The Term Structure of Interest Rates with Housing

Carles Vergara-Alert

IESE Business School. 21 Pearson Avenue, 08034 Barcelona, Spain.
Email: cvergara@iese.edu. Phone: +34 932534200. Fax: +34 932534343.

Abstract

This paper develops a general equilibrium model to study the link between the amount of capital invested in housing assets and the term structure of interest rates. In the model, the production of housing assets is irreversible and housing assets can be used as collateral for borrowing funds. Agents' decisions about consumption and investments in housing and non-housing assets generate a time-varying market price of risk that drives the dynamics of the term structure. The calibration to U.S. data using the simulated method of moments technique captures the dynamics of consumption, and the short- and long-term interest rates.

Keywords: Term structure; interest rates; housing; real investments; consumption.
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1. Introduction

Consumers and firms generally use short-term debt to finance the purchase of computers and most non-housing assets. Investments in computers represent a short-term investment because computers are assets characterized by high depreciation rates and a relatively short service life. In contrast, houses are usually financed with long-term debt. As opposed to computers, houses represent long-term investments with low depreciation rates and a long service life. Consequently, the aggregate amount of capital invested in non-housing assets, K, and the aggregate amount invested in housing assets, H, should be related to the short- and long-term interest rates.

In order to study this relationship, I develop a general equilibrium model in which the term structure of interest rates is endogenously determined by the representative agent's decisions on consumption and investments in housing and non-housing assets. The model is a fundamental extension of the Cox, Ingersoll, and Ross (CIR, 1985a and 1985b) model. As in Hirshleifer (1972) and Diamond and Dybvig (1983), the model considers two production technologies: (i) a fully reversible technology that offers constant returns to scale, as in the CIR model for the production of non-housing assets; and (ii) an irreversible technology for the production of housing assets. I assume that capital is the only input used in both production technologies. The assumption of perfect reversibility in non-housing capital and irreversibility in housing capital is based on the large adjustment costs in the housing markets and the unfeasibility to transform housing into non-housing assets.[1]

The model assumes an infinitely lived representative agent with preferences for non-housing and housing consumption. At each period, the agent must reassess how much capital to consume, how much capital to allocate to non-housing investments, and how much capital to allocate to housing investments. I study the equilibrium implications of these decisions on the endogenously generated term structure of interest rates. The ratio of housing to non-housing assets in the economy, H/K, is the state variable of this problem. To provide some insight into the key economic mechanism of the model, let us assume that the economy experiences a sequence of negative shocks. As a result, the economy has too little non-housing capital because the irreversibility constraint on long-term capital is binding. Consequently, the volatility of consumption growth will rise and the interest rate will fall.

The model and supporting empirical calibration suggest that the agent demands a high risk premium for holding housing assets for two reasons. First, the irreversibility of the housing production technology prevents her from incorporating new information about the economy over time. Therefore, undertaking housing investments requires her to give up the option value of delaying the investment decision, such that the rate of return must be high enough to compensate for that lost option value. Second, the agent is risk averse and cares about future non-housing consumption. Therefore, although she has utility for housing services, she dislikes the idea of owning “too much” housing and “not enough” non-housing assets.

The model is calibrated using the simulated method of moments (SMM) technique with

[1]The appendix shows a discussion of adjustment costs in the housing markets.

U.S. data from 1962 to 2016, and it provides estimates of consumption and the term structure of interest rates in terms of the ratio of housing to non-housing assets invested in the economy, H/K. I use aggregated U.S. economic data for fixed assets (i.e., stocks of capital), consumption and investments (i.e., flows of capital), and the term structure of interest rates. I separate the fixed assets and the investment accounts into housing and non-housing accounts. The model captures the first moments of consumption, and the short- and long-term interest rates, while it underestimates the second moments. Moreover, the model provides testable implications for consumption, and the short- and long-term interest rates. Empirical tests show that a 0.10% higher value in the ratio of housing to non-housing assets in the economy relates to a 0.57% (0.86%) lower value in the real (nominal) short-term interest rate and a 0.60% (0.89%) lower value in the real (nominal) long-term interest rate.

