td>
<td>508,533</td>
<td>95th percentile</td>
</tr>
<tr>
<td>10th percentile</td>
<td>613,959</td>
<td>99th percentile</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B</th>
<th>Bequest</th>
<th>Bequest</th>
</tr>
</thead>
<tbody>
<tr>
<td>Median</td>
<td>371,481</td>
<td>Failure</td>
</tr>
<tr>
<td>1st percentile</td>
<td>N/A</td>
<td>90th percentile</td>
</tr>
<tr>
<td>5th percentile</td>
<td>N/A</td>
<td>95th percentile</td>
</tr>
<tr>
<td>10th percentile</td>
<td>N/A</td>
<td>99th percentile</td>
</tr>
</tbody>
</table>

Recall that the target retirement portfolio in the base case is $1,142,637. Panel A shows two interesting things about this target. First, that the median retirement portfolio across the 10,000 scenarios in the simulation ($1,170,203) is indeed very close to the target portfolio in the plan. Second, that there is a very large dispersion, with the 5th (95th) percentile target portfolio being less than half of (over twice as large as) the median portfolio.

Panel B shows rather similar results for the bequest. The median across the 10,000 scenarios in the simulation ($371,481) is somewhat larger than the original target bequest ($300,000). In the upper percentiles of the distribution the bequest is many times larger than originally intended, but in the lower percentiles the portfolio fails; that is, it is depleted before the end of the retirement period.

---

5 Kolmogorov-Smirnov tests do not reject the null hypothesis of normality in the distribution of returns of the 60/40 and 40/60 allocations, with p-values of 0.998 and 0.743. Furthermore, the autocorrelation coefficients of the returns series of the 60/40 and the 40/60 allocations are 0.00 and 0.03.

6 But recall that in these simulations the bequest is whatever is left at the end of the retirement period.
end of the retirement period. In fact, the portfolio run out of money earlier than expected in almost 43% of the scenarios considered.

This large failure rate should not be entirely surprising. The $60,000 initial withdrawal from the $1,142,637 retirement portfolio represents an initial withdrawal rate of 5.3%, substantially higher than what is typically considered a safe initial withdrawal rate. The latter was originally found to be 4% by Bengen (1994), although this figure has been subject to a heated debate since then.\(^7\) Currently, the consensus seems to be that 4% is too high, implying that an individual that withdraws 4% of his portfolio on retirement day, and subsequently adjust his annual withdrawals by inflation, is likely to deplete the portfolio in less than 30 years; see, for example, Pfau (2010) and Estrada (2016).

### 3.4. More Sensitivity Analysis

The results just discussed are useful to illustrate both the many things that could happen over an individual’s lifecycle and ultimately the degree of uncertainty the individual would have to face. This analysis can be further complemented with a sensitivity analysis on some of the parameters underlying the Monte Carlo simulation. Exhibit 4 summarizes the results of 50,000 scenarios, equivalent to 10,000 scenarios for each of the five cases considered. The parameters are changed one at a time, with the rest remaining at their values in the base case.

Panel A focuses on the retirement portfolio (the end of the accumulation period), and panel B on the bequest (the end of the retirement period). In both panels the second column shows the impact of lowering the constant annual withdrawal by 10%, from $60,000 to $54,000; the third, the impact of lowering the initial withdrawal rate by 1.3 percentage points, from 5.3% to 4%; the fourth, the impact of selecting a more aggressive asset allocation during retirement, from 40/60 to 50/50; the fifth, the impact of selecting a more aggressive asset allocation during the working period, from 60/40 to 70/30; and the last, the impact of increasing the constant annual contribution by 10%, from $9,079 to $9,986.

