Positioning of Omnichannel Inventories to Protect Revenue

Alborz Hassanzadeh
HEC Montréal, Canada, al.zadeh@hec.ca

Victor Martínez-de-Albéniz
IESE Business School, Spain, valbeniz@iese.edu

Problem definition: Conventional inventory management strategies prove inadequate when confronted with the complexities of omnichannel retailing. Fulfillment from stores and transshipment emerged as alternative methods to supplant the traditional approaches, but unfortunately these methods are often too costly. How to share limited inventory between online and offline channels thus remains a challenging question. **Academic/practical relevance:** We characterize an omnichannel retailer's inventory sharing policy with limited inventory to be sold during a finite horizon. Inventory reserved for the online channel can later be transferred to the offline inventory pool, but once it is shipped to offline stores, it becomes out of reach for online orders. **Methodology:** We use an optimal control framework with limited inventory supply and stochastic demand processes. By recursively solving a series of differential equations, we characterize the optimal inventory positioning policy. **Results:** We demonstrate that a three-regime time-threshold policy governs the optimal inventory allocation policy; in the central regime, both channels are active and the retailer keeps minimal stock at the stores. During the beginning and end regimes, only one channel is open and has positive inventory. **Managerial implications:** While traditional inventory management techniques tend to favor a high fill rate and availability, our approach focuses on positioning inventory where it generates the most revenue, which depends on the remaining time in the season. Interestingly, this approach sometimes leads to strategic revenue protection decisions wherein it may be more profitable to preclude sales from a lower-margin channel with higher customer reach.

Key words: omnichannel, retail, revenue management, inventory positioning, optimal control

---

1. Introduction

Retail has been in flux for the past 25 years due to the ascent of online shopping; however, only recently have retailers begun to embrace the adoption of e-commerce truly. For example, Inditex founder Amancio Ortega recognized the significance of storage providers amid the thriving e-commerce landscape and is set to invest $905 million in warehouses in the US (Inditex 2022, Reuters 2022). Target announced an investment of $7 billion in e-initiatives to improve customers' online shipping experience (Target 2017, Repko 2023). Kohl's made a similar announcement declaring its commitment to making online shopping more convenient for customers (Wahba 2017). Walmart recognized the importance of online channels and acquired Bonobos and Jet to expand its customer reach and e-commerce growth (Walmart 2017a,b). They invested heavily and exerted great efforts to revamp the digital experience at the core to make e-shopping faster, easier, and more enjoyable (Ward 2023, Walmart 2023). Rodney McMullen, Chairman and CEO of Kroger, believes that a seamless experience will hold even greater significance in the future than it currently does; through their partnership with Ocado, Kroger aims to establish an intricate network of automated fulfillment centers aimed at transforming the company's operations and improving customers' experience (Silverstein 2023).

Incorporating online channels into their existing brick-and-mortar stores network, or striving to enhance existing online channels, is not limited to a select few brands. Rather, to survive in this dynamic and rapidly evolving environment, it has become an imperative practice among retailers across various sectors (Caro et al. 2020). However, not only are some retailers attempting to adopt e-commerce, but there are also those that have conventionally operated solely online and seek to integrate the traditional brick-and-mortar store into their network. For example, Amazon acquired Whole Foods and announced it was ready to "go big" on physical stores (Wingfield and de la Merced 2017, Lee 2023).

Omnichannel strategy – integrating online and offline channels into a single, cohesive, and seamless customer experience – is a concept developed over a decade ago (Brynjolfsson et al. 2013). It provides customers with the convenience of online shopping with the added benefits of in-store experiences. However, cross-channel activities and the integration of online and offline channels have made decision-making much more complex, thus triggering a rethinking of operational strategies.

In particular, traditional inventory planning methods designed for independent channels must be revised in this new environment with increasing competition (Guy 2015, Dekhne et al. 2020). The online channel presents retailers with a significant opportunity, as it has the potential to reach a much larger market and overcome geographical limitations. However, this channel's margins are typically lower than the offline channel. On the other hand, the offline channel tends to have higher margins but reaches a smaller customer base. These channel differences create a challenge for retailers, as they must weigh the online channel's potential reach against the offline channel's higher margins when making decisions about inventory allocation.

Retailers, therefore, require a delicate balancing policy to determine whether to reserve the limited inventory for online sales or otherwise to ship it to offline locations in order to generate store sales. Brands like Macy's and Best Buy have already suffered the consequences of their inventories being in the "wrong place" at the "wrong time" (Insider 2017). Others have come to recognize the necessity of rethinking their approach to inventory management to avoid the same consequences, and those who have improved their stock levels have reaped significant rewards. For example, Home Depot and Lowe's reported an increase of 83% and 121% in online orders due to their interconnected retail strategy and improved inventory management, respectively (Lowe's 2021, Home Depot 2021).

Another novel approach for managing inventory, which addresses stockout concerns and expedites delivery, involves leveraging retail stores as fulfillment centers (Snelling 2022). In a bid to tackle frequent stockouts facing online demand, Camper, a renowned Spanish footwear and shoe retailer, took on a more reactive approach in their inventory management and started their omnichannel journey with a project involving transshipment and fulfilling online demand from their stores (Martínez-de-Albéniz 2019). This fulfillment strategy requires detailed information on store inventory levels. Moreover, picking and packing online orders also can disrupt the staff's routine in-store operations. Finally, each online demand must be matched with a particular inventory position. The resulting matching problem is costly and technically complex, as illustrated by the implementation at Urban Outfitters (Andrews et al. 2019). Thus, it becomes evident that a preventive approach to inventory planning can be preferable, from a cost and a complexity perspective, to a reactive one.

For these reasons, to maximize sales while controlling delivery costs, we are interested in understanding how a retailer should position its inventory in the network. Specifically, given a certain inventory level, when should the retailer ship the inventory to the store, knowing that it is prohibitively expensive to ship it back to the distribution center (DC)? This work investigates these questions and characterizes the optimal positioning of omnichannel inventories. In doing so, service level in the online and offline channels is optimized with the objective of maximizing revenue.

For this purpose, we consider a simple two-echelon system (DC serving the online channel and store serving the offline channel) over a finite season. Inventory is perishable and available in a limited quantity, so it cannot be restocked during the season. This mimics, for example, electronic equipment, travel packages, fashion, apparel, or sporting goods. The retailer, referred to by her, must decide how to position its limited product stock between the two echelons. In contrast to conventional inventory management approaches, the DC is more than just a mere storage facility for products: online orders are fulfilled directly from the DC. This implies that when the DC is out of stock, the retailer can no longer sell online. Additionally, the retailer also operates an offline channel served from the physical store. The margin generated in the store may be higher than the online price, but demand is limited, so shipping too much to the store may result in unsold inventory at the end of the season.

As the selling season progresses, product depreciation presses our retailer to sell the inventory as expeditiously as possible while maintaining a higher margin. She thus faces a decision on how to share this limited inventory between the offline and online demand. This entails determining the quantity to reserve for online demand versus how many to send to the store to meet the offline demand while considering the shipping cost. The payoff is contingent upon various factors, such as the quantity available in both the DC and the store and the remaining time until the end of the selling season.

We formulate the inventory positioning decision of this omnichannel retailer as a continuous-time optimal control problem where expected revenue is maximized. By recursively solving the problem, we characterize the optimal inventory positioning policy. We show that the selling season is divided into three regions through a set of critical time thresholds. Toward the end of the season, it is best to refrain from shipping products to the store. The intuition behind this decision lies in the fact that as time dwindles toward the end of the season, sales opportunities inevitably decline. Therefore, even though the ticket price may be higher in the store, the short time left for sales renders the expected incremental value of having a product in the store lower than the shipping cost. This suggests that the retailer leverages the online channel as a safety net to bridge the gap when there is limited time left for sales but insufficient time to warrant the shipping expense.

As time becomes less of a constraint, the retailer juggles inventory across both channels in order to enlarge its potential demand. Interestingly, while operating an offline-online channel structure, the retailer employs a cautious approach to inventory management: at any time, she keeps only one unit in the store and reserves the DC stock for online demand.

However, as more and more time is available, we reach a time threshold when the retailer decides to ship the entire stock to the downstream store. In doing so, the retailer resists the lure of selling in the online channel, which offers a lower margin, albeit typically a broader reach. Therefore, our retailer safeguards revenue through proper omnichannel inventory positioning. This critical moment is driven not by service level considerations (since the online channel service level becomes zero – it is unavailable) but by a revenue protection motive: the retailer prefers to settle for a rigid approach where inventory is 'stuck' at the store, with the resulting risk of excess inventory, rather than to keep the flexibility of the DC, because flexibility is attached to the risk of lower margins. In a way, this strategy amounts to a commitment to sell at a high price and not accept lower-margin online sales.

Traditional inventory management techniques tend to prioritize a high fill rate and availability. However, our approach is modeled to generate revenue, which sometimes leads to strategic decisions wherein, based on the time left until the end of the season and inventory availability, it may be more profitable to preclude sales from a lower-margin channel.

In the remainder of the paper, we first provide an overview of the related literature in §2. We then model the inventory positioning problem of the retailer in §3. Section 4 presents a recursive scheme to use optimal control theory and characterizes the optimal replenishment policy. We also demonstrate our results and findings in a real-world case from luxury fashion retail in §5. We conclude our research with final discussions in §6. Proofs are contained in the Appendix.