This study makes two main contributions to the existing literature. First, the paper focuses on a stylized general equilibrium model that endogenously determines consumption, non-housing investments, housing investments, and the term structure of interest rates. The model's dynamics are driven by one state variable, which is the ratio of aggregate housing to non-housing capital invested in the economy. This state variable is not imposed but found in equilibrium from the structural model. In contrast to the reduced-form term-structure models (see Constantinides. 1992; Due and Kan, 1996; Dai and Singleton, 2002; Duffee, 2002; Ang and Piazzesi, 2003; and Lettau and Wachter, 2007), the model does not impose any statistical structures on the market price of risk in the economy. The time-varying market price of risk is solved endogenously. By endogenizing the market price of risk, the model provides rich economic intuition concerning the channels through which non-housing and housing investments and consumption jointly determine the dynamics of the term structure of interest rates. Therefore, this model offers a useful framework for studying how changes in interest rates affect real-estate investments and vice versa. This paper adds a theoretical framework to the empirical literature that connects business-cycle variables to real-estate investments, as in Iacoviello (2005), Davis and Heathcote (2005), Leamer (2007), and Del

Negro and Ortok (2007).

Second, I provide empirical tests to validate the model. Traditionally, equilibrium models of consumption have generally required high values for risk aversion to generate the low interest rates and high excess returns that I find in the data. However, Bekaert, Engstrom, and Grenadier (2010), Wachter (2006), and Buraschi and Jiltov (2007) have developed consumption-based term-structure models that produce realistic moments when they are calibrated to real data from both bond and stock markets. Their models are driven by the concept of external habit persistence introduced in Campbell and Cochrane (1999), which generates a time-varying market price of risk. More recently, Jermann (2013) and Kung (2015) have developed production-based models of the term structure that capture the dynamics that I observe in the data. However, their papers do not distinguish between non-housing (short-term) investments and housing (long-term) investments.

Figures 1 and 2 suggest that the distinction between housing and non-housing capital invested in the real economy is related to the dynamics of the term structure. Figure 1 compares the dynamics of the slope of the term structure to the ratio of aggregate housing to non-housing assets, H/K.[2] This comparison reveals that the slope of the term structure is closely related to the H/K ratio. The economic intuition is as follows. If the slope of the term structure is high, then the cost of capital for long-term investments relative to the cost of capital for short-term investments is also high. In this case, long term investments, such as investments in housing, become less attractive and the H/K ratio is low. Note that both the slope of the term structure and the H/K ratio are endogenous variables that are jointly determined equilibrium outcomes. Figure 2 shows that there is a link between the H/K ratio and the short-term interest rate. If the short-term rate is low, then the cost of capital for non-housing short-term investments is low and the capital allocated to this type

[2] To calculate H and K, I use data on fixed assets in the U.S. from the Bureau of Economic Analysis. I separate the different fixed-asset accounts into accounts related to housing and non-housing. Let H and K denote the accounts that aggregate all of these housing and non-housing accounts, respectively. Section 4 provides a detailed description of the data.

of investment should be high.[3] Note also that both the short-term interest rate and the H/K ratio are endogenous variables.

[FIGURES 1 AND 2 HERE]

The remainder of the paper is organized as follows. Section 2 covers the model's setup. In Section 3, I discuss the model's equilibrium and its economic implications, as well as functional forms for the short-term interest rate, the optimal consumption policy, the stochastic discount factor of the economy, and the rates horizons that determine the term structure at different. In Section 4, I provide details about the data, develop the model calibration, and show the empirical results. Finally, Section 5 concludes the paper.

2. The model

In this section, I set up the central planner's problemthe need to determine the optimal allocation of resources given the technological constraints. This problem is formally motivated by a general equilibrium economy in a decentralized production economy as in Lucas and Prescott (1970), and Cox, Ingersoll, and Ross (1985a).