In the second, third, and fourth columns, all of which consider changes in variables related to the retirement period, there is predictably little impact on the target retirement portfolio and a substantial impact on the bequest, in all cases relative to the values in Exhibit 3. The median bequest increases by almost 60% (from $371,481 to $589,950) when the 40/60 allocation in retirement is replaced by the 50/50 allocation; by over 90% (from $371,481 to $710,613) when withdrawals are lowered from $60,000 to $54,000; and by over 240% (from $371,481 to $1,279,304) when the initial withdrawal rate is lowered from 5.3% to 4%. In all these three cases

\(^7\) To be precise, Bengen (1994) found that a portfolio with a 50/50 stock/bond allocation, subject to an initial withdrawal rate of 4%, and subsequent annual withdrawals adjusted by inflation, never lasted less than 30 years, which is what he considered a minimum requirement for portfolio longevity.
the failure rate of 42.9% in Exhibit 3 decreases, in some cases substantially (to under 26% when lowering the initial withdrawal rate).

**Exhibit 4: Sensitivity Analysis on the Monte Carlo Simulation**

This exhibit summarizes the results of 10,000 scenarios for each of the five cases considered over an individual’s lifecycle, based on the statistical properties of the 60/40 and the 40/60 stock/bond allocations over the 1928-2019 period. Panel A considers the end of the accumulation period (the retirement portfolio) and panel B considers the end of the retirement period (the bequest). The second column (Lower W), shows the result of reducing the constant annual withdrawal by 10%; the third column (Lower IWR), the result of reducing the initial withdrawal rate by 1.3 percentage points; the fourth column (Higher R), the result of increasing the allocation to stocks during retirement by 10 percentage points; the fifth column (Higher S), the result of increasing the allocation to stocks during the working period by 10 percentage points; and the last column (Higher C), the result of increasing the constant annual contribution by 10%. ‘Failure’ is the percentage of scenarios in which the retirement portfolio was depleted before the end of the retirement period. All figures but ‘Failure’ in dollars.

<table>
<thead>
<tr>
<th>Panel A</th>
<th>Lower W</th>
<th>Lower IWR</th>
<th>Higher R</th>
<th>Higher S</th>
<th>Higher C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Median</td>
<td>1,164,871</td>
<td>1,183,060</td>
<td>1,181,517</td>
<td>1,307,785</td>
<td>1,286,243</td>
</tr>
<tr>
<td>1st percentile</td>
<td>361,001</td>
<td>373,071</td>
<td>359,312</td>
<td>348,357</td>
<td>396,888</td>
</tr>
<tr>
<td>5th percentile</td>
<td>506,093</td>
<td>508,136</td>
<td>495,709</td>
<td>499,644</td>
<td>555,407</td>
</tr>
<tr>
<td>10th percentile</td>
<td>598,404</td>
<td>607,665</td>
<td>598,791</td>
<td>615,908</td>
<td>663,209</td>
</tr>
<tr>
<td>90th percentile</td>
<td>2,321,623</td>
<td>2,343,782</td>
<td>2,352,519</td>
<td>2,935,031</td>
<td>2,562,628</td>
</tr>
<tr>
<td>95th percentile</td>
<td>2,840,918</td>
<td>2,864,485</td>
<td>2,887,476</td>
<td>3,679,903</td>
<td>3,111,627</td>
</tr>
<tr>
<td>99th percentile</td>
<td>4,096,042</td>
<td>4,151,768</td>
<td>4,230,587</td>
<td>5,489,660</td>
<td>4,599,491</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B</th>
<th>Lower W</th>
<th>Lower IWR</th>
<th>Higher R</th>
<th>Higher S</th>
<th>Higher C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Median</td>
<td>710,613</td>
<td>1,279,304</td>
<td>589,950</td>
<td>817,782</td>
<td>756,672</td>
</tr>
<tr>
<td>1st percentile</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>5th percentile</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>10th percentile</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>90th percentile</td>
<td>6,244,139</td>
<td>6,841,990</td>
<td>6,945,560</td>
<td>7,913,488</td>
<td>6,520,700</td>
</tr>
<tr>
<td>95th percentile</td>
<td>9,149,920</td>
<td>9,931,974</td>
<td>10,333,486</td>
<td>11,645,052</td>
<td>9,600,709</td>
</tr>
<tr>
<td>99th percentile</td>
<td>17,299,387</td>
<td>17,058,064</td>
<td>21,006,052</td>
<td>23,469,850</td>
<td>18,173,951</td>
</tr>
<tr>
<td>Failure</td>
<td>36.4%</td>
<td>25.7%</td>
<td>40.6%</td>
<td>37.8%</td>
<td>37.3%</td>
</tr>
</tbody>
</table>