2. Related Literature and Background

Our research contributes to three distinct areas of inquiry. First, this works contributes to advance omnichannel retail best practices. Second, we are connected to works about the strategic placement of inventory in various locations. Third, our prescriptions fall within the overarching framework of revenue management techniques.

2.1. Omnichannel

Brynjolfsson et al. (2013) provide a thorough review on integrating the offline and online channels dubbed as omnichannel. It is well established that online actions affect offline behaviors and vice versa. Some articles analyze this effect from customers' points of view. For example, Bell et al. (2018) demonstrate that showrooms serve primarily as information hubs rather than physical inventory storage for order fulfillment. The former thus helps generate profits by reducing online returns. Gallino and Moreno (2014) find that the 'buy-online, pick-up-in-store' (BOPS) strategy, in fact, decreases online sales by increasing pick-up visits and the channel-shift effect. Gao and Su (2017b) explored three mechanisms aimed at enhancing customer information provision: physical showrooms, virtual showrooms, and availability information. Their findings suggest that these mechanisms may not exhibit substantial complementarity in the absence of customer heterogeneity.

Other researchers studied customers' behavior. For example, Gallino et al. (2017) examine the effect of expanded variety resulting from access to out-of-stock products, revealing a rise in sales dispersion. The research of Bell et al. (2015) shows that a visit to a physical showroom, allowing for product display and experiential interaction, increases overall demand, including demand in online channels. Kumar et al. (2019) analyze the impact of opening new stores on online and offline sales. They identify (a) a convenience effect since there is an in-store return option for online purchases and (b) an exposure effect from higher engagement resulting from store interactions. Finally, Dzyabura and Jagabathula (2018), Hense and Hübner (2022), Lo and Topaloglu (2022) or Rooderkerk and Kök (2019) analyze how omnichannel retailers coordinate assortment decisions.

2.2. Inventory Management

Inventory management is one of the most complicated challenges for omnichannel retailers, particularly when it involves BOPS (Arslan et al. 2021, Meyersohn 2019). Gao and Su (2017a) analyzed this problem when the retailer faces a newsvendor problem in the store. In their work, the online channel is managed exogenously. They showed how revenue sharing between the online and offline channels leads to synchronized efforts. Saha and Bhattacharya (2021) further analyzed the implications of BOPS on store inventory management, assuming that the DC has unlimited capacity.

With a focus on optimal fulfillment strategies, Alishah et al. (2017) considered an omnichannel retailer with one store whereby the online demand can be fulfilled from the store at an additional cost. Consistent with the notion that store-only customers tend to have a higher profit margin, they showed that the optimal inventory positioning policy prioritizes store-only customers. Govindarajan et al. (2021) extended this setting to a retailer that operates multiple stores and DCs. They proposed a heuristic to guide replenishment decisions statically and inform fulfillment decisions dynamically. Bayram and Cesaret (2021) considered a similar setting as Govindarajan et al. (2021) and developed a cost-threshold policy to decide whether and how to fulfill online demand. However, as they point out, this fulfillment approach poses new challenges as the retailer must match an online order when it arrives at a specific location. In the work of Seifert et al. (2006), the retailer fulfills online demand from a DC, and in case of stockout, stores can fill this demand. They showed that the cost savings of this one-way transshipment might be significant. Finally, Andrews et al. (2019) provide order-store matching procedures and describe an implementation at Urban Outfitters.

Rather than expending resources on costly reactive approaches, our proposal is a preventive approach that offers a more straightforward policy that lends itself to simple implementation. To this end, our retailer must strategically share its limited inventory between its offline and online channels, thereby necessitating an optimal integration between these channels, bringing us to the second line of work: optimal inventory positioning.

Clark and Scarf (1960) observed that inventory allocation in a two-echelon network is especially complicated since we cannot reduce inventory levels in the second echelon through returns to the first layer or transshipments. Therefore, the optimal allocations depend on the inventory position at all locations. They showed that an echelon base-stock policy is optimal in a finite periodic review problem with no fixed ordering cost. Rosling (1989) showed the validity of this result in an assembly system. The result remains valid even when the finite-horizon assumption is relaxed as Federgruen and Zipkin (1984) showed. Axsäter and Rosling (1993) defined installation-stock policies as those that prescribe replenishment orders based on local inventory and demand forecasts and studied the efficiency of such a system compared to that of an echelon base-stock.

Tan (1974) characterized the structure of the optimal allocation policy in a particular setting with one warehouse and two retailers, while other works such as Jackson (1988), Jackson and Muckstadt (1989), Marklund and Rosling (2012), McGavin et al. (1993) or McGavin et al. (1997) are some examples of studies on the problem in a multi-retailer setting. Although the inventory allocation problem becomes computationally intractable in a multi-retailer setting or when the number of periods increases, the seminal work of Jackson (1988) shows that the inventory cost can be decreased if there are multiple replenishment throughout the selling season. Under different assumptions and settings, other researchers also studied the benefits of postponed replenishment, for example, Cheung and Lee (2002) or Lee and Tang (1997).

In contrast with these works, in our two-echelon model we allow demand to arrive directly to the upper echelon as well as the lower echelon. This introduces a new 'lost sales' consideration when inventory runs out at the DC, and has not been studied before in this literature.

2.3. Revenue Management

In our work, at each inventory level and time left in the season, the retailer has to decide on how many units to ship to the store; that is, we approach the retailer's inventory positioning problem through the lens of revenue management. There is a rich literature on this type of problem.

The seminal work of Gallego and Van Ryzin (1994) finds the optimal price given a time interval and leftover stock. They find that as the seller gets closer to the end of the selling season, it should decrease the price to increase demand unless there is a sale, at which point there is a jump in the optimal price. Similarly to the research of Gallego and Van Ryzin (1994), in a two-price model, Feng and Gallego (1995) and Feng and Xiao (1999, 2000) find that the optimal control policy is a time-threshold policy. Martínez-de-Albéniz and Talluri (2011) incorporate price competition and show that competitive price equilibrium is time- and inventory-dependent. While the literature primarily focuses on the optimal time for price change, in our work, the product price does not change throughout the selling season.

This work is closest to Martínez-de-Albéniz et al. (2022), who adopted a revenue-management framework and considered a supplier that decides whether to reserve its limited stock for its store or share it with an online platform. They find that the optimal inventory-sharing policy is a time-threshold policy. In contrast, our problem does not enjoy a binary decision space. However, the optimal replenishment policy has a similar structure in that we can recursively characterize a time-threshold shipping policy.

3. Model Formulation

Consider a retailer that is about to introduce and sell a product in a centralized supply chain. The selling season is a finite window of length T, for example, the Spring/Summer fashion season from early January until late June. Let t be the time to go until the end of the season; that is, we count time backward, $t = T$ denotes the start of the season, and $t = 0$ its end. During the season, the retailer offers the product through an offline channel, a store, for example, at a price r^{off} and an online channel, for example, her website, at a price r^{on}. For simplicity, we normalize the salvage value to zero.

In each channel $j \in \{\text{off}, \text{on}\}$, customers arrive following a Poisson process with rate λ^j. For example, λ^{off} might be the average footfall in the store multiplied by the conversion rate; λ^{on} might be the average number of clicks recorded from the website multiplied by conversion. For simplicity, we assume that the intensities of sales are constant throughout the selling season; however, if the demand process remains memoryless, our analysis can be extended to situations where λ^j varies over time.

Demand is transformed into sales only when there is inventory available for sale. As in most retail set-ups, we assume that online demand is fulfilled only from available inventory at the DC and offline demand from the store. These assumptions imply that there is no ship-from-store option at this retailer, reflecting that a majority of retailers find this option hard and expensive to implement, as discussed earlier. Note that our methodology can be extended to cases with the possibility of matching online orders to offline inventories, but the analysis is significantly more complex.

At the start of the season, a fixed amount of inventory x_T^{on} is available at the DC, while the store has x_T^{off} units available. We thus assume that the starting inventory is fixed and procurement for the DC is not feasible during the selling season. This is a common assumption in the literature and appears because of capacity and/or replenishment constraints, such as long lead-times in luxury fashion.

Unmet demand is unobserved and lost: the retailer observes only the realized sales and whether a stock-out has occurred. The retailer can ship some of the inventory available at the DC to the store so that inventory is also available for offline customers and hence serving both online and offline demands. Formally, at time $t \in [0,T]$, a decision must be made on the stocking level in the store, y_t^{off}, and in the DC, y_t^{on}, by shipping u_t units from the DC to the store, at a unit shipping cost of c. In order to make offline sales profitable, and in line with most fashion retail cases, we assume that $r^{\text{off}} - c \geq r^{\text{on}}$.

Furthermore, we do not allow shipments to occur from the store back to the DC. This is a prevalent practice in most fashion retail chains, and returned items are only shipped back at the end of the season. Again, this assumption can be relaxed but at the expense of tractability in the analysis. Finally, for tractability – to avoid a multi-echelon inventory management problem with lost sales, which is known to be intractable –, we assume that the shipping lead time is negligible, and hence we do not need to track in-transit inventories. A depiction of this setting is provided in Figure 1.