2.1. The production side: Non-housing and housing production technologies

Assume that there are two types of assets in this economy: housing assets and non-housing capital goods.[4] Consider also that there are two production technologies to produce these assets: an irreversible technology for producing housing assets, which exhibits constant returns to scale, and a perfectly reversible technology for producing non-housing assets, which

[3]Furthermore, when there are positive shocks in the non-housing production technology, investors will be willing to pay higher short-term interest rates because the returns on short-term investments increase accordingly. This is consistent with the finding that production-based factors have explanatory power for asset-price dynamics (see Cochrane, 1988; Jermann, 1998, 2013; and Tallarini, 2000).

[4]In this model, non-housing consumption goods are capital goods. In classical economic theories, capital is one of the three traditional factors of production. The others are land and labor. Goods are viewed as capital if: (i) they can be used in the production of other goods (they are a factor of production) and (ii) they were produced (e.g., they are not natural resources, such as land and minerals). In the rest of the paper, the concepts of *non-housing capital* and *non-housing capital goods* are used interchangeably.

has constant returns to scale and instantaneously produces consumption goods. The only input for production is non-housing capital.

Let K_t denote the amount of capital in the non-housing sector. This capital is available for immediate consumption or investment at any time t. Changes in K_t from time t to time $t+dt$ arise from four sources. The first source is the return from the fraction x_t of non-housing capital K_t that is allocated to non-housing production. Let $f_K(K_t)$ denote the reversible production function. For simplicity, let us assume that capital is the only factor of production, and it presents constant returns to scale, such that $f_K(K_t) = K_t \left(\mu_K dt + \sqrt{\sigma_K} dW_t^K\right)$. The second source is the return from the remaining fraction $(1 - x_t)$ that is allocated to investments at the short-term risk free rate r_t. The third source is consumption, $C_t dt$. The fourth source is the capital that is invested in housing production at each period, $I_t dt$. This investment is irreversible, so $I_t \geq 0$. Therefore, the stock of capital K_t evolves according to the following process:

$$dK_t = x_t K_t \left(\mu_K dt + \sqrt{\sigma_K} dW_t^K\right) + (1 - x_t) K_t r_t dt - C_t dt - I_t dt. \quad (1)$$

Let H_t denote the amount of capital in the housing sector at any time t. Changes in H_t from time t to time $t + dt$ arise from the returns from housing production. Let $f_H(H_t, I(t))$ denote the irreversible housing production function. For simplicity, let us assume that (housing and non-housing) capital is the only factor of production and that it presents constant returns to scale such that $f_H(H_t, I_t) = H_t \left(\mu_H dt + \sqrt{\sigma_H} dW_{H,t}\right) + I_t dt$. As a result, the stock of housing capital H_t evolves according to the following process:

$$dH_t = H_t \left(\mu_H dt + \sqrt{\sigma_H} dW_t^H\right) + I_t dt. \quad (2)$$

Note that the capital accumulated in the housing sector is illiquid in the sense that the agent is not able to transfer capital from the housing sector to the non-housing sector. Hence, the agent faces a trade-o between investing in a reversible (liquid) technology that supplies a consumption good and investing in an irreversible (illiquid) technology that supplies housing, which is a good that provides her with some utility Moreover, the following non-negativity and irreversibility constraints apply for all t:

$$K_t > 0, \quad C_t > 0, \quad H_t > 0, \quad and \quad I_t \geq 0. \tag{3}$$

2.2. The demand side: The representative agent

The economy is populated by identical competitive households. Assume that this economy can be modeled as a single representative agent who maximizes her expected utility of intertemporal consumption over non-housing and housing capital goods, has time-separable utility $U(C_t, H_t)$, and a patience rate parameter given by ρ. In each period, the agent must decide: (i) how much capital stock C_t she consumes, (ii) the fraction x_t of capital stock that she allocates to non-housing investments, and (iii) how much capital stock I_t she allocates to housing investments. Consequently, the agent solves the following problem:

$$\max_{C_t, x_t, I_t} \left\{ E_0 \left[\int_0^\infty e^{-\rho t} U(C_t, H_t) dt \right] \right\} \tag{4}$$

such that the conditions in equations (1), (2), and (3) hold. In equation (4), E_0 denotes the expectation operator.

