# Building and Maintaining an Assortment of Durable Goods

Mihalis G. Markakis* Victor Martínez-de-Albéniz† Xabier Barriola‡
Claudio Sánchez§

**Abstract**

**Problem Definition:** Product proliferation and renewal force consumers to continuously scan the market and selectively buy highly desired new items. At the same time, these products are typically durable, a fact that forces consumers to think about how a new addition will contribute to her already existing assortment. **Methodology/Results:** We develop a modeling framework based on dynamic programming, in which, given an assortment of products in the same category, a consumer decides in each period which product to use and how many to add to her assortment. As time evolves, older items may decay and lose their value, which triggers a need for assortment renewal. We provide a structural characterization of the value function and optimal policy in the general case, as well as a closed-form analytical solution in the case of exponentially-decaying product utilities. We then explore in more depth two distinguishing features of durable products, obsolescence and value uncertainty, and their implication on optimal assortments, firm revenue, consumer welfare, and generated waste. We find that, when consumers discount the future, firms have the incentive to offer products with faster obsolescence, generating conflicting interests between firms on the one hand and consumers and sustainability on the other hand. This alignment problem cannot be solved by taxing sales, as firms and (rational) consumers would internalize such taxation schemes, but can be mitigated through policies that reduce the maintenance cost of the assortment. In contrast, value uncertainty negatively affects consumers and firms alike, suggesting that it is better for everyone to offer goods with predictable future utility. **Managerial Implications:** Our results suggest that business models that help consumers maintain longer-lasting assortments may be more sustainable, financially and environmentally, than fast-fashion ones, where both obsolescence and uncertainty are high.

Keywords: durable goods; assortment management; obsolescence; value uncertainty; Dynamic Programming.

## 1 Introduction

The fast fashion industry, spearheaded by companies such as Inditex and H&M, has experienced a meteoric rise during the last three decades, offering consumers a seemingly endless variety of products at very affordable prices. Perhaps surprisingly, it is not uncommon for consumer surveys to reveal disappointment with the owned fast fashion assortments, despite the presumed variety

*IESE Business School, University of Navarra; MMarkakis@iese.edu
†IESE Business School, University of Navarra; VAlbeniz@iese.edu
‡INSEAD; xabier.barriola@insead.edu
§IESE Business School, University of Navarra; CSanchez@iese.edu

and affordability. For instance, a recent survey reported that the average American woman had 103 items in her closet, out of which 21% was declared to be "unwearable" and 12% had been never been worn. Additionally, 44% of the women surveyed said that they could not find an adequate piece of clothing inside their closet, so around 61% of them decided to buy new items. However, even after acquiring more items, 10% of the respondents felt depressed every time they opened their closet (Closet Maid 2016). At the same time, the industry has come under the scrutiny of policy makers for its unsustainable practices, in terms of wasteful production methods as well as huge amounts of finished-products waste. Examples are abundant: on the production side, indicatively, around 7,500 liters of water are used to produce a standard pair of jeans (United Nations 2019); on the consumer side, a recent report shows that the actual usage of fashion products has decreased by around 30% between 2002 and 2016 (Ellen MacArthur Foundation 2017). These trends suggest that there seems to be a contradiction between the short-term appetite for buying new items, and the longer-term disappointment with keeping them for a long period of time in one's closet. Moreover, the consequence of this apparent contradiction is a poor outcome in terms of social welfare, because of the high waste generated by the industry.

A critical observation towards understanding the origins, and the way out of this impasse, is the fact that fast fashion has, in essence, introduced a new category of products, whose utilities to the consumer vary significantly with time and available assortment. The life cycles of those apparel products are typically shorter than what existing literature would traditionally classify as durable goods (cars, furniture, large domestic appliances). On the other hand, fast fashion items are clearly not consumable goods either (food, drinks, medicine), because they are available for use during a few years. For this new breed of items, consumers have to actively manage their assortment, in terms of dynamic acquisition and disposal. In contrast to the extensive literature on assortment planning in Operations Management, in which assortments are optimized by a firm to maximize sales, here the assortment is built and maintained by the consumer, generating an expected utility from usage.

In this paper, we introduce a framework capturing the dynamics and decisions of assortment management, from the perspective of a rational consumer. Specifically, we consider a risk-neutral consumer, who optimizes her total expected net utility over a period of time, balancing acquisition cost, holding cost, and expected utility from an assortment of durable products, by deciding when and how much to purchase from a given product category. We provide a structural characterization of the value function and optimal policy for the resulting Dynamic Program in the general case, as well as a closed-form analytical solution in the case of exponentially-decaying product utilities.

Our objective for introducing and analyzing the above assortment management model is three-fold. First, we aspire to provide a micro-founded benchmark for rational decision making in the context of durable products, applicable not only to fast-fashion consumption, but also to other consumption types in which products are durable but sufficiently affordable so that multiple items are simultaneously owned and used over time. It can then be contrasted to qualitative or empirical findings in the literature on consumer behavior, in order to identify possible deviations, e.g., due to behavioral biases. Second, we wish to understand the incentives of rational consumers, and whether they are aligned or not to those of firms and policy makers. Importantly, our model provides one of the first frameworks to analyze the incentives of the fast fashion industry, where demand is endogenous to the decisions made by the firms. Third, the model introduced in this paper can lay the foundations for a future decision support system for consumers seeking to optimize their investments in durable products – their wardrobe, in particular.

The main insights of our work stem from the way that two distinguishing features of fast fashion products, obsolescence and value uncertainty, affect rational consumer behavior and, through that, firm profit as well as waste and societal welfare. We find that in the case of obsolescence, i.e., the rate at which the utility of fast fashion products decreases over time, the incentives of consumers are diametrically opposed to those of the firm when consumers discount the future: the firm's profit is maximized through products with fast decay, whereas the consumer's net utility is maximized through products of slow decay. This may not always be true when the discount factor is near one, although we still observe opposing interests frequently. Importantly, any tax on sales, intended to improve social welfare or reduce the rate of generated waste, is doomed to fail. This is in contrast to the findings of Plambeck and Wang (2009), which consider a similar problem in the context of consumer electronics, but from the firm's point of view and with exogenous demand. In contrast, we show that a "property" subsidy on the owned assortment, i.e., a reduction in the holding cost for old products that remain in use, is successful in aligning the incentives of consumers and the firm, primarily because product demand is endogenous in our case. In the case of value uncertainty, i.e., the unpredictable variability in the future utility to be received, the incentives of consumers are organically aligned to those of the firm, and lesser uncertainty is preferred by both sides. In other words, it would be better for everyone if the purported value of fast fashion products were more predictable, because this would increase consumer's future utility and incentivize larger assortments which would, in turn, be beneficial for the selling firm. This suggests that 'slower' fashion would be a win-win-win development for consumers, firms, and waste reduction, although this would have to be combined with more sustainable production and recycling practices.

The remainder of the paper is organized as follows. Section 2 reviews the relevant literature. We introduce a model of dynamic assortment management for a rational consumer, and provide a structural characterization of the value function and the optimal policy in §3. We derive managerial insights regarding the role of obsolescence and value uncertainty, and their impact on consumer behavior and firm incentives via numerical experiments in §4. Section 5 concludes the paper and presents avenues for future research.

## 2 Literature Review

The problem of consumer-side (rational) assortment management has not received much attention in the literature thus far. However, as we argue above, it is important to understand how consumers make decisions when they acquire durable goods, when they use them, and when they discard them. This becomes particularly pressing when the utility of these durable goods decays quickly over time (Coase 1972).

In contrast, there is extensive and, by now, relatively mature literature on firm-side assortment planning within Operations Management. Van Ryzin and Mahajan (1999) are the first to formulate the static assortment planning problem using a multinomial logit model. Their model captures the trade-off between offering variety and inventory costs. This seminal paper triggered many developments in assortment planning, see Kök et al. (2008) and Caro et al. (2020) for reviews of this literature. The study of assortment dynamics, closer to this paper's context, has also been quite active. Smith and Agrawal (2000), Mahajan and Van Ryzin (2001), Honhon et al. (2010) or Martínez-de Albéniz and Kunnumkal (2022) consider static assortments that evolve over time due to sales, and generate different substitutions across products once stock-outs appear. Caro and Gallien (2007) and Sauré and Zeevi (2013) consider assortments that can be updated from period to period, while firms learn about product's attractiveness. Bernstein and Martínez-de-Albéniz (2017) and Ferreira and Goh (2021) analyze dynamic assortments in the presence of strategic consumers and show that rotating the assortment can increase consumer visits to the store and sales. Caro et al. (2014) look at multi-period assortment planning, where product attractiveness decays over time, as in our paper, so it becomes important to spread product introductions over the sales season.