For each channel $j \in \{\text{off}, \text{on}\}$, let S_t^j be the stochastic counting process of sales with intensity λ^j. When $x_t^{\text{on}} > 0$, the inventory state of the retailer, hereafter shown by the couple $(x_t^{\text{off}}, x_t^{\text{on}})$, is then updated by

$$\mathrm{d}x_t^{\text{off}} = u_t - \mathrm{d}S_t^{\text{off}},$$
$$\mathrm{d}x_t^{\text{on}} = -u_t - \mathrm{d}S_t^{\text{on}}.$$

For notational brevity, we drop the subscript t from these variables in the remainder of the paper, which will be used as a system state variable of its own. We also note that it is sufficient to consider replenishment policies of the form $u = \mathbf{u}(t, x^{\text{off}}, x^{\text{on}})$, as the system is Markovian.

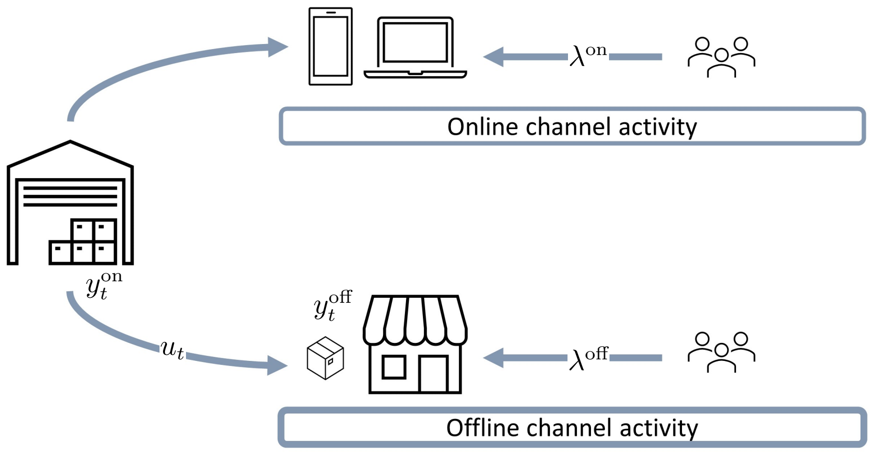


Figure 1 **Retailer's Omnichannel Inventory Allocation Problem**

At time t, suppose that our retailer decides to keep all the available units in the DC for online sales. In that case, during an infinitesimal time interval δ, a customer might arrive in the offline channel with probability $\delta\lambda^{\text{off}}$. If there is inventory available in the store, which we denote using the indicator function $\mathbb{1}_{\{x^{\text{off}}>0\}}$, a purchase is realized, and store stock level reduces by one unit. Otherwise, the offline demand is lost and unobserved. There also might be demand for one unit in the online channel with probability $\delta\lambda^{\text{on}}$. If an inventory is available at the DC, shown by $\mathbb{1}_{\{x^{\text{on}}>0\}}$, an online sale will be realized, which reduces inventory in the DC by one unit. Otherwise, online demand is lost and unobserved. Alternatively, there can be no arrivals in any of the channels with probability $1-\mathbb{1}_{\{x^{\text{off}}>0\}}\delta\lambda^{\text{off}}-\mathbb{1}_{\{x^{\text{on}}>0\}}\delta\lambda^{\text{on}}$. Given these events that might occur during δ, one can then informally derive the expected return of the *no-ship* decision ($u=0$) as

$$\begin{aligned} V\left(t,x^{\text{off}},x^{\text{on}}\right) = \quad & \mathbb{1}_{\left\{x^{\text{off}}>0\right\}}\delta\lambda^{\text{off}}\left(r^{\text{off}}+V\left(t-\delta,x^{\text{off}}-1,x^{\text{on}}\right)\right) \\ & +\mathbb{1}_{\{x^{\text{on}}>0\}}\delta\lambda^{\text{on}}\left(r^{\text{on}}+V\left(t-\delta,x^{\text{off}},x^{\text{on}}-1\right)\right) \\ & +\left(1-\mathbb{1}_{\left\{x^{\text{off}}>0\right\}}\delta\lambda^{\text{off}}-\mathbb{1}_{\{x^{\text{on}}>0\}}\delta\lambda^{\text{on}}\right)V\left(t-\delta,x^{\text{off}},x^{\text{on}}\right) \\ & +o\left(\delta\right), \end{aligned}$$

where $V(t,x^{\text{off}},x^{\text{on}})$ denotes the maximum expected profit obtained from t until the end of the season, with starting inventory $(x^{\text{off}},x^{\text{on}})$. Rearranging the terms and taking the limit as $\delta\to 0$, we obtain the famous Hamilton-Jacobian-Bellman (HJB) equation (Bertsekas 2012) as

$$\begin{aligned} \frac{\partial V\left(t,x^{\text{off}},x^{\text{on}}\right)}{\partial t} = & \mathbb{1}_{\left\{x^{\text{off}}>0\right\}}\lambda^{\text{off}}\left(r^{\text{off}}+V\left(t,x^{\text{off}}-1,x^{\text{on}}\right)-V\left(t,x^{\text{off}},x^{\text{on}}\right)\right) \\ & +\mathbb{1}_{\{x^{\text{on}}>0\}}\lambda^{\text{on}}\left(r^{\text{on}}+V\left(t,x^{\text{off}},x^{\text{on}}-1\right)-V\left(t,x^{\text{off}},x^{\text{on}}\right)\right). \end{aligned} \tag{1}$$

We have a terminal condition for this series of differential equations. When $t \to 0$, the leftover inventory is no longer valuable since there is no more sales opportunity, and the salvage value is zero. This implies that $V(0, x^{\text{off}}, x^{\text{on}}) = 0$.

If the retailer decides to ship one or more units, $u \in \{1, \ldots, x^{\text{on}}\}$, then the HJB equation (1) does not apply. Then, the expected return of the *shipping* decision is then

$$V\left(t, x^{\text{off}}, x^{\text{on}}\right) = V\left(t, x^{\text{off}} + u, x^{\text{on}} - u\right) - cu. \tag{2}$$

The omnichannel inventory positioning optimization problem can thus be formulated as:

$$V\left(t, x^{\text{off}}, x^{\text{on}}\right) = \max_{u \in \{0, \ldots, x^{\text{on}}\}} \left\{V\left(t, x^{\text{off}} + u, x^{\text{on}} - u\right) - cu\right\}. \tag{3}$$

To solve this problem, we first need to show in which circumstances $V(t, x^{\text{off}}, x^{\text{on}})$ is the solution to the HJB Equation (1) and in which it satisfies (2), along with the optimal value of u. We do so in the following section.

Before we proceed to the characterization of the optimal inventory positioning policy, we present an observation that facilitates further analysis.

Lemma 1. *Given inventory state $(x^{\text{off}}, x^{\text{on}})$, the value function V is increasing in t, for $t \in [0, T]$, $x^{\text{off}}, x^{\text{on}} \in \{0, \ldots\}$.*

The intuition and proof of this lemma hinge on the fact that when more time is available, the retailer can implement the same inventory positioning policy and still have extra time to sell any remaining items. Therefore, the value function must increase with time to go until the end of the season.

4. Solution Procedure

In this section, we devise a scheme to recursively construct the value function and characterize the optimal inventory positioning policy structure that protects revenue.

4.1. Construction of $V(t, x^{\text{off}}, 0)$

We first consider the case where no inventory is available in the DC. In other words, the online channel is closed. When there is also no inventory in the offline channel, $V(t, 0, 0) = 0$, as the retailer has nothing to sell. We characterize $V(t, x^{\text{off}}, 0)$ when $x^{\text{off}} \geq 1$.

When the online channel is inactive, no shipping decision needs to be made. In this case, we know that the HJB equation is valid:

$$\frac{\partial V\left(t, x^{\text{off}}, 0\right)}{\partial t} = \lambda^{\text{off}}\left(r^{\text{off}} - \Delta_0 V(t, x^{\text{off}})\right), \tag{4}$$

where

$$\Delta_0 V(t,x^{\text{off}}) \doteq V\left(t,x^{\text{off}},0\right) - V\left(t,x^{\text{off}}-1,0\right). \tag{5}$$

The terminal condition is $V(0,x^{\text{off}},0)=0$, since at the end of the season, there is no more sales opportunity, and the salvage value was normalized to zero. Solving the above differential equation renders a recursive formulation for the value function, that is

$$V\left(t,x^{\text{off}},0\right) = \int_0^t \lambda^{\text{off}}\left(r^{\text{off}} + V\left(s,x^{\text{off}}-1,0\right)\right)e^{-\lambda^{\text{off}}(t-s)}\,\mathrm{d}s. \tag{6}$$

We now characterize the properties of the value function, which facilitate further analysis.

Lemma 2. *At time t, the value function $V(t,x^{\text{off}},0)$ is increasing in the number of inventory units in the store, that is $V(t,x^{\text{off}},0) \geq V(t,x^{\text{off}}-1,0)$ for $t\in[0,T]$ and $x^{\text{off}}\in\{1,\ldots\}$.*

The intuition behind the lemma is that the retailer can implement the same inventory allocation policy with x^{off} units in the store as she does with $x^{\text{off}}-1$ units and still have one more unit left. Therefore, the value function increases with the store inventory level.