Finally, the agent faces a collateral constraint. She can borrow up to a fraction of the value of the housing stock defined by a constant loan-to-value ratio, LTV, with $0 < LTV < 1$. The value of housing stock is the amount of housing, H_t, times the shadow price of the housing stock, P_t.[5] The collateral constraint is given by the following inequality:

$$LTV\, H_t P_t \geq (x_t - 1) K_t, \tag{5}$$

which states that a proportion, LTV, of the housing value that she owns, $H_t P_t$, must be greater than the amount that she borrows, $(x_t - 1)K_t$. Note that x_t can be greater than 1.

[5]The shadow price of the housing stock is equivalent to Tobin's q and is given by the ratio of the market value of an additional unit of housing stock to its replacement cost. Therefore, it has a value of 1 whenever there is an investment in housing.

Therefore, the amount that she holds in the risk-free asset, $(1 - x_t)K_t$, could be negative, and it is equivalent to borrowing or holding debt.[6]

3. Equilibrium

This section studies the equilibrium of the representative agent problem described by equations (1)-(5). Specifically, I focus on the term structure of this economy provided by the equilibrium of the model. I examine the first-order conditions (FOCs) and provide the economic intuition behind the model.

3.1. Preliminary implications in equilibrium

Under the setup of the economy described in Section 1, a competitive equilibrium with dynamically complete markets is a set of processes $\{C_t^*, x_t^*, K_t^*, H_t^*, I_t^*, r_t, M_{t,s}\}$ such that the following statements hold:

1. Given r_t and the stochastic discount factor (SDF) of the economy, $M_{t,s}$, the set $\{C_t^*, x_t, I_t^*\}$ solves the representative household's problem defined in equation (4).
2. Given C_t^*, x_t^*, I_t^*, and the initial amounts of non-housing and housing capital goods in the economy (K_0 and H_0, respectively), the amounts of non-housing capital, K_t^*, and housing capital, H_t^*, solve the budget constraints in equations (1) and (2).
3. Markets clear. Therefore the clearing condition $x_t^* = 1$ holds at each time t.[7]
4. The stochastic process for r_t is such that $M_{t,s}^*$ is the unique SDF of this economy, and the following equation holds for each time t and s, with $s > t$: $\frac{1}{1+r_t} = \lim_{dt \to 0} E_t\left[M_{t-dt,t}\right]$.

The following lemma presents the existence of an equilibrium that satisfies the definition of competitive equilibrium above:

[6] Note that the cost of borrowing is equal to the risk-free rate, r_t. In other words, the agent may borrow and lend at the same rate.

[7] I also study the case of an open economy experiencing positive inflow from outside (e.g., foreign lenders exist). In this case, the representative agent of the economy can borrow in equilibrium, so that x_t can be greater than 1.

Lemma 1. *A competitive equilibrium with dynamically complete markets exists in which: (i) the set of processes* $\{K_t^*,\ H_t^*,\ C_t^*,\ I_t^*\}$ *is determined as the solution of the central planner's problem in equations (1)-(4); (ii) the optimal portfolio of the representative agent is determined by* $x_t = 1$*; and (iii) the SDF of this economy is given by* $M_{t,s} = e^{-\rho(s-t)} \frac{U_C(C_s^*, H_s^*)}{U_C(C_t^*, H_t^*)}$.

The first part of lemma 1 is based on Anderson and Raimondos (2008) findings, which provide conditions for ensuring that an equilibrium is dynamically complete. The second part states the market-clearing condition in equilibrium.[8] The third part remarks that the SDF of this economy presents the standard form of the SDF of classic consumption-based models. However, the dynamics of the SDF are different from the classic models, as the level and volatility of consumption depend on both Kt and Ht in this model.[9]

3.2. The value function and the Hamilton-Jacobi-Bellman equation

The model described by equations (1)-(4) leads to a two-state variable problem. Let J denote the value function for this problem:

$$J = J(K_t, H_t, t) = \max_{C_t, x_t, I_t} \left\{ E_0 \left[\int_0^\infty e^{-\rho t} U(C_t, H_t) dt \right] \right\}. \tag{6}$$