These works consider assortments over time, but focus exclusively on the consumer purchase process. The objective is to maximize the expected profit from the sequence of assortments, which is mainly driven by the sales probability, derived from a choice model. Hence, what happens after purchase is ignored. A notable exception, and perhaps the paper that comes the closest to the present work, is Fox et al. (2017), where a rational consumer buys a bundle of products that are sequentially consumed. This is similar to our model, except that in our case products remain in the assortment with a certain decay and variability around their expected utility, while in Fox et al. (2017) items disappear from the assortment. In other words, decay in Fox et al. (2017) is complete (retention of utility is zero) but contingent on usage, while in our model it is partial, but independent of usage. The authors find that carrying variety in the bundle is valuable, so as to exploit the value of broader variety at the time of consumption – which allows the consumer to pick the highest-utility item from all the remaining ones in the bundle. Our model reveals similar insights: carrying a broad assortment may be valuable so as to generate more variability in usage values, leading to higher utility. In contrast to Fox et al. (2017) though, we account for product partial decays and the possibility of renewing them whenever necessary, at any point of time during the horizon. These features make our model more appropriate for durable goods, and hence it can be applied to new contexts of interest. In addition, we advance the understanding of the problem at hand by analytically describing the assortment renewal process.

On the other hand, product usage has been the focus of a separate literature thread in Economics, studying durable goods. For these products, decay takes the spotlight, as consumers can use an item multiple times before its value disappears. The early focus on decay revolved around planned obsolescence, and how items would lose their value after a certain amount of time (Bulow 1986). Purohit (1992) focuses on the auto industry and shows that new vehicles have a negative effect on the price of used ones. This suggests that manufacturers phase out older items to highlight the fashionability of innovation. Gordon (2009) studies the PC processor industry to show that there is a negative connection between prices and replacements, and a positive relationship between quality and replacement. Therefore, consumers are willing to pay more for items whose quality is better. Several studies have focused on the dilemma about leasing vs. selling. In summary, leasing provides a higher usage rate (Gavazza 2011), which ultimately depends on the reliability of the product (Desai and Purohit 1999), although it might not be a good choice for the environment (Agrawal et al. 2012). Finally, the importance of technology and quality as the driving force behind new durable goods introduction has been studied in Fishman and Rob (2000), Gowrisankaran and Rysman (2012), and Lobel et al. (2016), among others.

Apart from the literature on durable goods, our aim is also to contribute to the burgeoning literature on sustainable operations. Related to the present work, Plambeck and Wang (2009) analyzes the effects of taxes on product waste in the consumer electronics industry. The authors find that sales decrease after incorporating an additional fee at the moment of purchase. Moreover, when the tax is included at the time of disposal, the manufacturer's profits are reduced. Long and Nasiry (2022) focuses on the sustainability of the fast fashion industry, and shows that the key driver of low product quality is the firm's incentive to offer variety. The authors also find that waste disposal policies and production taxes are effective in reducing the firm's leftover inventory. Chen et al. (2022) studies the agility of fast fashion retailers, and explores the optimal tax design for such firms that improves social welfare. We, too, investigate how taxes and subsidies can be used to align the incentives between the firm and consumers. We contribute to the literature by suggesting age-dependent subsidies that allow to align firm and consumer incentives.

# 3 Modeling and Analysis

## 3.1 Motivating Survey Data

Before undertaking the modeling task, and in order to obtain a better understanding of consumer behavior regarding durable goods purchasing and usage, we carried out a small-scale survey where we asked participants about their everyday use of clothes and shoes. The latter constitute a prime example of durable goods where value decay and uncertainty play an important role in the way consumers build and maintain their assortments dynamically. Out of 90 respondents, with an average age of 30.2 years, we found that they owned an average of 14.4 pairs of shoes, 20.9 lower-body apparel, and 42 upper-body apparel; and were spending an average of 831.2 euros in clothing per year. One interesting aspect of the survey is that regardless of the size of their assortments, respondents wore their "favorite" items more: on a monthly basis, they chose their favorite shoes 13.9 days, their favorite pants 8.4 days, and their favorite shirt 6.1 days. The small size of our survey notwithstanding, consumers tended to purchase many more items than the ones they used with regularity.

One participant in our survey, in particular, agreed to share with us the clothes that she selected to wear a period of 42 days. (We obtained the photographs for 31 days, and a detailed description of the items for the entire period.) The participant used 31 different upper-body garments, repeating 7 items twice, and 2 items three times. In terms of lower-body apparel, she used 14 different garments, repeating 3 items twice, 1 item three times, 2 items four times, and 1 item six times. Figure 1 shows how this participant decided to mix her assortment by combining different apparel over the course of the survey. She did not decide to repeat the same garment twice in a row, but she did repeat several lower-body items over a short period.


Figure 1: Assortment usage by a participant in our small-scale survey.

These findings motivate the modeling framework that we introduce below, and provide an anecdotal benchmark against which we can compare the insights resulting from our analysis.

### 3.2 A Model of Dynamic Assortment Management of a Rational Consumer

In this section, we introduce a consumer-side model of *dynamic assortment management*, for products within the *same product category*, over their life cycle: from acquisition, to use, and eventually disposal. As prime examples of the application scope of our modeling framework, one could think of clothes, shoes, and accessories; which, on the one hand, are clearly not consumable goods; but, on the other hand, devalue relatively quickly with time, in contrast to what the literature has traditionally considered as durable goods, such as cars, furniture, and large domestic appliances. It is precisely for such products that dynamic assortment management makes the most sense. We adopt the "canonical" viewpoint of a *rational, risk-neutral consumer*, whose objective is to maximize her total expected discounted net utility, unaffected in her decisions by any behavioral biases.

We assume that every product in the assortment can be in one of $n \geq 2$ different stages, indexed from 0 to $n-1$, which signify the number of time periods that the product has been in the assortment: products in stage 0 have just been acquired, and are thus considered "new," while products in stage $n-1$ are considered "old" and will become obsolete, and hence discarded from the assortment, in the very next period. The *state* of the assortment is captured by the generic vector $\mathbf{x} \in \mathbb{R}_+^n$, where $x_i$ denotes the number of distinct products in stage $i$, for $i = 0, \ldots, n-1$. For tractability, we assume that the composition of the assortment takes values in the nonnegative real numbers. However, our numerical experiments are conducted for the more realistic case of nonnegative integers.

The consumer uses exactly one of the products during each time period, as we assume that they belong to the same product category; the one that provides the highest expected utility. We assume that the utility that product $k$, in stage $i = 0, \ldots, n-1$, at time period $t = 0, \ldots, T-1$, provides to the consumer is comprised of a stage-specific component $\log(u_{t,i})$, common to all products in stage $i$ and period $t$, plus a product-specific random shock $\xi_{t,i,k}$. We assume that these random shocks are i.i.d., over products, stages, and periods, following the standard Gumbel distribution, i.e., with mean equal to zero, and scale parameter equal to one. The shocks capture item-specific preference changes, e.g., a dinner party that requires elegant clothing. This implies that, conditional on the parameters $u_{t,i}$, the probability that product $k$ delivers the highest utility is given by the well-known multinomial logit (MNL) formula $u_{t,i}/(1 + \sum_{j=0}^{n-1} u_{t,j} x_j)$. Moreover, the total expected utility received by the consumer, given an assortment $\mathbf{x}$, is equal to

$$\log\left(1 + \sum_{i=0}^{n-1} u_{t,i} x_i\right).$$

It is reasonable to assume that, in the beginning of period $t$, the parameters $u_{t,i}$ are known to the consumer. Before that period, however, e.g., at the time of acquisition of a product, these parameters may have a component that is deterministic as well as one that is random. The randomness could be the result of unpredictable variability in user preferences; for instance, in an apparel context, changes in the prevailing fashion styles, or a colder-than-usual season that mandates warmer clothes. Therefore, we define the (instantaneous) reward function of the consumer as follows:

$$U_t(\mathbf{x}) \doteq \mathbb{E}_{u_{t,i}} \log \left( 1 + \sum_{i=0}^{n-1} u_{t,i} x_i \right). \tag{1}$$

The introduction of uncertainty on the parameters $u_{t,i}$, so that the reward function is defined as the expectation over their joint distribution, allows us to capture the fact that consumers, at the time of purchase, need to pay a known price for a product with intrinsically unknown value, during the various stages of its life cycle. Furthermore, uncertainty would be extremely helpful to account for product heterogeneity, when our model is estimated from real data; see the related discussion in §5.

The formulation above allows us to capture several important phenomena related to durable goods, and fashion items in particular. First, we are able to introduce seasonal values; for instance, if a period denotes a quarter, then we could set $\log(u_{t,i}) = \log\left[1 + \cos\left(\frac{2\pi i}{4}\right)\right]$, so that, one year after buying a product (in stage $i = 4$), its value has been restored to its starting level. Second, our formulation allows to incorporate random fashion trends that increase the utility of old items similar to the current trend, e.g., $\log(u_{t,i}) = f\left(\|Z_{t-i} - TREND_t\|\right)$, where $Z_{t-i}$ is a multi-dimensional vector of attributes of the products sold in period $t - i$, $TREND_t$ are the preferred attributes for the market, and $f(\cdot)$ is a decreasing function. Third, our formulation allows us to introduce products whose utilities decay with age; for example $\log(u_{t,i}) = \alpha_{t-i} + \log(\beta)i$, with $\beta \in [0, 1]$, and $\alpha_\tau$ providing the base utility of products bought in period $\tau$. This specification implies that $u_{t+1,i+1} = u_{t,i}\beta \leq u_{t,i}$. Finally, the model is sufficiently flexible to incorporate all these effects together, e.g., $\log(u_{t,i}) = \log\left[1 + \cos\left(\frac{2\pi i}{4}\right)\right] + f\left(\|Z_{t-i} - TREND_t\|\right) + \log(\beta)i$.