Lemma 3. *At time t, the marginal revenue $\Delta_0 V(t,x^{\text{off}})$ is decreasing in store inventory level, that is $\Delta_0 V(t,x^{\text{off}}) \geq \Delta_0 V(t,x^{\text{off}}+1)$ for $t\in[0,T]$ and $x^{\text{off}}\in\{1,\ldots\}$.*

This result shows that the value function is concave in the number of units in the store. Intuitively, since the potential demand is finite, increasing inventory reduces the sales probability, thus having a diminishing effect on the marginal revenue.

Lemma 4. *For $x^{\text{off}}\in\{1,\ldots\}$, the marginal revenue $\Delta_0 V(t,x^{\text{off}})$ increases with time, i.e., $\partial\Delta_0 V(t,x^{\text{off}})/\partial t \geq 0$.*

This result is also intuitive: as more time is available, the probability of selling each unit in the store increases. Therefore, each additional unit becomes more valuable, thus increasing the marginal revenue function.

These three results can be complemented with an explicit characterization of the value function.

Theorem 1. *For $x^{\text{off}}\in\{1,\ldots\}$, the value function $V(t,x^{\text{off}},0)$ is structured as*

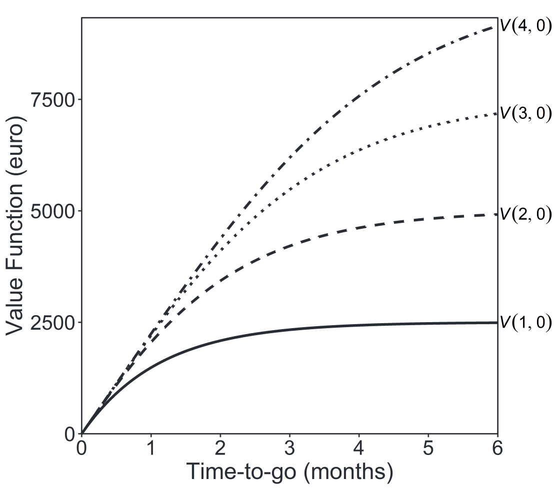
$$V\left(t,x^{\text{off}},0\right) = r^{\text{off}}\left(x^{\text{off}} - \sum_{i=0}^{x^{\text{off}}-1}\frac{x^{\text{off}}-i}{i!}\left(\lambda^{\text{off}}t\right)^i e^{-\lambda^{\text{off}}t}\right). \tag{7}$$

This theorem finalizes the value function's characterization and provides a close-form solution when the online channel is closed. As more time is available, the sales probability increases, thus increasing the expected return.

We demonstrate our findings using data from a prestigious luxury brand that retails a high-end handbag in its flagship store in Paris and on its website during the 6-month Spring/Summer fashion season. Due to high seasonality, short selling windows, and rapid product depreciation due to new trends introductions, luxury fashion retail fits seamlessly with our model (Caro and Martínez-de-Albéniz 2015). We measure time in months, so $T=6$. The season starts in early January and ends in late June.

Without loss of generality, the salvage value is assumed to be zero. Therefore the net profit obtained by selling a unit in the store, excluding the shipping cost, is $r^{\text{off}} = €2,500$ while $r^{\text{on}} = €1,750$ if the handbag is sold through the website, due to online channel commissions, e.g., the cut that a luxury marketplace like Farfetch may take. The demand rate in the stores is estimated to be $\lambda^{\text{off}} = 0.9$ units per month, whereas the demand rate on the website is approximately $\lambda^{\text{on}} = 5.7$ units per month. Finally, the unit shipping cost is estimated as $c = €40$.

We provide a depiction of the structure of the value function without online sales in Figure 2. We observe from the plots that when all the inventory is placed at the store, $V(t, x^{\text{off}}, 0)$ increases with time and store inventory level (Lemma 2 and Theorem 1). We also observe that the value of an additional unit decreases with store inventory level (Lemma 3); however, it increases as more time is available (Lemma 4). Notice that as time is scarce – t close to zero – the distinction between possessing two or four units becomes inconsequential since there is limited opportunity to sell even one unit. However, there is a stark contrast between the expected return of stocking two versus four units in the store early in the season when there is plenty of time for sales – t close to $T=6$ months – in which the expected demand is $0.9 \times 6 = 5.4$ units.



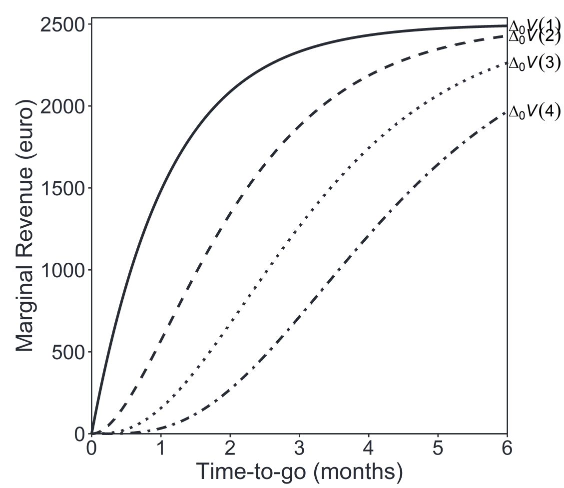


Figure 2 The Value Function (Left) and the Marginal Revenue (Right) When the Online Channel Is Closed

4.2. Construction of $V(t, x^{\text{off}}, 1)$

Calculation of $V(t, x^{\text{off}}, 0)$ brings us one step closer to solving the general case. However, we must first analyze the case with one unit available in the DC. Generally, when the DC has inventory available, at any time t, the retailer has to choose how many units to ship. She does so given the information on hand that is the time to go until the end of the season and the inventory state $(x^{\text{off}}, x^{\text{on}})$; this is an optimal stopping time problem.

To solve this problem with one unit in the DC, we identify under which circumstances $V(t, x^{\text{off}}, 1) = V(t, x^{\text{off}} + 1, 0) - c$ and when $V(t, x^{\text{off}}, 1)$ is the solution to the HJB equation (1).

With a reminder that $V(t, x^{\text{off}}, 0)$ can be calculated using (7), we first define

$$\Delta V\left(t, x^{\text{off}}, x^{\text{on}}\right) \doteq V\left(t, x^{\text{off}}, x^{\text{on}}\right) - V\left(t, x^{\text{off}} - 1, x^{\text{on}} + 1\right). \tag{8}$$

Now consider the case when the offline channel is closed, so the inventory state is $(0, 1)$. For $t \to 0$, since the selling horizon is too short, the expected revenue of selling the remaining unit in the store falls short in compensating the shipping cost c, that is $V(t, 1, 0) - c < 0$, since $V(t, 1, 0) \to 0$. This implies that for $t \to 0$, the no-ship decision is optimal. As a result, $V(t, 0, 1)$ satisfies the HJB equation (1). Solving this equation given the terminal condition $V(0, 0, 1) = 0$, yields, for t close to zero,

$$V(t, 0, 1) = \int_0^t \lambda^{\text{on}} r^{\text{on}} e^{-\lambda^{\text{on}}(t-s)}\, \mathrm{d}s = r^{\text{on}}\left(1 - e^{-\lambda^{\text{on}} t}\right). \tag{9}$$

Note from (9) that $V(t, 0, 1)$ is increasing in t. At the same time, from Theorem 1, we know that $V(t, 1, 0)$ is also increasing in t. Combined, these results imply that from $t = 0$, as t grows, there might be a point where $V(t, 0, 1)$ becomes smaller than $V(t, 1, 0) - c$. We denote this point by $\tau(1, 0)$ which we formally define as

$$\tau(1, 0) = \inf\left\{0 \le t \le T : \Delta V(t, 1, 0) = V(t, 1, 0) - V(t, 0, 1) = c\right\}; \tag{10}$$

therefore, for $t < \tau(1, 0)$, the no-ship decision is optimal and $\Delta V(t, 1, 0) < c$.

Letting $\tau = \tau(1, 0)$, at time $\tau + \delta$ (again δ being an infinitesimal duration), the optimal stopping time problem to solve is

$$V(\tau + \delta, 0, 1) = \max\left\{V(\tau + \delta, 1, 0) - c,\, \delta\lambda^{\text{on}} r^{\text{on}} + (1 - \delta\lambda^{\text{on}})\, V(\tau, 0, 1)\right\}. \tag{11}$$

Now suppose shipping is the optimal decision at $\tau + \delta$. Since it is also optimal to ship at τ, then it must be the case that

$$V(\tau + \delta, 1, 0) - V(\tau, 1, 0) \ge \delta\lambda^{\text{on}}\left(r^{\text{on}} - \left(V(\tau, 1, 0) - c\right)\right), \tag{12}$$

or equivalently

$$r^{\text{on}} - \frac{1}{\lambda^{\text{on}}} \cdot \frac{\partial V(t,1,0)}{\partial t}(\tau) \leq V(\tau,1,0) - c. \tag{13}$$

The structure of this inequality is intuitively appealing: $\frac{\partial V(t,1,0)}{\partial t}(\tau)$ is the marginal gain in revenue as more time is available, and $1/\lambda^{\text{on}}$ is the average time it takes to sell the one unit online. Therefore, the left side of (13) is the net revenue of selling the unit online, and the right side is the expected profit in the event of shipping to the store.