Note that this value function depends on both the non-housing capital account, K_t, and the housing capital account, H_t. The state space of the problem $\{K_t, H_t\}$ is divided into two regions: a no-investment region and an investment region.[10] When the pair $\{K_t, H_t\}$ is inside the no-investment region, the agent consumes, but makes no new housing investments. When the pair $\{K_t, H_t\}$ is inside the investment region, the agent consumes and makes housing investments. Let J_K and J_H denote the first derivative of the value function with respect to K_t and H_t, respectively. The inequality $J_K > J_H$ holds in the no-investment region, while

[8] If we assume that the net supply of risk-free bonds is at zero, then the equilibrium will require setting the interest rates at a level at which the representative agent will choose not to invest in them. Hence, $x_t = 1$.

[9] Note that the dynamics of consumption in most consumption-based models are imposed or are not governed by at least two processes that depend on endogenous variables.

[10] See Kogan (2001, 2004), Mamaysky (2001), and Tuzel (2010) for similar models with two sectors and irreversible investments.

the equality $J_K = J_H$ holds in the investment region.

The solution of the agent's optimal control problem, which is defined above, satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\max \Bigg\{ \underbrace{\sup_{C_t, x_t, I_t} \left\{ E_0 \left[\widehat{dJ^*} + e^{-\rho t} U(C_t, H_t) dt \right] \right\}}_{No-investment\ region}, \underbrace{J_K - J_H}_{\substack{Investment \\ region}} \Bigg\} = 0 \tag{7}$$

where $\widehat{dJ^*}$ is represented by the following expression:

$$\widehat{dJ^*} = J_t + \left[x_t \mu_K K_t + (1 - x_t) K_t r_t - c_t - I_t \right] J_K + \frac{1}{2} \left[x_t^2 \sigma_K K_t^2 \right] J_{KK} +$$

$$+ \left[\mu_H H_t + I_t \right] J_H + + \frac{1}{2} \left[H_t^2 \sigma_H \right] J_{HH} + \left[x_t \sqrt{\sigma_K \sigma_H} \rho_{KH} K_t H_t \right] J_{KH}.$$

These no-investment and investment regions correspond to the two parts of the maximization function in equation (7). When the first part of this maximization problem is binding, the pair $\{K_t, H_t\}$ is inside the no-investment region and, therefore, $I_t = 0$. If the second part of this maximization problem is binding, then the pair $\{K_t, H_t\}$ is inside the investment region and $I_t > 0$.

3.3. Obtaining the first-order conditions

There are three FOCs in equilibrium. First, I obtain the following FOC when I take the derivative of the HJB equation with respect to C_t:

$$J_{K_t} = e^{-\rho t} U_C(C_t, H_t). \tag{8}$$

This envelope condition establishes the equilibrium trade-o between consumption today and consumption in the next period. In other words, there is an equilibrium between (i) the discounted marginal gain or loss in utility $e^{-\rho t} U_C(C_t, H_t)$ from consuming one more unit of non-housing goods at time$t$, and (ii) the marginal gain or loss J_K from investing this unit either in non-housing or housing assets, which will influence future utility. Consequently, the representative agent must be indifferent among the following decisions in equilibrium: (i) consuming an extra unit of non-housing goods today; (ii) investing an extra unit of capital in the non-housing technology in order to consume some extra non-housing capital in the next period; and (iii) investing an extra unit of capital in the housing technology to be able to consume some extra housing capital in the future.

Second, I take derivatives of the HJB equation with respect to x_t to obtain the short-term interest rate in equilibrium:

$$r_t = \underbrace{\mu_K}_{\substack{\textit{Drift of the short}\\ \textit{term process}}} + \underbrace{\sigma_K \left[\frac{J_{KK}}{J_K} K_t\right] x_t}_{\textit{Risk aversion}} + \underbrace{\sqrt{\sigma_K \sigma_H} \rho_{KH} \left[\frac{J_{KH}}{J_K} H_t\right] x_t}_{\substack{\textit{Hedging term for}\\ \textit{housing investments}}} . \tag{9}$$