The last two columns consider changes in variables related to the accumulation period and their impact on both the retirement portfolio and the bequest. When the 70/30 allocation replaces the 60/40 allocation, the median retirement portfolio increases by more than 11% (from $1,170,203 to $1,307,785) and the median bequest by over 120% (from $371,481 to $817,782); when annual contributions are increased from $9,079 to $9,986, the median retirement portfolio increases by almost 10% (from $1,170,203 to $1,286,243) and the median bequest by almost 104% (from $371,481 to $756,672). In both cases, the failure rate of 42.9% in Exhibit 3 decreases.

This type of analysis on the parameters of a Monte Carlo simulation makes it possible to evaluate the empirical impact of a change in any of the relevant variables of the model. The difference between the sensitivity analyses in Exhibits 2 and 4 is that, in the former, results are obtained from closed-form solutions based on empirical parameters; in the latter, results are obtained from a large number of potential scenarios based on the statistical behavior of the series considered. Importantly, both types of analysis are insightful and complement each other.
3.5. Further Discussion

The retirement planning model introduced here properly begins from the end (the retirement period and its relevant variables); then it works backward to determine the target retirement portfolio; and then it works backward once again to the beginning (the working period and its relevant variables). Although this is the only plausible way to plan for retirement, the model rests on the rather implausible assumption that a young individual can accurately foresee his life in retirement, determine a sequence of withdrawals to sustain it, decide the bequest he wants to leave, and select an asset allocation for the retirement period. This may be too much to ask from any young individual, however foresighted he may be.

However, this does not invalidate the model or imply that it cannot be plausibly implemented. It is indeed the case that without the inputs (exogenous variables) relevant for the retirement period neither a target portfolio nor the contributions to be made during the working years can be determined. But a realistic way to get around this problem would be for an individual at the beginning of his working years to make his best guess about the values of the exogenous variables, and then to update his guess over time, thus obtaining new solutions for the target portfolio and the contributions to be made during the rest of his working years.

The only difference with the approach previously proposed is that instead of going from no portfolio to a target portfolio over the whole accumulation period, the individual would instead go from the portfolio he has at the time of updating his guess to a different target portfolio over a shorter period of time. This would change expression (6) slightly but the rest of the process would remain unchanged.

To illustrate, assume that after working for 25 years and with 15 working years to go an individual now has a different view of the withdrawals he will need during retirement, or the bequest he wants to leave, or the asset allocation he wants to implement. He would then use the updated values of the relevant (exogenous) variables for the retirement period and find his new target portfolio by using expression (3), just as before. And he would then find the constant annual real contribution over his remaining 15 working years, beginning from the portfolio he has accumulated over the previous 25 years. This would require a slight modification to expression (6), which in the case would be

\[
C^* = - \frac{P^* - P_{25}(1+5)^{15}}{(1+5)-(1+5)^{15}}
\]

where \(P_{25}\) is the value of the portfolio accumulated over the first 25 working years and \(P^*\) would be the new target portfolio.

Introducing adjustments to a retirement plan should be considered the rule rather than the exception. These adjustments may follow from revised choices about the retirement period,
as in the example above; or from the impact of a wide variety of factors affecting the working period, such as changes in income or the introduction of shorter-term goals for which the individual will have to save, to name but two. Estrada (2019, 2020a) highlights the value of flexibility when planning for retirement, reflected in an individual’s ability to adjust contributions, withdrawals, or asset allocations when reality disrupts a financial plan.