We show that (13) holds for any $t > \tau$. For that, define

$$\begin{aligned} f(t) &\doteq V(t,1,0) - c - r^{\text{on}} + \frac{1}{\lambda^{\text{on}}} \cdot \frac{\partial V(t,1,0)}{\partial t} \\ &= r^{\text{off}} - c - r^{\text{on}} + \left(\frac{\lambda^{\text{off}}}{\lambda^{\text{on}}} - 1\right) r^{\text{off}} e^{-\lambda^{\text{off}} t}. \end{aligned}$$

Note that $f(t)$ is continuously differentiable and inequality (13) implies that $f(\tau) \geq 0$.

Suppose $\lambda^{\text{off}} \geq \lambda^{\text{on}}$. Since $r^{\text{off}} - c \geq r^{\text{on}}$, $f(t) \geq 0$. On the other hand, $\lambda^{\text{off}} < \lambda^{\text{on}}$ leads to

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} = -\left(\frac{\lambda^{\text{off}}}{\lambda^{\text{on}}} - 1\right) \lambda^{\text{off}} r^{\text{off}} e^{-\lambda^{\text{off}} t} \geq 0, \tag{14}$$

which in turn means that $f(t)$ is increasing in t. Therefore, from τ, $f(t)$ only increases and $f(\tau) \geq 0$ implies that $f(t) \geq 0$ for all $t > \tau$. This implies that if time threshold $\tau(1,0)$ exists, it is unique and for $t \geq \tau(1,0)$, shipping is the optimal decision and $\Delta V(t,1,0) = c$.

At this point, we proceed to solve the problem when the offline channel is open ($x^{\text{off}} \in \{1,\ldots\}$), and there is one unit in the DC. Similarly to before, for $t \to 0$, the no-ship decision is optimal, therefore, $V(t, x^{\text{off}}, 1)$ is the solution to the HJB equation (1) with the terminal condition $V(0, x^{\text{off}}, 1) = 0$. We now know that from $t = 0$, as more time is available, both $V(t, x^{\text{off}}, 1)$ and $V(t, x^{\text{off}} + 1, 0)$ increase (Lemma 1). Therefore, there might be a point where $V(t, x^{\text{off}}, 1)$ intersects $V(t, x^{\text{off}} + 1, 0) - c$. We formally define this point as

$$\tau\left(x^{\text{off}} + 1, 0\right) = \inf\left\{0 \leq t \leq T : \Delta V\left(t, x^{\text{off}} + 1, 0\right) = c\right\}. \tag{15}$$

Lemma 5 generalizes the structure identified for $x^{\text{off}} = 0$ to $x^{\text{off}} \geq 0$.

Lemma 5. *Given inventory state $(x^{\text{off}}, 1)$, for $x^{\text{off}} \in \{1,\ldots\}$,*

- *Time thresholds $\tau(x^{\text{off}} + 1, 0)$ are unique and increase in the store inventory level, that is $\tau(x^{\text{off}} + 1, 0) \geq \tau(x^{\text{off}}, 0)$,*
- *Shipping is always the optimal decision for $t \geq \tau(x^{\text{off}} + 1, 0)$.*

The findings of Lemma 5 were the last remaining elements we required to wholly define the optimal inventory positioning policy, which we summarize in the following theorem.

Theorem 2. *When there is one unit of inventory in the DC, the optimal inventory positioning policy is characterized by epochs $\tau(x^{\text{off}}+1,0)$: the no-ship decision is optimal for $t<\tau(x^{\text{off}}+1,0)$ and it is optimal to ship all the inventory to the offline channel otherwise.*

For $x^{\text{off}} \in \{1,\ldots\}$, Theorem 2 thus guides the inventory positioning decision through a set of time-thresholds $\tau(x^{\text{off}}+1,0)$. The retailer ships the remaining unit from the DC to the store if and only if the time left until the end of the season is greater than $\tau(x^{\text{off}}+1,0)$ and keeps the unit still otherwise. These time-thresholds can numerically be computed by solving $\Delta V(t,x^{\text{off}}+1,1)=c$. If there is no finite solution to this equation, one can set $\tau(x^{\text{off}}+1,0)=\infty$.

Figure 3 demonstrates these results using information from the real scenario described in §4.1. We observe in the graphs that the optimal shipping policy divides the selling season into two intervals. When the store has no inventory, the retailer maintains the online-only channel structure in the first interval. In contrast, in the second interval, she only operates the offline channel by shipping the remaining unit from the DC to the store and closing the online channel. On the other hand, given an already active offline channel, at any time before the time threshold $\tau(x^{\text{off}}+1,0)$, the retailer operates both channels simultaneously. In contrast, she exclusively operates the offline channel in the second interval.

The results in Theorem 2 and Figure 3 are intuitively appealing as well. Early in the season, when there is plenty of time left and the opportunities for sales are higher, the retailer commits to the channel with a higher margin, therefore taking on an offline-only strategy. As the clock ticks down and the probability of selling each unit shrinks, the retailer is not incentivized to ship. This is because the anticipated revenue falls short of justifying the shipping cost.

Similarly to before, early in the season, there is a notable dissimilarity between the expected return of stocking one and three units in the store. In contrast, this dissimilarity fades away toward the end of the season. Moreover, if there is only one unit in the DC and no inventory available in the store after May 18, the retailer keeps the unit in the DC still. However, if there are one, two or three units available in the store, the retailer refrains from committing all units to the store and operates both channels at any time after April 2, February 22, and January 16, respectively (Lemma 5). If there are three or more units available in the store, the retailer always operates both channels throughout the season, that is $\tau(x^{\text{off}},0)\geq T=6$ for $x^{\text{off}} \in \{4,\ldots\}$.

Finally, as more time is available for sale, the retailer may use the same inventory positioning policy and still have additional time to sell the remaining inventory. As a result, the value function increases with time.

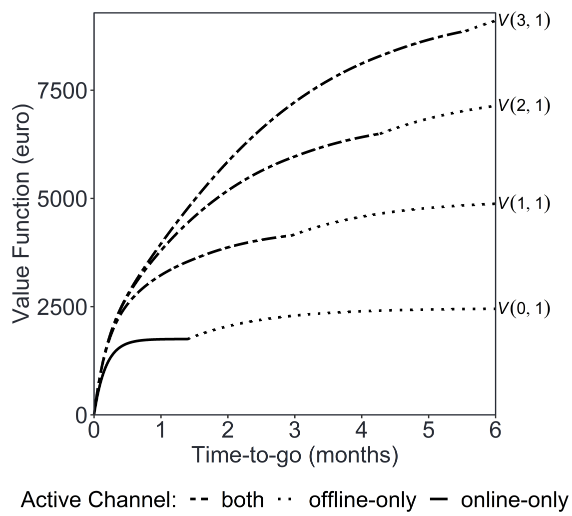


Figure 3 The Value Function When There Is One Unit in the DC

4.3. Construction of $V(t, x^{\mathbf{off}}, x^{\mathbf{on}})$

Calculation of $V(t, x^{\text{off}}, 1)$ paves the way for analyzing the general case and identifying the structure of the optimal inventory positioning policy. From Lemma 1, we know that starting from $t = 0$, as t grows, there might be a point where it is optimal to switch from inventory state $(0, x^{\text{on}})$ to $(1, x^{\text{on}} - 1)$, by shipping one unit from the DC. Formally, we define this point in time as

$$\tau(1, x^{\text{on}} - 1) = \inf\left\{0 \le t \le T : \Delta V(t, 1, x^{\text{on}} - 1) = V(t, 1, x^{\text{on}} - 1) - V(t, 0, x^{\text{on}}) = c\right\}. \tag{16}$$

At this point, we present an observation that is crucial to the remainder of the analysis.

Lemma 6. *Given inventory state $(x^{\text{off}}, x^{\text{on}})$, for $x^{\text{off}} \in \{1, \ldots\}$ and $x^{\text{on}} \in \{1, \ldots\}$, an optimal time-threshold policy divides the selling season into two intervals; at any time before $\tau(x^{\text{off}} + x^{\text{on}}, 0)$, the retailer is always better off keeping all the units in the DC still, and for $t \ge \tau(x^{\text{off}} + x^{\text{on}}, 0)$, it is always optimal to commit all inventory to the store.*

This lemma implies that if there exists a time epoch $\tau(x^{\text{off}} + x^{\text{on}}, 0)$, at any time before this threshold, the retailer does not stock more than one unit in the store. Intuitively, the retailer can save shipping costs by postponing any shipment to the moment when the store runs out of inventory.

We continue our analysis by providing another result that proves to be useful.

Lemma 7. *When no inventory is available in the store, for $x^{\text{on}} \in \{1, \ldots\}$,*

- *Time thresholds $\tau(1, x^{\text{on}})$ are unique and decrease in the inventory level at the DC, that is $\tau(1, x^{\text{on}} - 1) \ge \tau(1, x^{\text{on}})$,*
- *Shipping is always the optimal decision for $t \ge \tau(1, x^{\text{on}})$.*

These results were the last pieces of information that we needed to fully characterize the optimal inventory positioning policy and protect revenue.