Apart from the usage value of an assortment, there are also costs associated with building and maintaining it. New products can be bought at a price $p_t$ per unit in period $t$. The fact that we assume a single price for all products purchased during a period is, again, motivated by our assumption that all products belong to the same category: while in practice there will invariably be price variations, competition should keep them relatively close. If the consumer decides to buy new products, they are made available to her in the next period as new, i.e., in stage $i = 0$, at period $t + 1$. Moreover, we assume that, for each product in stage $i$ and period $t$, the consumer incurs an (instantaneous) inventory holding cost $h_{t,i}$. The latter cost could be capturing actual cost of maintaining the assortment, but more importantly, the opportunity cost of having "invested" in the assortment and not on alternative opportunities.

Summarizing, the net reward at time $t$, at state $\mathbf{x}$ and under purchasing decision $q$, is equal to

$$\pi_t(q, \mathbf{x}) \doteq U_t(\mathbf{x}) - \sum_{i=0}^{n-1} h_{t,i} x_i - p_t q. \tag{2}$$

Observe that, while we have described $U_t(\cdot)$ as grounded in the MNL choice model, our formulation and later results are only expressed as function of $U_t(\cdot)$. This suggests that one may consider alternative primitives for the usage process. The only condition for our general results to hold true is that $U_t(\cdot)$ is jointly concave in $\mathbf{x}$. This condition is satisfied when $U_t(\cdot)$ follows (1).

We can formulate the dynamic assortment management problem of a rational consumer as a Dynamic Program (DP). The value function at state $\mathbf{x} = (x_0, \dots, x_{n-1}) \in \mathbb{R}_+^n$ is given by the recursion:

$$J_t(\mathbf{x}) = \max_{q \geq 0} \left\{ \pi_t(q, \mathbf{x}) + \delta J_{t+1}(q, x_0, \dots, x_{n-2}) \right\}, \tag{3}$$

where $t = 0, \dots, T-1$, with $J_T(\cdot) \doteq 0$ and $0 \leq \delta \leq 1$ is a discount factor. We denote the optimal solution to the above recursion in period $t$ by $q_t^*$, which corresponds to the optimal purchasing quantity.

In the model above, dynamic assortment management has a single operational lever, product purchasing, as the discounted long-term utility and cost of products are endogenized in the purchasing decision. One could consider an extension to our benchmark model with a second decision, the dynamic disposal of items, before they become obsolete at the end of their life cycle. Such a model would be particularly meaningful under storage capacity or budget constraints. Nevertheless, for the types of products that motivate our work (clothes, shoes, gadgets, consumer electronics, books) these constraints do not seem to be a first-order consideration.

### 3.3 Structure of Value Function and Optimal Policy

We begin our analysis by providing a structural characterization of the value function and the optimal policy of the DP in Eq. (3). We note that the analysis is only made possible by the assumption that state and decision variables are continuous; assuming them taking integer values would complicate the analysis significantly, making it unlikely to be tractable. We relegate all proofs of the results presented in the remainder of the section to Appendix A.

**Proposition 1** *The value function $J_t(\cdot)$ is concave in $\mathbf{x}$.*

Synonymous to concavity is that the marginal value of an item at stage $k$ is decreasing, i.e., $J(x_0, \dots, x_{k-1}, x_k + 1, x_{k+1}, \dots, x_{n-1}) - J(x_0, \dots, x_{k-1}, x_k, x_{k+1}, \dots, x_{n-1})$ decreases in $x_k$. The proposition, thus, establishes that there are decreasing returns with respect to the number of items carried in the assortment.

Given the concavity, one may expect that $J_t(\cdot)$ is also submodular. Such condition would imply that items at different stages are largely substitutes, i.e., that the marginal value $J(x_0, \ldots, x_{k-1}, x_k + 1, x_{k+1}, \ldots, x_{n-1}) - J(x_0, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_{n-1})$ decreases in $x_j$, $j \neq k$. While we show that this is the case when there is no uncertainty in the parameters $u_{t,i}$, perhaps surprisingly, we also prove that this property does not hold in general.

**Proposition 2** *If $Var(u_{t,i}) = 0$ for all $t$ and $i$, and $q_t^* > 0$ for all $t$, then $J_t(\cdot)$ is submodular.*

Establishing the submodularity of $J_t(\cdot)$ is cumbersome: in the proof, we use the implicit condition $q_t^* > 0$ together with the structure of function $U_t(\cdot)$ without uncertainty, to simplify the recursive computation of $\frac{\partial^2 J_t}{\partial x_i \partial x_j}$ and prove the result. We also demonstrate that when there is uncertainty in $u_{t,i}$, and especially in the presence of negative correlations, $J_t(\cdot)$ may not be submodular. Indeed, if random utilities are negatively correlated in time, then it is plausible that the marginal value of a product is affected in different ways by the availability of products in the assortment in different stages of their life cycle; that is, $J(x_0, \ldots, x_{k-1}, x_k + 1, x_{k+1}, \ldots, x_{n-1}) - J(x_0, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_{n-1})$ may decrease in some $x_j$'s, $j \neq k$, but may increase in others.

The structural characterization of the value function of the DP in Eq. (3) follows, naturally, a characterization of the optimal policy.

**Proposition 3** *Let the optimal purchasing quantity at period $t$, $q_t^*(\mathbf{x})$, be strictly positive. Then, it satisfies:*

$$q_t^*(\mathbf{x}): \ \delta \frac{\partial J_{t+1}}{\partial x_0}(\mathbf{x}) = p_t. \tag{4}$$

*Furthermore, if $J_{t+1}(\cdot)$ is submodular, then the sensitivity of the optimal purchasing quantity with respect to the existing assortment satisfies:*

$$\frac{\partial q^*}{\partial x_i}(\mathbf{x}) \leq 0, \qquad i = 0, \ldots, n-2. \tag{5}$$

The first part of Proposition 3 shows that the optimal purchasing quantity, in any period, is determined by the familiar condition that the marginal value, $\frac{\partial J_{t+1}}{\partial x_0}$, has to be equal to the marginal cost, $p_t$. The *uniqueness* of the optimal policy is guaranteed by the fact that $\frac{\partial J_{t+1}}{\partial x_0}(q, \mathbf{x})$ decreases in $q$ from Proposition 1. The second part of Proposition 3, valid under the conditions of Proposition 2, shows that the optimal purchasing quantity decreases in the existing assortment; an intuitive result if the products are largely substitutes (under submodularity).

### 3.4 A Tractable Case: Stationary, Exponential Rewards and Long Horizon

The DP in Eq. (3) provides a general formulation for the dynamic assortment management problem of a rational consumer that, unfortunately, does not admit a closed-form solution. Put differently, at this level of generality, the most that one can hope for, on the theory side, is the structural characterization provided in §3.3. Here, we investigate a special case with time-stationary parameters, exponential utilities and costs, and infinite horizon, where a tractable analysis and a closed-form solution are possible.

Specifically, we consider a scenario where both $T$ and $n$ are very large; infinite in our mathematical model. The utility parameters $u_{t,i}$ are time-homogeneous and decay exponentially in $i$, so that $u_{t,i} = u_t\beta^i$, where $\beta \in (0, 1)$ and $\{u_t\}$ is a sequence of i.i.d. copies of a nonnegative random variable $u$. Moreover, we assume that the holding costs are also homogeneous and also decay exponentially in $i$, i.e., $h_{t,i} = h\beta^i$; and that prices do not change, i.e., $p_t = p$. In this setting, the DP in Eq. (3) can be solved in closed-form, as shown next.

**Proposition 4** *Consider the dynamic assortment management problem of §3.2, with $T = n = \infty$, $u_{t,i} = u_t\beta^i, h_{t,i} = h\beta^i$, and $p_t = p$. The optimal policy is a time-stationary buy-up-to policy. More specifically, letting $s = \sum_{i=0}^{\infty}\beta^i x_i$, there exists $\bar{q}$ such that it is optimal to purchase $q^* = \max\{\bar{q} - \beta s, 0\}$; and $\bar{q}$ is satisfies:*

$$\mathbb{E}\left[\frac{1}{1/u + \bar{q}}\right] = p\left(\frac{1}{\delta} - \beta\right) + h. \tag{6}$$

The tractability of this special case stems from the fact that the high-dimensional state space, in the general case, can be reduced to a single-dimension state here, $s = \sum_{i=0}^{\infty}\beta^i x_i$, as can been seen in the proof. This implies that, under the optimal policy, the single-dimension state becomes $s = \bar{q}$ and the quantity bought in every period is $q^* = \bar{q} - \beta\bar{q} = (1 - \beta)\bar{q}$.

This result sheds light on the impact of the model parameters on $\bar{q}$, which relates to the "target" assortment size; more precisely to $s$, the weighted sum of the products in the assortment. Since the left-hand side of (6) is decreasing in $\bar{q}$, it follows that the assortment size decreases in $p$ and $h$, as one would expect, because, as the assortment maintenance costs increase, the appropriate response is to reduce it. We also see that as $\beta$ and $\delta$ increase, i.e., as a product becomes more durable (higher $\beta$) or the future becomes more valuable (higher $\delta$), it is worthwhile investing in broader assortments, because they cost the same but deliver more future value.