This result is a generalization of the equilibrium interest rate in the Cox, Ingersoll, and Ross (CIR; 1985a) term-structure model. Although the J_K and J_{KK} variables in this model and in CIR (1985a) are equivalent, the J_{KH} variable does not account for the housing investment in CIR (1985a). The result presented here also generalizes the equilibrium interest rate of the model of durable goods in Mamaysky (2001).[11]

Given equation (9), the short-term interest rate r_t presents the following characteristics: (i) the more productive the non-housing technology is (i.e., the higher the μ_K), the higher the r_t; (ii) $\frac{J_{KK}}{J_K} K_t$ and $\frac{J_{KH}}{J_K} H_t$ have negative signs, and they are measures of risk aversion towards non-housing and housing investments, respectively; (iii) the higher the uncertainty in the non-housing sector σ_K^2, the lower r_t will be; and (iv) r_t is related to the standard deviations σ_K and σ_H of the non-housing and housing processes, respectively, as well as the correlation ρ_{KH} between these processes through the hedging term for the non-housing versus housing risk $\frac{J_{KH}}{J_K} H_t$. Note that this hedging term is zero when non-housing and housing processes are uncorrelated ($\rho_{KH} = 0$). Because the ratio $\frac{J_{KH}}{J_K} H_t$ is negative, this hedging term is positive for a negative correlation between K and I (i.e., $\rho_{KH} < 0$), and negative for a positive

[11] The model in Mamaysky (2001) presents only one source of uncertainty: shocks in the capital stock of nondurable goods (i.e., non-housing). As a result, the hedging term for housing investments in equation (9) that Mamaysky (2001) obtains is zero. Moreover, this model does not include collateral constraints.

correlation (i.e., $\rho_{KH} > 0$).

Third, I obtain the following FOC when I take derivatives of the HJB equation with respect to I_t:

$$J_K \geqslant J_H. \tag{10}$$

Note again that the state space of the problem $\{K_t, H_t\}$ is divided into two regions: the no-investment region ($I_t = 0$) in which $J_K > J_H$, and the investment region ($I_t > 0$) in which $J_K = J_H$ holds. Initially, the following inequality holds in the investment region: $J_H > J_K$. However, the agent allocates capital to housing investments, which increases the $\frac{H_t}{K_t}$ ratio until J_H is equal to J_K. Hence, agents allocate capital to housing investments when the ratio of housing to non-housing capital, $\frac{H_t}{K_t}$, becomes sufficiently low.

3.4. Equilibrium conditions in terms of the housing to non-housing ratio H_t/K_t

By using the left part of the HJB equation in (7), the FOCs, and Lemma 1, I obtain the following two-dimensional ordinary differential equation (ODE), which applies to the no-investment region:

$$\begin{aligned} 0 = {} & e^{-\rho t} U_C(C_t, H_t) + J_t + J_K \left[x_t K_t \mu_K + (1 - x_t) K_t r_t - C_t\right] + J_H \left[H_t \mu_H\right] + \\ & + 0.5 J_{KK} \left[x_t^2 K_t^2 \sigma_K\right] + 0.5 J_{HH} \left[H_t^2 \sigma_H\right] + J_{KH} \left[x_t K_t H_t \sqrt{\sigma_K \sigma_H} \rho_{KH}\right], \end{aligned} \tag{11}$$

such that the conditions in equation (8) and the equality in (10) hold. I also reduce the dimensionality of the ODE from two dimensions to one. First, consider the utility function $U(C_t, H_t) = \frac{\left(C_t^{\beta} H_t^{(1-\beta)}\right)^{1-\gamma}}{1-\gamma}$. As the production function is homogeneous of degree 1 and the utility function is homogeneous of degree $1-\gamma$, the value function is homogeneous of degree one. This implies that the ratio of outstanding long-term capital outstanding to short-term capital invested in the economy is sufficient to characterize this economy. Let us define $g(\omega_t)$ as part of the value function such that:

$$J(K_t, H_t, t) = e^{-\rho t} \frac{H_t^{1-\gamma}}{1-\gamma} g(\omega_t), \tag{12}$$