THEOREM 3. *A three-regime time-threshold policy governs the optimal inventory positioning decision:*

- *when* $t \le \tau(1, x^{\text{on}} - 1)$*, it is optimal not to ship, i.e.,* $u^* = 0$*;*
- *when* $\tau(1, x^{\text{on}} - 1) \le t \le \tau(x^{\text{off}} + x^{\text{on}}, 0)$*, it is optimal to ship one unit* $u^* = 1$ *if* $x^{\text{off}} = 0$ *and* $x^{\text{on}} > 0$*, and* $u^* = 0$ *otherwise;*
- *when* $t \ge \tau(x^{\text{off}} + x^{\text{on}}, 0)$*, it is optimal to ship the entire inventory to the store, i.e., if* $u^* = x^{\text{on}}$*.*

In other words, at optimality, both channels are operational in the central regime, while only one channel remains active during the beginning and end regimes.

When the offline channel is closed, for $x^{\text{on}} \in \{2, \ldots\}$, Theorem 3 identifies a set of time-thresholds $\tau(1, x^{\text{on}} - 1)$ that can numerically be computed by solving $\Delta V(t, 1, x^{\text{on}} - 1) = c$. If there is no finite solution to this equation, one can set $\tau(1, x^{\text{on}} - 1) = \infty$. These time thresholds, together with the time epochs characterized by Theorem 2 divide the selling season into three intervals: at any time before $\tau(1, x^{\text{on}} - 1)$, the retailer is better off keeping the units at the DC still, while for $t \in [\tau(1, x^{\text{on}} - 1), \tau(x^{\text{on}} - 1, 0)]$, it is optimal to keep only one unit in the store. In contrast, for $t \ge \tau(x^{\text{on}} - 1, 0)$, the retailer must send all the remaining inventory from the DC to the store. The structure of the resulting value function, using data from the real case described in §4.1, is shown in Figure 4.

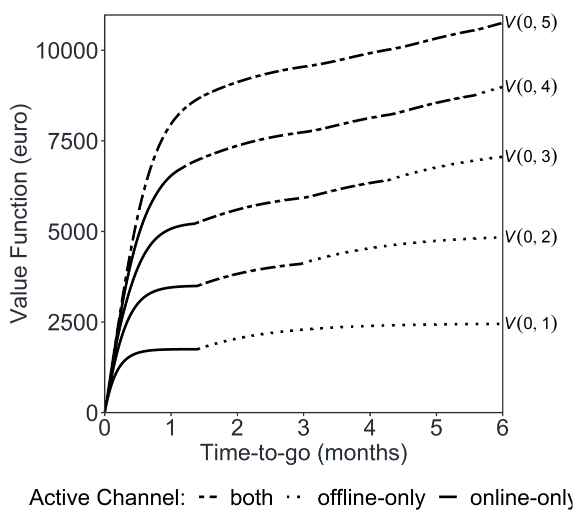


Figure 4 The Value Function When the Offline Channel Is Closed

The graph shows revenue rises when there is more time available for sales. Given different inventory levels in the DC, we observe two critical time epochs. When $\Delta V(t, 1, x^{\text{on}} - 1)$ reaches

c, it is optimal to switch from an online-only channel structure to one where both channels are open (Theorem 3). Note that in this middle region, the optimal stock level in the store is one unit (Lemma 6). When the retailer reaches the second threshold, she will shift focus from operating online and offline channels to fulfilling only offline demand. This divides the selling season into three intervals.

Theorem 3 indicates that the retailer reserves all the inventory for online sales when the inventory level is low relative to the time left, thereby closing the offline channel toward the end of the selling season. Intuitively, this is because the anticipated revenue falls short of justifying the cost of shipping. Notably, when the offline channel is closed and the relative inventory to the time left is high, the retailer adopts a conservative approach: maintaining a single unit in-store and replenishing upon sale.

Contradictory to the tempting and alluring prospect of limitless reach of the online channel, we find that retailers commit all the inventory to the offline channel at the beginning of the season, thereby taking on an offline-only strategy. These results show that inventory management serves as a *revenue protection* agent, especially by shipping all products to the offline channel to avoid sales from the lower-margin online channel.

5. Managerial Insights

In this section, we further analyze the omnichannel inventory positioning problem using the luxury fashion retail benchmark from §4.1. Applying the optimal inventory positioning policy in Theorems 2 and 3 results in the channel structure depicted in Figure 5. The graph shows that as more stock is available, the optimal inventory positioning policy postpones using the safety net, that is, the online channel as we get closer to the end of the season. For example, if there are two units in the DC, the retailer halts all shipments after May 19; had she had five units in the DC, she could afford to wait for one more month and then stop all shipments from June 28 onwards. On the other hand, with a lower stock level in the DC and more time left until the end of the season, the optimal policy favors the higher-margin channel (the store) for a more extended period: the threshold beyond which the retailer operates both channels is shifted closer to the end of June. For example, under the time threshold policy outlined in this paper, with two units in the DC, it is optimal for the retailer to implement an offline-only strategy at any time before April 2. However, if she had four units in the DC, the optimal policy dictates that the retailer should continuously operate both channels until May 25, after which she halts all store replenishment.

Performing an algebraic sensitivity analysis for our inventory positioning problem is daunting due to its recursive nature. Therefore, we aim to gain insights into effective managerial practices in our environment through numerical sensitivity analysis. We focus on the situation at the beginning of the season where $x^{\text{off}} = 0$ and $x^{\text{on}} > 0$.

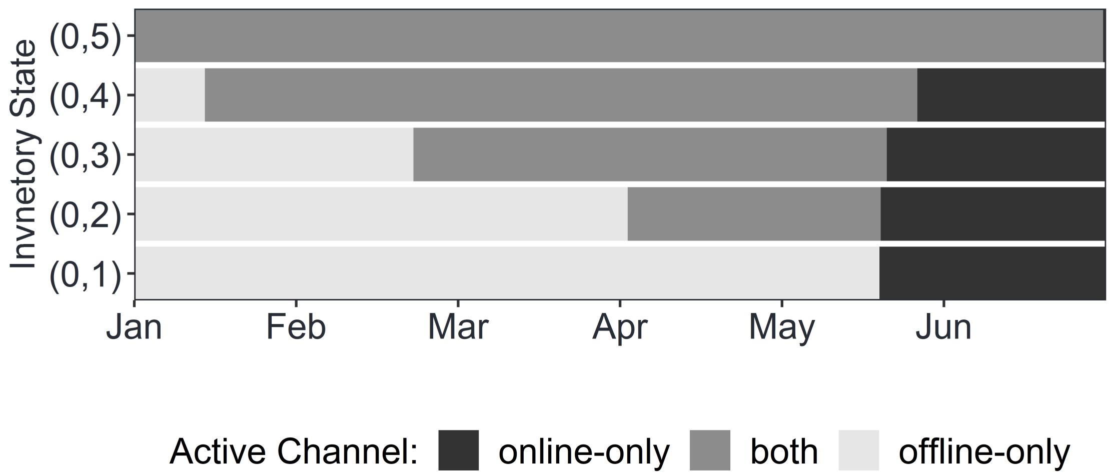


Figure 5 **Optimal Inventory Positioning Policy**

The shipping cost is a crucial factor in inventory positioning problems, and companies consistently try to reduce this cost (IKEA 2022, Martínez-de-Albéniz 2021, Milne 2022). Nevertheless, in industries such as luxury fashion and home appliances retail, the shipping cost remains a critical determinant in inventory positioning decisions. Figure 6 demonstrates a numerically-based sensitivity analysis of the optimal channel structure with respect to this key parameter. The left plot illustrates the variation in the time epochs $\tau_{\text{stop offline}} := \tau(1, x^{\text{on}} - 1)$, after which the retailer exclusively reserves DC inventory for online demand. We notice that with an increase in the shipping cost, it is better to advance the halt of replenishing the offline channel. Indeed, as the shipping cost rises, the retailer requires more time to compensate for this additional fee, ultimately leading to an acceleration of the closure of the offline channel.

The right plot displays the change in the time epochs $\tau_{\text{only offline}} := \tau(x^{\text{on}}, 0)$, which is the point before which the retailer allots all the inventory to the store. Similarly to before, as the shipping cost increases and the retailer requires more time to compensate for the added cost, $\tau_{\text{only offline}}$ increases. However, notice that toward the end of the season, an increase in the shipping cost leads to a considerably greater increase in $\tau_{\text{stop offline}}$ as opposed to $\tau_{\text{only offline}}$ early in the season. Hence, a decline in the shipping cost favors the higher-margin channel significantly more toward the season's end than early on, where ample time remains.