4. Assessment

Retirement planning, like starting a diet or visiting the dentist, is something that many people always find reasons to postpone; dealing with these issues never appears to be imperative and short-term problems that must be taken care of usually take priority. In particular, young individuals at the beginning of their careers can hardly be blamed for lacking the motivation and the information to think about their retirement. But the inevitable result of all this is that many people approach the end of their careers unprepared, with little time to react and few options to explore.

To be sure, some people do intend to plan, but often at the wrong time or the wrong way. Seeking advice shortly before retirement on how to hit a target portfolio is doing it too late; not knowing whether the target portfolio will indeed sustain the desired lifestyle in retirement is doing it the wrong way. Proper retirement planning must be done early and solved backward. It must be done early because it is the best way to minimize the savings that need to be made, and to maximize the range of corrective options available; and it must be solved backward because it is the only way to ensure that the target portfolio is designed to sustain the desired retirement.

The approach introduced here starts from the end; that is, from assessing the desired retirement, the withdrawals from the portfolio necessary to sustain it, and the bequest to be left. These variables, together with an asset allocation for the withdrawal period (hence an expected return for the portfolio in retirement), jointly determine the target portfolio. And the latter, together with an asset allocation for the accumulation period (hence an expected return for the portfolio during the working years), jointly determine the periodic contributions during an individual’s career that need to be made to hit the target.

The analytical framework introduced here yields one closed-form solution for the target retirement portfolio as a function of the bequest, the periodic withdrawals, and the return expected from the portfolio in retirement; and another for the contributions to be made during the accumulation period as a function of the target retirement portfolio and the return expected from the portfolio during the working years. It also yields just one closed-form solution to determine the contributions to be made during the accumulation period as a function of all the relevant variables of the model; and it provides a tool to perform insightful sensitivity analysis to evaluate many of the relevant trade-offs that individuals planning for retirement have to face.
The example discussed illustrates how to apply the analytical framework, derive the value of the relevant variables, outline the expected path of the portfolio over the lifecycle, perform sensitivity analysis, and explore the relevant uncertainties by means of Monte Carlo simulations. It also highlights the exogenous variables that need to be decided before the analysis, and the endogenous variables determined by the model.

Understanding what needs to be done to plan properly for retirement is not very difficult; actually doing it is the problem. That said, it is in fact critical to have the correct analytical framework to point individuals and financial advisors in the right direction. Hopefully this article will contribute toward that worthy goal.
Appendix

This appendix discusses the analytical framework behind expressions (1) through (7) in the text. For the reasons already discussed, it starts with the retirement period, then continues with the working period, and concludes with the lifecycle. Exhibit A1 provides the timeline.

Exhibit A1: Timeline

This exhibit shows the timeline on which the analysis is based, with a 40-year working period and a 30-year retirement period. The individual makes 39 contributions ($C_t$) to, and 30 withdrawals ($W_t$) from, his retirement portfolio. The bequest ($B$) is what is left after taking the last withdrawal and one more year of compounding.

<table>
<thead>
<tr>
<th>$C_t$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$C_{38}$</th>
<th>$C_{39}$</th>
<th>$W_1$</th>
<th>$W_2$</th>
<th>$W_3$</th>
<th>$W_{28}$</th>
<th>$W_{29}$</th>
<th>$W_{30}$</th>
<th>$B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>$\ldots$</td>
<td>38</td>
<td>39</td>
<td>40</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>$\ldots$</td>
<td>28</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>$\ldots$</td>
<td>38</td>
<td>39</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>$\ldots$</td>
</tr>
</tbody>
</table>