where ω_t is defined as $\omega_t = \log\left(\frac{1}{X_t}\right)$ and $X_t = \frac{H_t}{K_t}$. Given this state variable ω_t, the no-investment region will be given by $(-\infty, \omega^*]$, where ω^* is determined as part of the agent's control problem.[12] The process for ω_t is obtained using the two-dimensional version of Ito's lemma:

$$d\omega_t = \left[(x_t\mu_K + (1-x_t)r_t - \mu_H) - 0.5(x_t^2\sigma_K - \sigma_H) - \widehat{C}_t - \Lambda_t\right] dt + x_t\sqrt{\sigma_K} dW_t^K - \sqrt{\sigma_H} dW_t^H \tag{13}$$

where $\widehat{C}_t = C_t/K_t$ and $\Lambda_t = \left[\frac{I_t}{K_t}(e^{\omega_t}+1)\right]$. Therefore, Λ_t is a function of the ratios $\widehat{I}_t = I_t/K_t$ and $e^{\omega_t} = K_t/H_t$ or, equivalently, $\Lambda_t = \Lambda_t(\widehat{I}_t, \omega_t)$. Note that $\Lambda_t = 0$ when $I_t = 0$.

From equations (8) and (10), I obtain the following expressions for the optimal consumption policy, $\widehat{C}_t$, and the smooth pasting condition at the boundary, ω^*, respectively:

$$\widehat{C}_t = e^{-\omega_t}\left(\frac{g'(\omega_t)}{(1-\gamma)e^{\omega_t}}\right)^{\frac{1}{\beta(1-\gamma)-1}} \tag{14}$$

$$(e^{-\omega^*}+1)g'(\omega^*) = (1-\gamma)g(\omega^*)\ . \tag{15}$$

The super-contact condition $J_{KH} = J_{KK}$ or, equivalently, $J_{HK} = J_{HH}$, introduced in Dumas (1991) must also hold at the boundary of the investment region. If I use the form of the value function in equation (12), then the super-contact condition becomes:

$$((1-\gamma)e^{\omega^*}-1)g'(\omega^*) = (e^{\omega^*}+1)g''(\omega^*)\ . \tag{16}$$

Remarkably, the consumption policy that I obtain from equation (14), the smooth pasting condition in equation (15), and the super-contact condition in equation (16) only depend on the state variable ω_t or the realization of the state value at $\omega_t = \omega^*$. Finally, I need to obtain the form of the function $g(\omega)$ in order to determine the equilibrium conditions. Consider a

[12] The inequality $J_K > J_H$ holds in the investment region. Then the agent would allocate capital into long-term investments until $J_K = J_H$. The trigger ω^* is the value of the state variable ω_t, such that $J_K = J_H$.

constant $\bar{x}$, with $\bar{x} \geq 1$, such that the agent borrows up to a constant fraction $(1-\bar{x})$ of K_t.[13] The following theorem describes the functional form for $g(\omega)$.

Theorem 1. *The function $g(\omega)$ is the solution of the following ODE:*

$$0 = 0.5\beta_1 g''(\omega) + \beta_2 g'(\omega) + \beta_3 \left(\frac{g'(\omega)}{e^{\omega}}\right)^{\frac{\beta(1-\gamma)}{\beta(1-\gamma)-1}} + \beta_4 g(\omega), \tag{17}$$

where:

$$\beta_1 = (2-\bar{x})\bar{x}\sigma_K + \sigma_H - (1-0.5\bar{x})\bar{x}\sqrt{\sigma_K\sigma_H}\rho_{KH}, \tag{18}$$

$$\beta_2 = \mu_K - (1-0.5\bar{x})\sigma_K\bar{x} - (\mu_H + 0.5(2-\gamma)\sigma_H) + (1-\gamma)(2-\bar{x})\bar{x}\sqrt{\sigma_K\sigma_H}\rho_{KH}, \tag{19}$$

$$\beta_3 = \gamma\left(1-\gamma\right)^{\frac{-\beta(1-\gamma)}{\beta(1-\gamma)-1}}, \tag{20}$$