Another trend that merits further analysis is the relative size of the offline and online markets. In our baseline, $\lambda^{\text{on}} = 5.7$ units per month, while $\lambda^{\text{off}} = 0.9$ units per month. This reflects that the average store is much smaller than the online channel, which aggregates large geographies into a single sales channel. Given that this luxury retail chain has 25 stores in the country, this means that the share of online total retail sales is equal to $5.7/(5.7 + 25 \times 0.9) = 20\%$. This is similar to other fashion retailers. For example, Mango was selling about 36% of revenue online in 2022

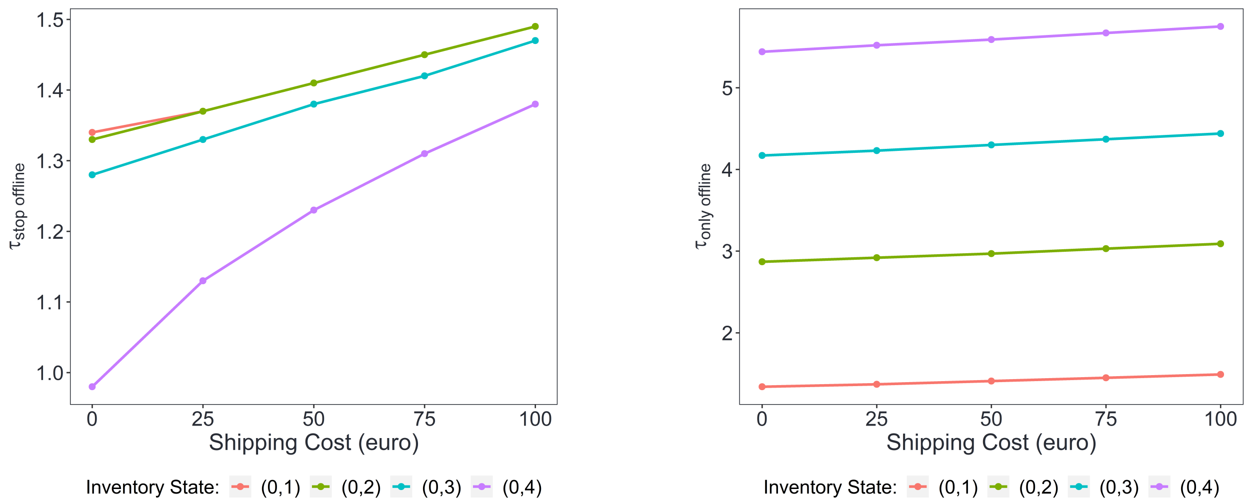


Figure 6 **(Color Online) Sensitivity Analysis of Time Epochs $\tau_{\text{stop offline}}$ (Left) and $\tau_{\text{only offline}}$ (Right) with Respect to the Shipping Cost**

(Mango 2023). For luxury products specifically, Bain & Company projects that the online channel will become the most potent, accounting for 30% of the global market (Impact Retail 2023). In contrast, Inditex only sells about 22% of revenue in online channels (Inditex 2022).

We perform a sensitivity analysis to analyze how the ratio of online sales impacts omnichannel inventory positioning decisions and present the results in Figure 7.

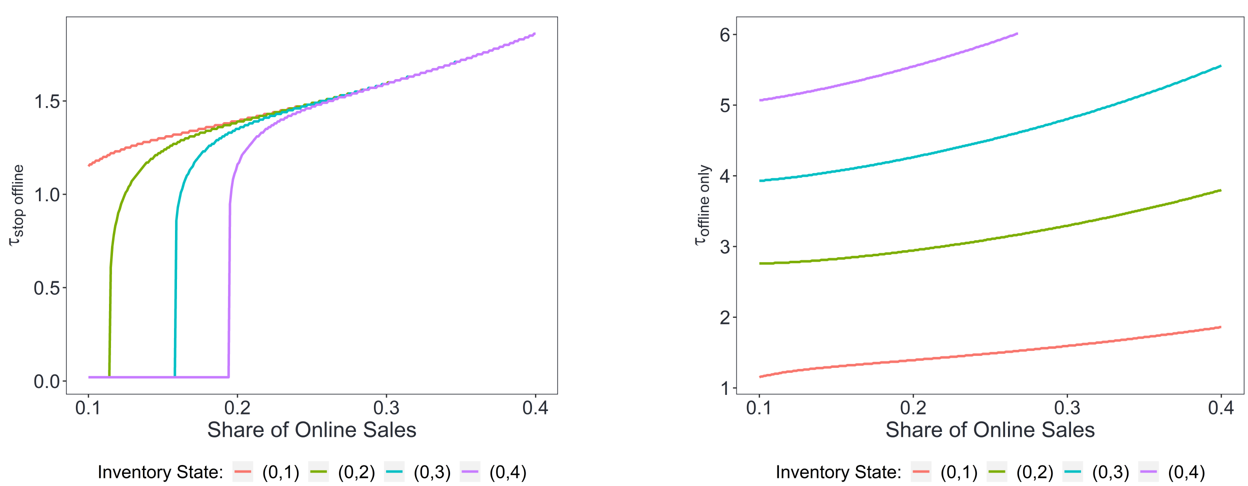


Figure 7 **(Color Online) Sensitivity Analysis of Time Epochs $\tau_{\text{stop offline}}$ (Left) and $\tau_{\text{only offline}}$ (Right) with Respect to the Proportion of Online Sales**

In the left plot in Figure 7, we observe that when the share of online product sales increases, the retailer is generally better off expediting the closure of the offline channel since she can sell more online. The graph on the right in Figure 7 shows that when the proportion of online to total sales increases, $\tau_{\text{only offline}}$ increases. Indeed, when the share of online sales increases, the online channel becomes more attractive. The retailer thus has less incentive to commit all inventory to the offline channel.

6. Conclusion

In this work, we analyze an omnichannel inventory positioning problem. Using optimal control theory, we recursively solve the problem and analyze the shipping decisions of retailers that operate an online channel, such as a website, and an offline channel, a store, for example. The optimal policy is a time-threshold policy as characterized in Theorems 2 and 3. When the inventory level to the time left until the end of the season is high, retailers opt not to replenish the store's stock. On the other hand, with plenty of time left for sale early in the season, retailers commit all inventory to offline demand, relinquishing potential sales in the channel with a lower margin. Finally, in the middle region, we find that retailers operate both channels. However, it is best to follow a conservative approach and keep no more than one unit in the store. As discussed in section 1, strategies such as online demand fulfillment from stores and transshipments are often costly. Our results provide a concrete preventive framework to leverage inventory positions to protect revenue.

There are several directions worth exploring. For example, in our model, demand arrivals and purchases are equivalent. Although many researchers adopt the same assumption, an exciting extension would be distinguishing between arrivals and purchases so that only some arrivals necessarily translate into a purchase. One can then employ a Bayesian framework so the retailer "learns" from her observations and updates her belief about the demand distribution, in the spirit of Chen et al. (2017). Furthermore, our framework focuses on a setting whereby the retailer operates her online channel. However, many retailers opt for using a marketplace platform since it can be as easy as registering on a website with few commitments. In such platforms, a commission fee is retained if a unit is sold via the platform. Given this setting, it would be intriguing to delve deeper into the inventory management problem when it is transferred to the platform.

References

Alishah EJ, Moinzadeh K, Zhou YP (2017) Store fulfillment strategy for an omni-channel retailer. Technical report, Working paper, Microsoft Corporation, Redmond, WA.

Andrews JM, Farias VF, Khojandi AI, Yan CM (2019) Primal–dual algorithms for order fulfillment at urban outfitters, inc. *INFORMS Journal on Applied Analytics* 49(5):355–370.

Arslan AN, Klibi W, Montreuil B (2021) Distribution network deployment for omnichannel retailing. *European Journal of Operational Research* 294(3):1042–1058.

Axsäter S, Rosling K (1993) Installation vs. echelon stock policies for multilevel inventory control. *Management Science* 39(10):1274–1280.

Bayram A, Cesaret B (2021) Order fulfillment policies for ship-from-store implementation in omni-channel retailing. *European Journal of Operational Research* 294(3):987–1002.

Bell D, Gallino S, Moreno A, et al. (2015) Showrooms and information provision in omni-channel retail. *Production and Operations Management* 24(3):360–362.

Bell DR, Gallino S, Moreno A (2018) Offline showrooms in omnichannel retail: Demand and operational benefits. *Management Science* 64(4):1629–1651.

Bertsekas D (2012) *Dynamic programming and optimal control: Volume I*, volume 1 (Athena scientific).

Brynjolfsson E, Hu YJ, Rahman MS (2013) Competing in the age of omnichannel retailing. *MIT sloan management Review* .

Caro F, Kök AG, Martínez-de-Albéniz V (2020) The future of retail operations. *Manufacturing & Service Operations Management* 22(1):47–58.

Caro F, Martínez-de-Albéniz V (2015) Fast fashion: Business model overview and research opportunities. *Retail supply chain management: Quantitative models and empirical studies* 237–264.

Chen L, Mersereau AJ, Wang Z (2017) Optimal merchandise testing with limited inventory. *Operations Research* 65(4):968–991.

Cheung KL, Lee HL (2002) The inventory benefit of shipment coordination and stock rebalancing in a supply chain. *Management science* 48(2):300–306.

Clark AJ, Scarf H (1960) Optimal policies for a multi-echelon inventory problem. *Management science* 6(4):475–490.

Dekhne A, Lange T, Magnus KH, Scheringer I, Vincken S (2020) Better service with connected inventory. https://www.mckinsey.com/industries/retail/our-insights/better-service-with-connected-inventory, Accessed March, 2023.

Dzyabura D, Jagabathula S (2018) Offline assortment optimization in the presence of an online channel. *Management Science* 64(6):2767–2786.

Federgruen A, Zipkin P (1984) Computational issues in an infinite-horizon, multiechelon inventory model. *Operations Research* 32(4):818–836.