An interesting insight is derived with respect to the uncertainty of $1/u$: since $1/(x+\bar{q})$ is convex in $x$, as $\sigma_{1/u}$ increases (keeping the expected $1/u$ unchanged) the term $\mathbb{E}\left[\frac{1}{1/u+\bar{q}}\right]$ increases, and hence it becomes optimal to increase $\bar{q}$. In other words, more utility uncertainty leads to larger assortments. This is intuitive as consumer utility is obtained by picking the largest realization of the individual product utilities. More variability of these utilities is more valuable with a larger assortment. Thus, the incentive to carry a larger assortment increases with the variability of $1/u$. In a way, this can be interpreted as a real option, which becomes more valuable as underlying risks are amplified; see Trigeorgis (1996). Note however that this does not mean that the consumer is better off with more uncertainty. In fact, expected utility is decreasing with the variability of $u$, as we observe in §4.2. But a way to protect the consumer against the drop in utility is to opt for a broader assortment.

### 3.5 Tractable Lower and Upper Bounds

The tractable analysis for the case of time-stationary, exponential rewards, motivates the developments in this section. In short, we show that even if the rewards are not exponential but can rather be bounded from above and below by exponential functions, then tractable upper and lower bounds can be established for the corresponding value function that hold for *finite horizon and product life cycle.*

More concretely, we assume that there exist universal constants $0 < \beta_{LB} \leq \beta_{UB} \leq 1$, as well as nonnegative sequences of random variables $\{u_{LB,t}\}$ and $\{u_{UB,t}\}$, independent of $\{u_{t,i}\}$, such that

$$u_{LB,t} \cdot \beta_{LB}^i \leq u_{t,i} \leq u_{UB,t} \cdot \beta_{UB}^i \qquad \text{with probability 1, for all } i \text{ and } t. \tag{7}$$

Further, we define

$$h_{UB,t} \doteq \max_i \left\{ \frac{h_{t,i}}{\beta_{LB}^i} \right\} \text{ and } h_{LB,t} \doteq \min_i \left\{ \frac{h_{t,i}}{\beta_{UB}^i} \right\}, \text{ for all } t,$$

and through the above, the functions $\mathbb{R}_+^2 \to \mathbb{R}_+$:

$$\pi_{LB,t}(q,s) \doteq \mathbb{E}\left[\log\left(1 + u_{LB,t} \cdot s\right)\right] - h_{UB,t} \cdot s - p_t q,$$

and

$$\pi_{UB,t}(q,s) \doteq \mathbb{E}\left[\log\left(1 + u_{UB,t} \cdot s\right)\right] - h_{LB,t} \cdot s - p_t q.$$

These lead to formulating the related one-dimensional Dynamic Programs:

$$J_{j,t}(s) = \max_{q \geq 0} \left\{ \pi_{j,t}(q,s) + \delta J_{j,t+1}\big(q + \beta_j s\big) \right\}, \qquad j \in \{LB, UB\}, \tag{8}$$

with $J_{j,T}(\cdot) \doteq 0$. Since the reward functions $\pi_{LB,t}(\cdot)$ and $\pi_{UB,t}(\cdot)$ are designed to bound, from below and above respectively, the reward function of the original Dynamic Program in Eq. (2):

$$\pi_{LB,t}\big(q, s_{LB}(\mathbf{x})\big) \leq \pi_t(q,\mathbf{x}) \leq \pi_{UB,t}\big(q, s_{UB}(\mathbf{x})\big), \qquad \forall (q,\mathbf{x}) \in \mathbb{R}_+^{n+1}, \qquad \forall t,$$

where

$$s_j(\mathbf{x}) \doteq \sum_{i=0}^{n-1} \beta_j^i x_i, \qquad j \in \{LB, UB\},$$

it should come as no surprise that the value functions in Eq. (8) provide similar bounds to $J_t(\cdot)$.

**Proposition 5** *Consider the dynamic assortment management problem of §3.2, with finite horizon, $T$, and product life cycle, $n$; and with the random utilities satisfying the exponential-decay condition in Eq. (7). Then, the value function of the Dynamic Program in Eq. (3) is bounded as follows:*

$$J_{LB,t}\big(s_{LB}(\mathbf{x})\big) \leq J_t(\mathbf{x}) \leq J_{UB,t}\big(s_{UB}(\mathbf{x})\big), \qquad \forall \mathbf{x} \in \mathbb{R}_+^n,$$

*for every $t \in \{0, \ldots, T-1\}$.*

We note that these bounds are tight, in the following sense: if the utilities provided by the different products decay (precisely) exponentially with their age, then the lower and the upper bound coincide. Moreover, it is straightforward to show that $J_{j,t},\ j \in \{LB, UB\}$, are concave and increasing one-dimensional functions.

# 4 The Impact of Value Decay and Uncertainty on Optimal Assortments

The modeling framework of the previous section allows us to understand how a consumer should manage an assortment of durable goods. Here, we investigate how different model parameters affect consumer welfare, firm revenue, and generated waste, in a scenario where the demand is "rational" and endogenous. We seek to identify situations where the incentives of firms and consumers are inherently aligned; or in conflict, where the deciding party – the firm in most cases – will opt for outcomes that make the other party worse off. Indeed, a well-documented problem with durable goods is that firms prefer to release lower quality products or to speed up introduction times to induce replacement (Nair and Hopp 1992, Lobel et al. 2016, Koca et al. 2021). Such strategies force consumers to buy more items that are used less often which, in turn, leads to higher volumes of generated waste. For instance, consumers in Italy disposed, on average, 7.7 kg of clothing in 2016; in the same year, UK consumers disposed 3.1 kg of clothing (Labfresh 2017). Our modeling framework provides a micro-foundation of the process of waste creation by assuming that durable goods become obsolete after a certain number of periods.

Our numerical experiments are focused on exploring the role of value decay and value uncertainty, both key parameters for managing an assortment of durable good, on firm revenues, consumer welfare, and generated waste. In cases where the incentives of firms and consumers are not inherently aligned, we investigate regulatory interventions that bridge the gap between the two sides of the market, and push towards more sustainable outcomes.

In the remainder of the section, we use the following baseline parameter values. We set the product's lifespan to four semesters ($n = 4$ periods), to mimic a two-year life cycle of Spring-Summer and Fall-Winter clothes. The per-product utility $u$ is set at 1.5, i.e., $\log(u) = 0.4055$.

Price varies between $p = 0.04$ and 0.57, and discount factor takes values from $\delta = 0.5$ to 1. The holding cost is 0.05, unless otherwise specified. We vary the decay parameter $\beta$ between 0.1 and 0.9, and the level of value uncertainty is controlled with a noise term $\epsilon \in [0, 1]$ around the mean. To provide some context to these unit-less values, we can assign a monetary value of \$200 to each unit of cost and utility. As a result, a value of $p = 0.34$ represents a one-time expense of \$68 and a holding cost of $h = 0.05$ represents a cost of \$2 per year. Similarly, a utility of $\log(1 + u) = 0.91$ represents an average usage value of \$36 per year; this means that a wardrobe of 10 items delivers an average usage value of $\log(1 + 10 \times u) = 2.77$; equivalently, \$110 per year.

The computation of the value function in steady state is done by solving the DP with 100 periods (50 years, at 2 semesters per year), and by keeping the policy and performance obtained after removing the first 30 periods (warm-up effect) and the last 30 periods (end-of-horizon effect).

## 4.1 Value Decay

First, we study the impact of the decay rate of product value with time, assuming the utility function $u_{t,i} = u\beta^i$. The reason is that, for exponential-decay utilities, Proposition 4 provides a useful analytical benchmark: the total number of items kept in the assortment increases in $\beta$ and $\delta$; in other words, the size of the assortment increases as the value of products becomes longer-lasting and as consumers discount future rewards less relatively to current ones. However, Proposition 4 assumes that products have infinite durability. Here, we investigate the case of products with finite life cycle. Specifically, we cap the life cycle of products to 4 periods; i.e., after 4 periods product value becomes zero.

For each parameter combination, we solve the DP and retrieve the optimal number of products purchased, which typically varies over time, and compute its average. We also report the total average number of items maintained in the assortment, as well as the corresponding consumer utility and firm revenue. Figure 2 illustrates one instance of these numerical experiments. As we can see in this example, higher $\beta$'s (i.e., lower values of decay rate) increase the per-period utility because older products retain more of their value; resulting in larger purchases, bigger assortments, and higher consumer utility. At the same time, this also benefits the firm, which obtains higher revenue due to higher sales. This is in line with Proposition 4.