Accumulation/Working Period

Withdrawal/Retirement Period

Lifecycle

The Retirement Period. Consider a representative individual at the beginning of his working years. He contemplates his retirement, which he expects to last $N$ years, and believes that it can be sustained by $N$ withdrawals ($W_1, \ldots, W_N$) from his retirement portfolio $P$, made at the beginning of each of his years in retirement. He aims to keep his purchasing power constant over time, which implies that $W_1 = \ldots = W_N = W$, where $W$ is the constant annual real (inflation-adjusted) withdrawal. He aims to leave a bequest $B$ and expects an annualized real return $R$ from his portfolio during retirement. At the beginning of his first three years in retirement the individual’s portfolio will be

Year 1: $P - W$

Year 2: $(P - W)(1 + R) - W = P(1 + R) - W(1 + R) - W$

Year 3: $[P(1 + R) - W(1 + R) - W](1 + R) - W = P(1 + R)^2 - W(1 + R)^2 - W(1 + R) - W$

It is clear then that at the beginning of his last year in retirement his portfolio will be

Year $N$: $P(1 + R)^{N-1} - W(1 + R)^{N-1} - W(1 + R)^{N-2} - \ldots - W(1 + R) - W$

This portfolio will compound over one more year (the individual’s last year in retirement) and become his bequest; hence

$\{P(1 + R)^{N-1} - W(1 + R)^{N-1} - W(1 + R)^{N-2} - \ldots - W(1 + R) - W\}(1 + R) = B$

and, therefore,

$P(1 + R)^N - W(1 + R)^N - W(1 + R)^{N-1} - \ldots - W(1 + R)^2 - W(1 + R) = B$

which can be rewritten as

$P \cdot (1 + R)^N - W \cdot \sum_{t=1}^{N} (1 + R)^t - B = 0$  \hspace{1cm} (A1)

This expression can be solved for the individual’s target portfolio ($P^*$), which is given by

$P^* = \frac{B + W \cdot \sum_{t=1}^{N} (1 + R)^t}{(1 + R)^N}$  \hspace{1cm} (A2)
Recall that the expression for the sum of a geometric series is given by

$$\sum_{t=0}^{T-1} a \cdot g^t = a \cdot \frac{1-g^T}{1-g}$$  \hspace{1cm} (A4)$$

where $a$ is a constant, $g$ is the periodic rate at which the series grows, and $T$ is the number of compounding periods.

Note that the sum term in (A2) can be rewritten as

$$\sum_{t=1}^{N} (1+R)^t = (1+R) \cdot \sum_{t=0}^{N-1} (1+R)^t$$  \hspace{1cm} (A5)$$

Using (A4), the sum term on the right-hand side of (A5) can be rewritten as

$$\sum_{t=0}^{N-1} (1+R)^t = \frac{(1-(1+R)^N)}{(1-(1+R))}$$  \hspace{1cm} (A6)$$

Thus, inserting (A6) into (A5) we obtain

$$\sum_{t=1}^{N} (1+R)^t = -(1+R) \cdot \frac{(1-(1+R)^N)}{R}$$  \hspace{1cm} (A7)$$

Finally, inserting (A7) into (A2) yields

$$P^* = \frac{B-W-(1+R)}{(1+R)^N}$$  \hspace{1cm} (A8)$$

which is the same as expression (3) in the text; that is, the individual's target portfolio as a function of his bequest, his constant annual real withdrawal, and the expected return of his portfolio over the $N$ years he expects to be retired.