Feng Y, Gallego G (1995) Optimal starting times for end-of-season sales and optimal stopping times for promotional fares. *Management science* 41(8):1371–1391.

Feng Y, Xiao B (1999) Maximizing revenues of perishable assets with a risk factor. *Operations Research* 47(2):337–341.

Feng Y, Xiao B (2000) Optimal policies of yield management with multiple predetermined prices. *Operations Research* 48(2):332–343.

Gallego G, Van Ryzin G (1994) Optimal dynamic pricing of inventories with stochastic demand over finite horizons. *Management science* 40(8):999–1020.

Gallino S, Moreno A (2014) Integration of online and offline channels in retail: The impact of sharing reliable inventory availability information. *Management Science* 60(6):1434–1451.

Gallino S, Moreno A, Stamatopoulos I (2017) Channel integration, sales dispersion, and inventory management. *Management Science* 63(9):2813–2831.

Gao F, Su X (2017a) Omnichannel retail operations with buy-online-and-pick-up-in-store. *Management Science* 63(8):2478–2492.

Gao F, Su X (2017b) Online and offline information for omnichannel retailing. *Manufacturing & Service Operations Management* 19(1):84–98.

Govindarajan A, Sinha A, Uichanco J (2021) Joint inventory and fulfillment decisions for omnichannel retail networks. *Naval Research Logistics (NRL)* 68(6):779–794.

Guy S (2015) Retailers wrestle with consumers' desires to shop anywhere and anytime. https://www.digitalcommerce360.com/2015/09/23/retailers-wrestle-consumers-desires-shop-multiple-ways/, Accessed April, 2023.

Hense J, Hübner A (2022) Assortment optimization in omni-channel retailing. *European Journal of Operational Research* 301(1):124–140.

Home Depot (2021) The Home Depot Inc.'s CEO Craig Menear on Q4 2020 results - earnings call transcript. https://seekingalpha.com/article/4408269-home-depot-inc-s-hd-ceo-craig-menear-on-q4-2020-results-earnings-call-transcript, Accessed February, 2023.

IKEA (2022) The IKEA road to decarbonising transports and logistics. https://about.ikea.com/en/behind-scenes/commitments/2022/07/07/decarbonising-transports-and-logistics-to-create-a-sustainable-future, Accessed April, 2023.

Impact Retail (2023) Defining luxury: The top trends worth monitoring in 2023. https://www.linkedin.com/pulse/defining-luxury-top-trends-worth-monitoring-2023/, Accessed April, 2023.

Inditex (2022) Inditex group annual report 2022. https://static.inditex.com/annual_report_2022/pdf/Inditex-group-annual-report-2022.pdf, Accessed April, 2023.

Insider (2017) Best Buy struggles with omnichannel. https://www.businessinsider.com/best-buy-struggles-with-omnichannel-2017-3, Accessed April, 2023.

Jackson PL (1988) Stock allocation in a two-echelon distribution system or "what to do until your ship comes in". *Management Science* 34(7):880–895.

Jackson PL, Muckstadt JA (1989) Risk pooling in a two-period, two-echelon inventory stocking and allocation problem. *Naval Research Logistics (NRL)* 36(1):1–26.

Kumar A, Mehra A, Kumar S (2019) Why do stores drive online sales? evidence of underlying mechanisms from a multichannel retailer. *Information Systems Research* 30(1):319–338.

Lee D (2023) Amazon chief vows to 'go big' on physical stores. https://www.ft.com/content/f132299d-3b31-4248-8ddf-81182c5930fc, Accessed March, 2023.

Lee HL, Tang CS (1997) Modelling the costs and benefits of delayed product differentiation. *Management science* 43(1):40–53.

Lo V, Topaloglu H (2022) Omnichannel assortment optimization under the multinomial logit model with a features tree. *Manufacturing & Service Operations Management* 24(2):1220–1240.

Lowe's (2021) Lowe's reports fourth quarter 2020 sales and earnings results. https://corporate.lowes.com/newsroom/press-releases/lowes-reports-fourth-quarter-2020-sales-and-earnings-results-02-24-21#:~:text=(NYSE%3A%20LOW)%20today%20reported,the%20fourth%20quarter%20of%202019., Accessed March, 2023.

Mango (2023) Mango makes a profit of 81 million euros and achieves record sales in 2022. https://www.mangofashiongroup.com/en/w/mango-gana-81-millones-de-euros-en-2022-y-alcanza-su-r%C3%A9cord-hist%C3%B3rico-de-ventas, Accessed April, 2023.

Marklund J, Rosling K (2012) Lower bounds and heuristics for supply chain stock allocation. *Operations Research* 60(1):92–105.

Martínez-de-Albéniz V (2019) Omnichannel strategy at Camper. *IESE Business School case study* .

Martínez-de-Albéniz V (2021) Distribution strategy at Mango. *IESE Business School case study* .

Martínez-de-Albéniz V, Pinto C, Amorim P (2022) Driving supply to marketplaces: Optimal platform pricing when suppliers share inventory. *Manufacturing & Service Operations Management* 24(4):2367–2386.

Martínez-de-Albéniz V, Talluri K (2011) Dynamic price competition with fixed capacities. *Management Science* 57(6):1078–1093.

McGavin EJ, Schwarz LB, Ward JE (1993) Two-interval inventory-allocation policies in a one-warehouse n-identical-retailer distribution system. *Management Science* 39(9):1092–1107.

McGavin EJ, Ward JE, Schwarz LB (1997) Balancing retailer inventories. *Operations Research* 45(6):820–830.

Meyersohn N (2019) Retail's Amazon antidote: Buy online, pickup in store. https://edition.cnn.com/2018/12/29/business/walmart-target-best-buy-amazon-online-pickup/index.html, Accessed February, 2023.

Milne R (2022) IKEA cuts prices as material and shipping costs ease. https://www.ft.com/content/10c7eac3-c774-4932-b4be-4b6f4e64b6b6, Accessed April, 2023.

Repko M (2023) Target bets on e-commerce by investing $100 million in hubs to speed up delivery. https://www.cnbc.com/2023/02/22/target-bets-on-e-commerce-invests-in-delivery-hubs.html, Accessed April, 2023.

Reuters (2022) Inditex founder amancio ortega to buy u.s. warehouses for $905 million. https://www.reuters.com/business/retail-consumer/inditex-founder-amancio-ortega-buy-us-warehouses-905-mln-2022-09-13/, Accessed April, 2023.

Rooderkerk RP, Kök AG (2019) Omnichannel assortment planning. *Operations in an omnichannel world* 51–86.

Rosling K (1989) Optimal inventory policies for assembly systems under random demands. *Operations Research* 37(4):565–579.

Saha K, Bhattacharya S (2021) 'buy online and pick up in-store': Implications for the store inventory. *European Journal of Operational Research* 294(3):906–921.

Seifert RW, Thonemann UW, Sieke MA (2006) Relaxing channel separation: Integrating a virtual store into the supply chain via transshipments. *IIE Transactions* 38(11):917–931.

Silverstein S (2023) Kroger sees e-commerce as future profit engine. https://www.grocerydive.com/news/kroger-sees-e-commerce-as-future-profit-engine/645201/, Accessed April, 2023.

Snelling B (2022) Why retailers are converting stores into fulfillment centers. https://www.forbes.com/sites/brinsnelling/2022/04/04/why-retailers-are-converting-stores-into-fulfillment-centers/, Accessed March, 2023.

Tan FK (1974) Optimal policies for a multi-echelon inventory problem with periodic ordering. *Management Science* 20(7):1104–1111.

Target (2017) Investing to grow: Target commits more than $7 billion to adapt to rapidly evolving guest preferences. https://corporate.target.com/article/2017/02/financial-community-meeting, Accessed March, 2023.

Wahba P (2017) Why Kohl's CEO still thinks having more stores is better. https://fortune.com/2017/03/21/kohls-stores-closings/, Accessed March, 2023.

Walmart (2017a) Five big reasons Walmart bought Jet.com. https://corporate.walmart.com/newsroom/business/20160919/five-big-reasons-walmart-bought-jet-com#:~:text=A%20lot%20of%20folks%20ask,app%20to%20site%20to%20store., Accessed March, 2023.

Walmart (2017b) Walmart to acquire bonobos and appoint Andy Dunn to oversee exclusive consumer brands offered online. https://corporate.walmart.com/newsroom/2017/06/16/walmart-to-acquire-bonobos-and-appoint-andy-dunn-to-oversee-exclusive-consumer-brands-offered-online, Accessed March, 2023.

Walmart (2023) A big year in the books: Walmart eCommerce by the numbers. https://corporate.walmart.com/newsroom/2023/02/22/a-big-year-in-the-books-walmart-ecommerce-by-the-numbers, Accessed April, 2023.

Ward T (2023) Walmart turns up the excitement and discovery with brand-new Walmart.com experience. https://corporate.walmart.com/newsroom/2023/04/03/walmart-turns-up-the-excitement-and-discovery-with-brand-new-walmart-com-experience, Accessed April, 2023.

Wingfield N, de la Merced MJ (2017) Amazon to buy Whole Foods for $13.4 billion. https://www.nytimes.com/2017/06/16/business/dealbook/amazon-whole-foods.html, Accessed March, 2023.