However, this is not always representative of the situation: depending on the combination of price $p$, baseline utility $u$, holding cost $h$, and discount factor $\delta$, the firm's revenue may be increasing or decreasing in $\beta$. Indeed, as $\beta$ increases, the utility derived from each product also increases. Due to decreasing marginal returns (concavity of the value function), this would suggest that the need for additional products is reduced. It turns out though that this effect is partially offset by the discount factor, which values current utility more than future ones. Specifically, consider the direct effect of a variation of $q_t = x_{t,0} = x_{t+1,1} = x_{t+2,2} = x_{t+3,3}$. The consumer utility,

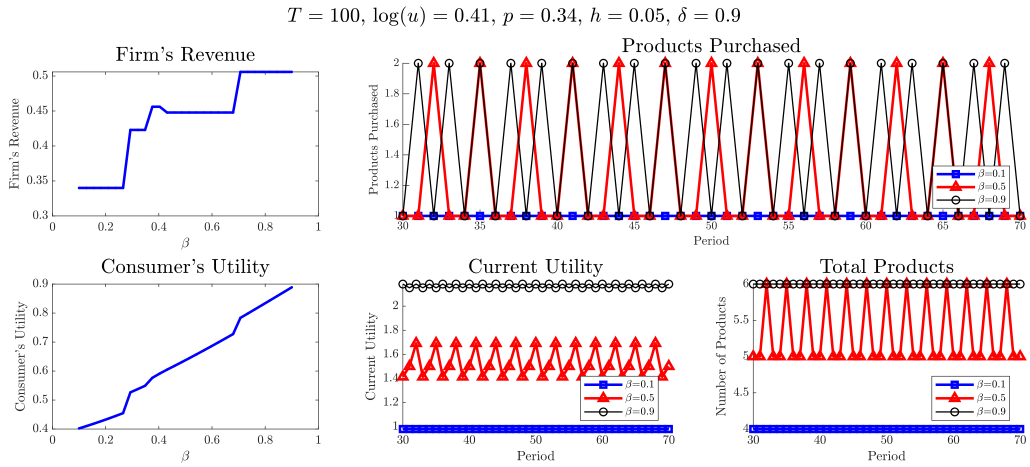


Figure 2: Solution of the DP with baseline parameter values, for different levels of value decay rate $\beta$ (continuous variation on the left; discrete values $0.1, 0.5$ and $0.9$ on the right, for clarity), and mild discounting of future rewards ($\delta = 0.9$).

$U := \sum_{t'} \delta^{t'} \log(1 + \sum_{i=0}^{n-1} u\beta^i x_{t',i})$, is such that

$$\frac{\partial U}{\partial q_t} = \delta^t \sum_{i'=1}^{n} \delta^{i'} \frac{u\beta^{i'}}{1 + \sum_{i=0}^{n-1} u\beta^i x_{t+i',i}},$$

which may increase or decrease in $\beta$ depending on the value of $\delta$. (Of course, the total effect will also depend on the variation on $q_{t'}, t' \neq t$.) At optimality, this should be equal to the increase in purchase and holding cost, $\delta^t(p + h\sum_{i=1}^{n} \delta^i)$, which is independent of $\beta$. The interplay of these opposing effects is captured in Figure 3, where we show the average number of products in the assortment, as a function of the value decay rate $\beta$ and the price $p$, for different discount factors $\delta$. When the discount factor approaches 1, the increasing term dominates, resulting in upward-sloping curves. Conversely, the further the discount factor is removed from 1, the more the increasing effect diminishes, leading to downward-sloping curves.

To further understand the interplay between these opposing forces driven by $\beta$, $\delta$, and $p$, we compute the optimal $\beta$ that the firm would choose in order to maximize its revenues, $\beta^*$, for a given the discount factor $\delta$. It is not uncommon to have multiple solutions in this optimization problem, so we identify the optimal interval for $\beta$, and then select $\beta^*$ to be the smallest optimum; this should be, presumably, the lowest-cost product design for the firm. We repeat this process, for different values of $\delta$ and one hundred price points, equally spaced from 0.1 to 1. Finally, we compute the distribution of $\beta^*$ for five different percentiles. Figure 4 reports the results: we observe that, for $\delta = 0.5$, in more than half of the scenarios it is optimal to set $\beta$ at its lowest value 0.1. Even at $\delta = 0.9$, in about half of the scenarios $\beta^* < 0.5$.

An unforced conclusion of the above numerical experiments is that, when there is no discounting of future rewards (i.e., $\delta \geq 1$), both consumers and firms prefer high-durability products. This

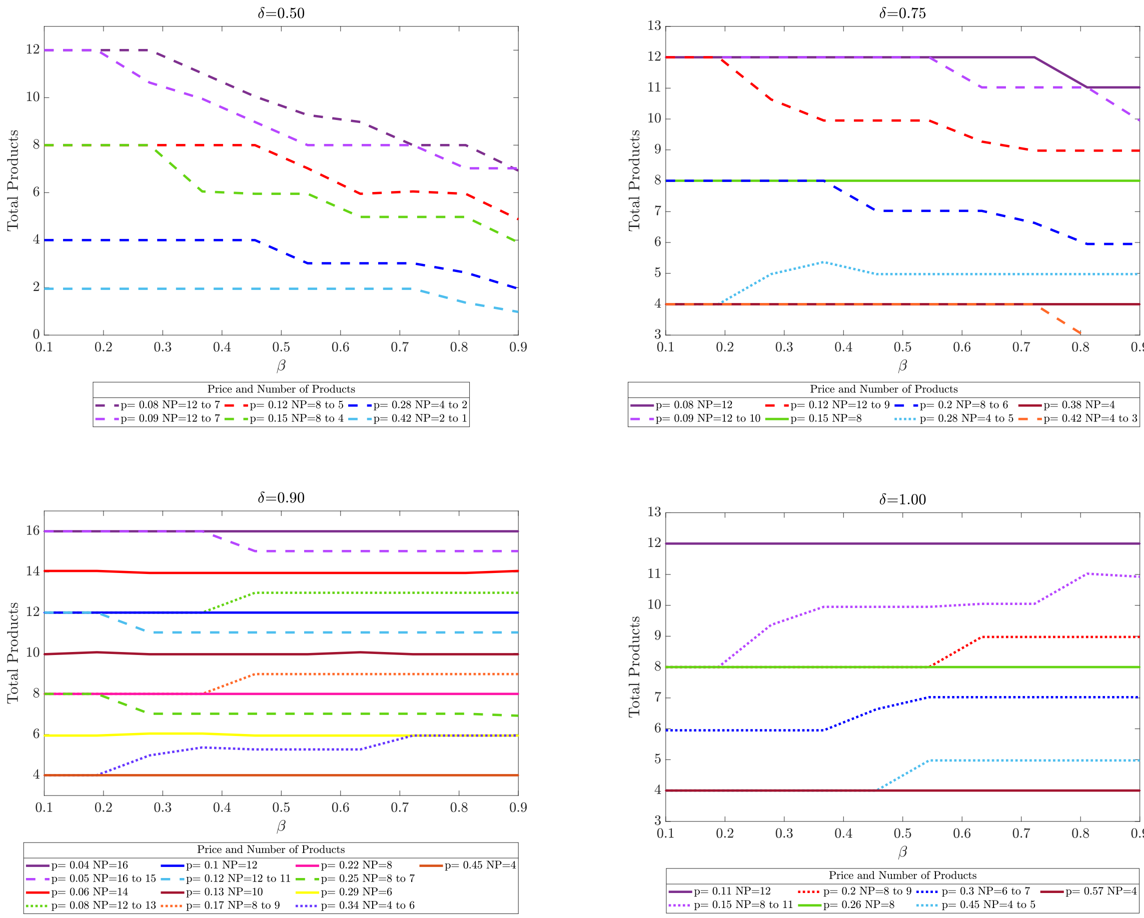


Figure 3: Average number of products in the assortment, as a function of the value decay rate $\beta$ and the price $p$, for different discount factors $\delta = 0.5, 0.75, 0.9$ and 1.

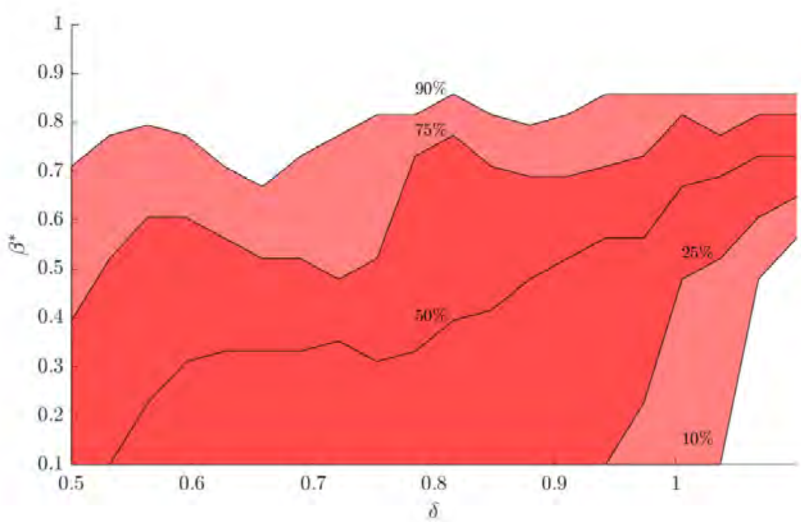


Figure 4: Optimal $\beta$ chosen by the firm.

means higher production costs and ultimately more waste, but these appear to be necessary sacrifices in order to sustain high levels of social welfare, both at the consumer and the firm side. On the other hand, when there is sufficient discounting of the future (e.g., $\delta < 0.9$), there is an incentive alignment problem: the firm earns more revenue by offering low-durability goods, which implies lower consumer utility. Indeed, with higher-durability goods, the consumer would end up keeping a smaller assortment – since the consumer could obtain a higher usage value for a longer time, with fewer items – and the firm would lose sales. This misalignment is highly problematic. As the firm is the one that selects the value of $\beta$ in practice, by controlling the process of product design, it is more likely to release products with a low $\beta$, so that the consumer purchases a larger number of items. Obviously, this strategy has terrible consequences both from a consumer welfare and from an environmental perspective: many low-durability items are produced and purchased, which are disposed not long after. This echoes well-known critiques of the fast-fashion business model (Caro and Martínez-de-Albéniz 2015, Long and Nasiry 2022).