**The Working Period.** Having determined his target portfolio and expecting to work for $M$ years, the representative individual needs to determine the annual contributions he needs to make to his retirement portfolio to hit the target $P^*$ on his retirement date. He expects to make $M-1$ contributions ($C_1, \ldots, C_{M-1}$) to his portfolio, the first at the end of his first working year and the last one year before retirement. He aims to keep the contributions constant in real terms, which implies that $C_1 = \ldots = C_{M-1} = C$, where $C$ is the constant annual real (inflation-adjusted) contribution. Finally, during his working years he expects an annualized real return $S$ from his portfolio. Thus, at the end of the first three years of his working period the individual's portfolio will be

<table>
<thead>
<tr>
<th>Year</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$C$</td>
</tr>
<tr>
<td>2</td>
<td>$C(1+S) + C = C(1+S) + 1$</td>
</tr>
<tr>
<td>3</td>
<td>$C<a href="1+S">(1+S) + 1</a> + C = C[(1+S)^2 + (1+S) + 1]$</td>
</tr>
</tbody>
</table>

It is clear then that at the end of his next-to-last working year his portfolio will be

<table>
<thead>
<tr>
<th>Year</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M-1$</td>
<td>$C[(1+S)^{M-2} + (1+S)^{M-3} + \ldots + (1+S)^2 + (1+S) + 1]$</td>
</tr>
</tbody>
</table>

This portfolio will compound over one more year (the individual's last working year) and hit his target portfolio on the retirement date; that is,

<table>
<thead>
<tr>
<th>Year $M$</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$</td>
<td>${C[(1+S)^{M-2} + (1+S)^{M-3} + \ldots + (1+S)^2 + (1+S) + 1]}(1+S) = P^*$</td>
</tr>
<tr>
<td></td>
<td>$C[(1+S)^{M-1} + (1+S)^{M-2} + \ldots + (1+S)^2 + (1+S)] = P^*$</td>
</tr>
</tbody>
</table>
which can be rewritten as

\[ P^* = C \cdot \sum_{t=1}^{M-1} (1 + S)^t \]  
(A9)

This expression can be solved for the constant annual real contribution \( C^* \), which is given by

\[ C^* = \frac{P^*}{\sum_{t=1}^{M-1} (1 + S)^t} \]  
(A10)

Note that the denominator of (A10) can be rewritten as

\[ \sum_{t=1}^{M-1} (1 + S)^t = (1 + S) \cdot \sum_{t=0}^{M-2} (1 + S)^t \]  
(A11)

Thus, using (A4) again, the sum term on the right-hand side of (A11) can be rewritten as

\[ \sum_{t=0}^{M-2} (1 + S)^t = \left( \frac{1 - (1 + S)^{M-1}}{1 - (1 + S)} \right) = -\frac{1 - (1 + S)^{M-1}}{S} \]  
(A12)

Replacing (A12) into (A11) we obtain

\[ \sum_{t=1}^{M-1} (1 + S)^t = (1 + S) \cdot \sum_{t=0}^{M-2} (1 + S)^t = -\left( \frac{1 - (1 + S)^{M-1}}{S} \right) \]  
(A13)

and rearranging (A13) we get

\[ \sum_{t=1}^{M-1} (1 + S)^t = (1 + S) \cdot \sum_{t=0}^{M-2} (1 + S)^t = -\left( \frac{(1+S)-(1+S)^{M}}{S} \right) \]  
(A14)

Finally, replacing (A14) into (A10) we obtain

\[ C^* = -\frac{P^*}{\frac{(1+S)-(1+S)^{M}}{S}} \]  
(A15)

which is the same as expression (6) in the text; that is, the individual’s constant annual real contribution as a function of his target portfolio and the expected return of his portfolio over the \( M \) years he aims to work.

The Lifecycle. As a final and unifying step note that the individual may skip setting a target portfolio and simply consider how much he needs to contribute during his working years to sustain the expected withdrawals during his retirement. Thus, replacing (A8) into (A15) we obtain

\[ C^* = -\frac{B-W \cdot (1+R) \frac{1-(1+R)^N}{R}}{\frac{(1+S)-(1+S)^{M}}{S}} \]  
(A16)

which is the same as expression (7) in the text; that is, the individual’s constant annual real contribution as a function of his bequest, his constant annual real withdrawal, and the annualized returns of his portfolio during his \( N \) years in retirement and his \( M \) working years.
References