How to remedy this unsatisfying outcome? Plambeck and Wang (2009) have suggested a tax on sales to increase the cost of production and, in turn, the time between two consecutive purchases. In our case, where multiple items co-exist in the assortment and $\beta$ affects their future usage value, a sales tax does not change the sign of the slope of firm revenue. Moreover, in our model, product renewal and assortment breadth are endogenous consumer decisions. Our conclusion is that the firm would still look to release lower-durability products, despite any taxation on sales, while the consumer would still be better off with higher-durability ones. In other words, a tax on sales would be internalized by the firm, and would not help in aligning its incentives towards more sustainable consumption.

Effectively, one needs a tax or a subsidy that affects the firm differently at different levels of durability/value decay rate. In other words, one needs a scheme that makes the firm worse off when $\beta$ is low, and better off when it is high. Intuitively, we would like the consumer to keep consuming new items at higher levels of $\beta$, as opposed to substituting many low-durability items for few high-durability ones. This can be achieved via a $\beta$-dependent subsidy on the holding cost or, alternatively, a subsidy on the usage of older items.

For instance, one can offer a subsidy to the holding cost when $\beta > \hat{\beta}$, reducing it to a fraction $\gamma$. We see the results in Figure 5, when the subsidy is 50% and 80% of the holding cost, at two threshold values of $\hat{\beta} = 0.5$ and 0.8. We observe that, despite revenue still having a decreasing shape above and below the threshold, it becomes optimal for the firm to set $\beta^* > \hat{\beta}$. One may wonder how such scheme could be implemented. One such way is for the subsidy to directly cover maintenance costs of the items, such as cleaning and repairing older clothes. One could also think about using technology that would credit the subsidy when older items are used; recall that the use probability of an item of age $i$ is $u_{t,i}x_{t,i}/(1 + \sum_{i'=1}^{n-1} u_{t,i'}x_{t,i'})$, so usage of older products is directly related to the value of $\beta$.

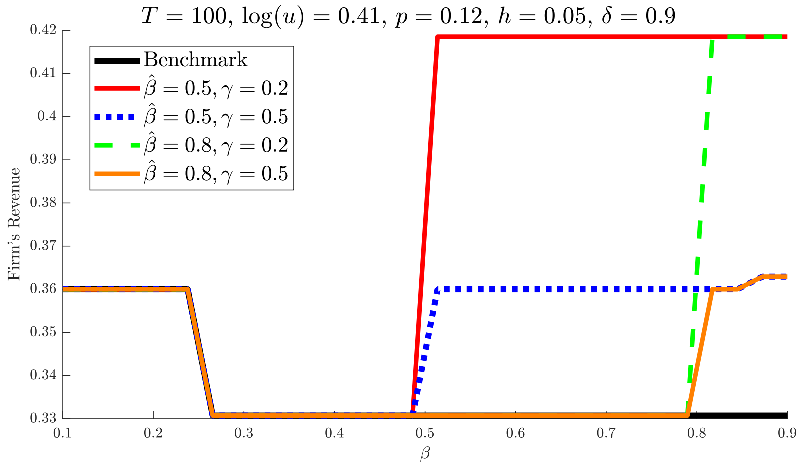


Figure 5: Firm revenue when holding cost is subsidized, and the consumer faces an effective holding cost $h_{t,i}\big(1-(1-\gamma)1_{\beta>\hat{\beta}}\big)$, for different values of $\hat{\beta}$ and $\gamma$.

### 4.2 Value Uncertainty

Uncertainty regarding the value obtained from using a durable good is another key parameter for a consumer deciding how to manage her assortment, as the consumer pays a price, set by the firm, without perfect knowledge of the product's intrinsic value; akin to risk vs. reward considerations in a portfolio selection problem. We have introduced uncertainty in Eq. (1) to define the reward function as the expectation over their joint distribution. In this section, we regulate the amount of value uncertainty by considering $u_{t,i} = u(1+\tilde{\varepsilon})$ with $\tilde{\varepsilon} = \pm\varepsilon$ with probability 50%. From Eq. (1), Jensen's inequality implies that larger uncertainty, i.e., higher $\varepsilon$, reduces consumer utility. This is in contrast to the 'real option' value of having more choice, in which higher variance of $\xi_{t,i,k}$ (the shock to individual items in each period), increases the expected utility – because the consumer picks the item that delivers the highest realized utility. Here, uncertainty in $u_{t,i}$ penalizes expected utility more in downward shocks, in comparison to increases in upward ones, leading to $U_t$ being decreasing in $\varepsilon$ overall.

Of course, this direct effect applies under a given assortment. Interestingly, the same argument should apply to marginal utility, because $\partial U_t/\partial x_{t,i} = \mathbb{E}u_{t,i}/(1+\sum_{i'=0}^{n-1} u_{t,i'}x_{t,i'})$ is also concave in $u_{t,i}$; although concave in $u_{t,i'}$ for $i' \neq i$. As a result, the number of products purchased should also decrease with increasing uncertainty. However, as shown in Figure 6, it is possible that the number of products purchased may increase at high price levels. In that case, we have $x_{t,i} = 0$ for some $t$ and $i$, which eliminates the convexity of $\partial U_t/\partial x_{t,i}$ in $u_{t,i}$ and magnifies the concavity in $u_{t,i'}$. Finally, we also observe that consumer utility may not be monotonic in variability. This is because utility decreases with fixed $x_{t,i}$, but may have sharp increases when the optimal $x_{t,i}$ increases.

Regardless of these different situations, we observe that incentives are aligned between firm and

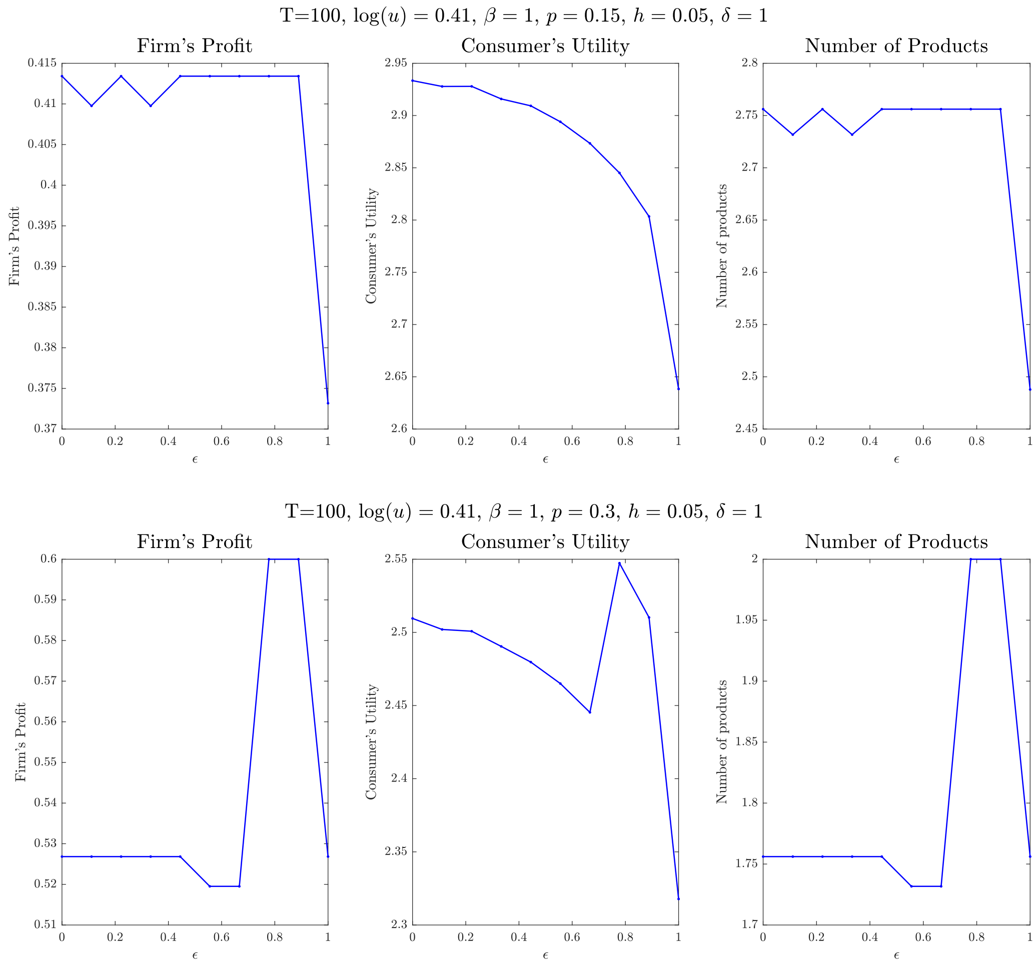


Figure 6: Firm profit (left), consumer utility (middle) and average number of products purchased (right) as a function of value uncertainty levels $\varepsilon$, for low-price scenario (top) and high-price scenario (bottom).

consumers. With the exception of high prices, both prefer lower variability, as more predictable future incentivizes larger assortments, consumer utility and firm profits. Going back to the fast fashion business model, this suggests that firms operating in the mass market, e.g., Zara or H&M, should prefer to sell items of more basic nature, making its future value more certain, and stay away from more radical fashion with the same average utility but more variability.

# 5 Conclusion

The present paper constitutes a first attempt towards micro-founding the way that rational consumers build and maintain an assortment of durable goods over time. Our analysis sheds light on the interaction between assortment renewal (purchases) and usage in a dynamic context. We provide a structural characterization of the value function of the corresponding Dynamic Program – concave and, in the case of no utility uncertainty, submodular – as well as a closed-form solution in the case of infinite horizon and exponential utilities; a buy-up-to policy, reminiscent of base-stock policies in inventory management.

We accompany our analytical results by an extensive numerical study, which highlights the impact of value decay and uncertainty on consumer choice and the resulting assortments; two crucial characteristics of fast fashion goods, which have been the main motivation behind our work. Regarding value decay, we find that, unless the factor by which consumers discount the future is sufficiently high, there is an inherent misalignment between the incentives of firms and consumers: firms prefer products whose value decays fast in order to incentivize new purchases, while consumers would rather have longer-lasting products. Given that firms are the ones who ultimately decide on product design, the outcome is products with shorter effective life cycles, usage concentrated around recently bought items, and large amounts of generated waste; as corroborated by current business practice. We find that taxation on sales cannot align the incentives of firms and customers. However an age-dependent usage subsidy would succeed in incentivizing firms to produce longer-lasting goods, while keeping consumption high. Regarding value uncertainty, our results show that, with the exception of high-price regimes, both firms and consumers are aligned in preferring goods with lower value uncertainty. Together, these insights suggest that business models that help consumers maintain longer-lasting assortments may be more sustainable, financially and environmentally, than fast-fashion ones, where both obsolescence and value uncertainty are high.

We hope that our work can serve as a building block for more comprehensive models of dynamic product acquisition and usage over time. In particular, it would be interesting to understand how firms should design their go-to-market strategies to 'enter' the assortment of a rational consumer. This could mean, for example, determining the utility and price positioning (high value and price vs. lower ones), as well as the adequate revenue model (high upfront payment vs. pay-per-use or other servitization approaches). It would also be useful to understand the competitive implications of our assortment model. Specifically, when single products are bought, competition between firms is well understood (Vives 1999). However, when many products constitute the consumer's assortment, there is usually a 'first-sale' advantage, in which the first product purchased makes the bar for the second one higher, and this creates inter-temporal effects in the pricing game.

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# Appendix A

## Proof of Proposition 1

Let $\mathbf{x} \in \mathbb{R}_+^n$ and $q \in \mathbb{R}_+$. We prove the concavity of the value function, by means of mathematical induction. For $t = T$ the result clearly holds, establishing the basis of the hypothesis. Let us assume that $J_{t+1}(\cdot)$ is concave. Note that the objective function on the right-hand side of Eq. (3) is jointly concave on $\mathbb{R}_+^n \times \mathbb{R}_+$: $\pi_t(q, \mathbf{x})$ is jointly concave as a sum of concave functions, and $J_{t+1}(\cdot)$ is concave by the induction hypothesis. Moreover, joint concavity is a joint extension of concavity on $\mathbb{R}_+^n \times \mathbb{R}_+$; see Example 2 in Section 6.1 in Smith & McCardle (2002). This implies that $J_t(\cdot)$ is concave; see Proposition 4 in Smith & McCardle (2002).

## Proof of Proposition 2

We prove the submodularity of the value function $J_t(\cdot)$ when the parameters $u_{t,i}$ are deterministic, for all $i$ and $t$. Under this condition, we have that

$$\frac{\partial^2 U_t}{\partial x_i \partial x_j}(\mathbf{x}) = -\frac{u_{t,i} u_{t,j}}{\left(1 + \sum_{k=0}^{n-1} u_{t,k} x_k\right)^2} \leq 0. \tag{9}$$

From the expression above, it is also straightforward to verify that, for all $i \neq j \neq 0$,

$$\frac{\partial^2 U_t}{\partial x_i \partial x_j}(\mathbf{x}) \frac{\partial^2 U_t}{\partial x_0^2}(\mathbf{x}) - \frac{\partial^2 U_t}{\partial x_i \partial x_0}(\mathbf{x}) \frac{\partial^2 U_t}{\partial x_j \partial x_0}(\mathbf{x}) = 0. \tag{10}$$

Our proof strategy relies on showing that the utility function $U_t(\cdot)$ endows the properties in Eqs. (9)-(10) to $J_t(\cdot)$; this implies directly the submodularity of the value function. Specifically, we prove by induction that, for all $i \neq j$,

$$\frac{\partial^2 J_t}{\partial x_i \partial x_j}(\mathbf{x}) = \frac{\partial^2 U_t}{\partial x_i \partial x_j}(\mathbf{x}).$$

(Note that $J_t(\cdot)$ is twice differentiable because $U_t(\cdot)$ is twice differentiable). For $t = T$, we have that $J_T = 0$, so the basis of the induction is true. Assume that the property above is true for $t+1$, i.e.,

$$\frac{\partial^2 J_{t+1}}{\partial x_i \partial x_j}(\mathbf{x}) = \frac{\partial^2 U_{t+1}}{\partial x_i \partial x_j}(\mathbf{x}).$$

We show that this property remains true for $t$. By the Envelope Theorem, we have that

$$\frac{\partial J_t}{\partial x_i}(\mathbf{x}) = \frac{\partial \pi_t}{\partial x_i}\big(q^*(\mathbf{x}), \mathbf{x}\big) + \frac{\partial J_{t+1}}{\partial x_{i+1}}\big(q^*(\mathbf{x}), \mathbf{x}\big) = \frac{\partial U_t}{\partial x_i}(\mathbf{x}) - h_{t,i} + \frac{\partial J_{t+1}}{\partial x_{i+1}}\big(q^*(\mathbf{x}), \mathbf{x}\big).$$

This implies that

$$\frac{\partial^2 J_t}{\partial x_i \partial x_j}(\mathbf{x}) = \frac{\partial^2 U_t}{\partial x_i \partial x_j}(\mathbf{x}) + \frac{\partial^2 J_{t+1}}{\partial x_{i+1} \partial x_{j+1}}\big(q^*(\mathbf{x}), \mathbf{x}\big) + \frac{\partial^2 J_{t+1}}{\partial x_{i+1} \partial x_0}\big(q^*(\mathbf{x}), \mathbf{x}\big) \frac{\partial q^*}{\partial x_j}(\mathbf{x}).$$

The (implicit) derivative of $q^*(\mathbf{x})$ with respect to $x_j$ can be computed via the Implicit Function Theorem, applied to the optimality equation

$$\frac{\partial J_{t+1}}{\partial x_0}\big(q^*(\mathbf{x}), \mathbf{x}\big) = \frac{p_t}{\delta}$$

(see Proposition 3), which is valid when $q^* > 0$, and hence true by assumption in this result. We have that

$$\frac{\partial^2 J_{t+1}}{\partial x_{j+1}\partial x_0}(\mathbf{x}) + \frac{\partial^2 J_{t+1}}{\partial x_0^2}(\mathbf{x})\frac{\partial q^*}{\partial x_j}(\mathbf{x}) = 0 \implies \frac{\partial q^*}{\partial x_j}(\mathbf{x}) = -\frac{\partial^2 J_{t+1}}{\partial x_{j+1}\partial x_0}(\mathbf{x})\Big/\frac{\partial^2 J_{t+1}}{\partial x_0^2}(\mathbf{x}),$$

and hence,

$$\frac{\partial^2 J_{t+1}}{\partial x_{i+1}\partial x_{j+1}} + \frac{\partial^2 J_{t+1}}{\partial x_{i+1}\partial x_0}\frac{\partial q^*}{\partial x_j} = \left(\frac{\partial^2 J_{t+1}}{\partial x_{i+1}\partial x_{j+1}}\frac{\partial^2 J_{t+1}}{\partial x_0^2} - \frac{\partial^2 J_{t+1}}{\partial x_{i+1}\partial x_0}\frac{\partial^2 J_{t+1}}{\partial x_{j+1}\partial x_0}\right)\Big/\frac{\partial^2 J_{t+1}}{\partial x_0^2}.$$

Now, by combining the induction property and Eq. (10), we have that the numerator of this expression is equal to zero:

$$\frac{\partial^2 J_{t+1}}{\partial x_{i+1}\partial x_{j+1}}\frac{\partial^2 J_{t+1}}{\partial x_0^2} - \frac{\partial^2 J_{t+1}}{\partial x_{i+1}\partial x_0}\frac{\partial^2 J_{t+1}}{\partial x_{j+1}\partial x_0} = \frac{\partial^2 U_{t+1}}{\partial x_{i+1}\partial x_{j+1}}\frac{\partial^2 U_{t+1}}{\partial x_0^2} - \frac{\partial^2 U_{t+1}}{\partial x_{i+1}\partial x_0}\frac{\partial^2 U_{t+1}}{\partial x_{j+1}\partial x_0} = 0.$$

Therefore,

$$\frac{\partial^2 J_t}{\partial x_i \partial x_j}(\mathbf{x}) = \frac{\partial^2 U_t}{\partial x_i \partial x_j}(\mathbf{x}) \le 0,$$

which concludes the proof by induction of the submodularity of $J_t(\cdot)$.

On the other hand, when the parameters $u_{t,i}$ are (non-deterministic) random variables, $J_t(\cdot)$ may not be submodular. We provide a simple example: consider $t = T-2$ and $u_{T-2,i}(\cdot) = 0$ for all $i$. In this special case, the analysis above implies that

$$\frac{\partial^2 J_t}{\partial x_i \partial x_j}(\mathbf{x}) = \left(\frac{\partial^2 J_{t+1}}{\partial x_{i+1}\partial x_{j+1}}\frac{\partial^2 J_{t+1}}{\partial x_0^2} - \frac{\partial^2 J_{t+1}}{\partial x_{i+1}\partial x_0}\frac{\partial^2 J_{t+1}}{\partial x_{j+1}\partial x_0}\right)\Big/\frac{\partial^2 J_{t+1}}{\partial x_0^2}.$$

Recalling that $J_{t+1}(\cdot) = U_{T-1}(\cdot)$, and that $\frac{\partial^2 J_{t+1}}{\partial x_0^2} \le 0$ by the concavity of the value function, $J_t(\cdot)$ is submodular if and only if at $\mathbf{y} = (q^*, \mathbf{x})$:

$$\frac{\partial^2 U_{t+1}}{\partial x_{i+1}\partial x_{j+1}}(\mathbf{y})\frac{\partial^2 U_{t+1}}{\partial x_0^2}(\mathbf{y}) - \frac{\partial^2 U_{t+1}}{\partial x_{i+1}\partial x_0}(\mathbf{y})\frac{\partial^2 U_{t+1}}{\partial x_{j+1}\partial x_0}(\mathbf{y}) \ge 0, \qquad \forall i \ne j.$$

Exploiting the structure from Equation (1), and letting $S_k \doteq u_{t+1,k}/(1 + \sum_{j=0}^{n-1} u_{t+1,j}x_j)$ be the (random) choice probability for product $k$ in period $t+1 = T-1$, the condition above takes the form:

$$\mathbb{E}\Big[S_{i+1}S_{j+1}\Big]\mathbb{E}\Big[S_0^2\Big] \ge \mathbb{E}\Big[S_{i+1}S_0\Big]\mathbb{E}\Big[S_0 S_{j+1}\Big].$$

This may not be satisfied when $S_0$ is independent of $S_{i+1}$ and $S_{j+1}$, but $S_{i+1}$ and $S_{j+1}$ themselves are negatively correlated. For instance, suppose that $i+1 = 1, j+1 = 2, n = 3$, $y_1 = y_2 = 1$, $u_1$ be uniform in $[0,1]$, $u_2 = 1 - u_1$, and $u_0 = 1$ ($t+1 = T-1$ omitted). Then, $S_0 = x_0/(1 + x_0 + 1)$, $S_1 = u_1/(1 + x_0 + 1)$, and $S_2 = (1-u_1)/(1 + x_0 + 1)$, which imply that

$$\mathbb{E}\Big[S_1 S_2\Big]\mathbb{E}\Big[S_0^2\Big] < \mathbb{E}\Big[S_1\Big]\mathbb{E}\Big[S_2\Big]\left(\mathbb{E}\Big[S_0\Big]\right)^2 = \mathbb{E}\Big[S_1 S_0\Big]\mathbb{E}\Big[S_0 S_2\Big].$$

## Proof of Proposition 3

Assume that $q_t^*(\mathbf{x}) > 0$, i.e., that the optimal purchasing quantity at period $t$ is strictly positive, and that the value function $J_{t+1}(\cdot)$ is submodular. Since the objective function of the DP in Eq. (3) is concave, and $q_t^*(\mathbf{x})$ is its unconstrained maximizer,

$$q_t^*(\mathbf{x}): \ \frac{\partial J_{t+1}}{\partial x_0}(\mathbf{x}) = -\frac{\partial \pi_t}{\partial q}(\mathbf{x}) = p_t$$

is a necessary and sufficient condition for optimality. As already discussed in the proof of Proposition 2, the Implicit Function Theorem implies that

$$\frac{\partial q^*}{\partial x_i}(\mathbf{x}) = -\frac{\partial^2 J_{t+1}}{\partial x_{i+1}\partial x_0}(\mathbf{x}) \Big/ \frac{\partial^2 J_{t+1}}{\partial x_0^2}(\mathbf{x}),$$

given that $q_t^*(\mathbf{x}) > 0$. From concavity, we have that

$$\frac{\partial^2 J_{t+1}}{\partial x_0^2}(\mathbf{x}) \leq 0.$$

Therefore,

$$\frac{\partial q^*}{\partial x_i}(\mathbf{x}) \leq 0 \iff \frac{\partial^2 J_{t+1}}{\partial x_{i+1}\partial x_0}(\mathbf{x}) \leq 0,$$

which holds because $J_{t+1}(\cdot)$ is submodular.

## Proof of Proposition 4

The problem in Section 3.4 is a time-stationary, infinite-horizon discounted DP. Hence, the sequence of value functions converges to a steady-state one, $J_t(s) \to J(s)$; see (Bertsekas 2000). Then, the result follows as a corollary of Proposition 5 (noting that we need to ensure convergence of sum of discounted costs, which is a result of Lebesgue's dominated convergence theorem): in the case of exponential rewards, it can be verified that upper and lower bound coincide, and the steady-state version of the DP in Eq. (8) gives:

$$J\big(s\big) = \max_{q \geq 0} \left\{ \pi(q, s) + \delta J\big(q + \beta s\big) \right\}.$$

The unconstrained maximum of right-hand side of the equation above is characterized by the first-order optimality condition:

$$\pi'(q, s) + \delta J'(q + \beta s) = 0 \implies p = \delta J'(q + \beta s).$$

Letting $\bar{q}$ such that $p = \delta J'(\bar{q})$, we must have that $q^* + \beta s = \bar{q}$ if $q^* \geq 0$, and zero otherwise. This implies that

$$q^* = (\bar{q} - \beta s)^+.$$

Furthermore, since the problem is time-stationary, we can characterize $\bar{q}$ in closed-form. Indeed, for $s \leq \bar{q}$, in which case $q^* = \bar{q} - \beta s$, we have that

$$J(s) = \pi(\bar{q} - \beta s, s) + \delta J(\bar{q}).$$

As a result,

$$J'(s) = \frac{\partial \pi}{\partial s}(\bar{q} - \beta s, s) = \beta p + \frac{\partial U}{\partial s}(s) - h,$$

which implies that

$$J'(\bar{q}) = \beta p + \mathbb{E}\left[\frac{u}{1 + u\bar{q}}\right] - h.$$

The fact that $p = \delta J'(\bar{q})$ implies Eq. (6).

## Proof of Proposition 5

Eq. (7) implies that

$$u_{LB,t} \leq \min_i \left\{\frac{u_{t,i}}{\beta^i_{LB}}\right\}, \qquad \text{w.p.1}, \qquad \forall t.$$

and

$$u_{UB,t} \geq \max_i \left\{\frac{u_{t,i}}{\beta^i_{UB}}\right\}, \qquad \text{w.p.1}, \qquad \forall t.$$

Combined with the fact that

$$h_{UB,t} \doteq \max_i \left\{\frac{h_{t,i}}{\beta^i_{LB}}\right\} \text{ and } h_{LB,t} \doteq \min_i \left\{\frac{h_{t,i}}{\beta^i_{UB}}\right\}, \forall t,$$

we have:

$$\pi_{LB,t}\big(q, s_{LB}(\mathbf{x})\big) \leq \pi_t(q, \mathbf{x}) \leq \pi_{UB,t}\big(q, s_{UB}(\mathbf{x})\big), \qquad \forall (q, \mathbf{x}) \in \mathbb{R}^{n+1}_+, \qquad \forall t,$$

where

$$s_j(\mathbf{x}) \doteq \sum_{i=0}^{n-1} \beta^i_j x_i, \qquad j \in \{LB, UB\}.$$

Since the instantaneous expected rewards $\pi_{LB,t}\big(q, s_{LB}(\mathbf{x})\big)$ and $\pi_{UB,t}\big(q, s_{UB}(\mathbf{x})\big)$ bound from below and above, respectively, the actual reward, $\pi_t(q, \mathbf{x})$, at all states and periods, then proving that the value functions of the corresponding DPs follow a similar ordering can be proved via a simple mathematical induction.