$$\beta_4 = -\rho + (1-\gamma)(\mu_H - 0.5\gamma\sigma_H), \tag{21}$$

under the conditions that must hold at the optimal boundary ω^* shown in equations (15) and (16). In addition, the following boundary condition accounts for the states in which ω becomes very small:

$$\lim_{\omega\to-\infty} g(\omega) = +\infty. \tag{22}$$

Proof. See appendix. ■

By expressing equation (9) for the short-term interest rate in terms of the state variable $g(\omega_t)$, I obtain:

$$r_t = \mu_K + \underbrace{\sigma_K\left[\frac{g''(\omega_t) - g'(\omega_t)}{g'(\omega_t)}\right]x_t}_{\textit{Risk aversion}} + \underbrace{\sqrt{\sigma_K\sigma_H}\rho_{KH}\left[\frac{-g''(\omega_t) + (1-\gamma)g'(\omega_t)}{g'(\omega_t)}\right]x_t}_{\substack{\textit{Hedging term for}\\ \textit{long term investments}}}. \tag{23}$$

The following proposition shows the differential form for the short-term interest rate. Most

[13]Note that Lemma 1 accounts for the particular case $\bar{x} = 1$, in which the markets clear or the agent does not borrow. Values of $\bar{x}$ greater than one imply that the agent borrows.

of the parametric models of the term structure of interest rates are of the form described in this proposition, as discussed in Duffie and Kan (1994).

Proposition 1. *The short rate process for this problem is the solution of the stochastic differential equation of the form:*

$$dr_t = [\alpha_1 + \alpha_2 r_t]\, dt + \left[\alpha_1^K + \alpha_2^K r_t\right] dW_t^K + \left[\alpha_1^H + \alpha_2^H r_t\right] dW_t^H, \tag{24}$$

where α_1, α_2, α_1^K, α_2^K, α_1^H, *and* α_2^H *are functions of* t, ω_t, Λ_t, *and* $g(\omega_t)$, *and depend on the parameters of the model* μ_K, μ_H, σ_K, σ_H, ρ_{KH}, β, *and* γ.

Proof. See appendix. ■

The short rate process obtained from the model developed in this paper and shown in equation (24) has a form similar to the two-factor CIR (1985b) model and the set of two-factor ane term-structure models (ATSMs) in Dai and Singleton (2000). However, there is one main dierence between these models and my model. In the model presented here, the time-dependent coefficients α_1, α_2, α_3, α_1^K, α_2^K, α_1^H, and α_2^H are neither deterministic nor defined by an affine structure. Instead, these coefficients depend on $g(\omega_t)$ and Λ_t, whose functional forms are provided by the optimization problem described in Section 2. Finally, note that the power θ of the diffusion terms is 0.5 as in the CIR model and the ATSM, while θ is 1.0 in Merton (1973), Vasicek (1977), Brennan and Schwartz (1979), and Black, Derman and Toy (1990), and 1.5 in Ahn and Gao (1999). Table I compares the forms of the short rate processes for these classic term structure models.

[TABLE I HERE]

Denote the equilibrium bond prices $B(\omega_t, t, T)$ as the date t securities that deliver one unit of the consumption good at date T. The following theorem shows how to calculate the price of any bond $B(\omega_t, t, T)$.

Theorem 2. *The equilibrium price at time t of a zero-coupon bond that expires at time T, $B(\omega_t, t, T)$, is the solution of the following partial differential equation (PDE):*

$$0 = B_t - r_t B + \left[(\bar{x}\mu_K + (1-\bar{x})r_t - \mu_H) - 0.5(\bar{x}^2\sigma_K - \sigma_H) - \widehat{C}(\omega_t) - \widehat{\Lambda}(\omega_t)\right] B_\omega$$

$$+0.5\left[\bar{x}^2\sigma_K + \sigma_H + \bar{x}\rho_{KH}\sqrt{\sigma_K\sigma_H}\right] B_{\omega\omega}, \tag{25}$$

subject to the following boundary conditions:

$$B(\omega_T, T, T) = 1, \tag{26}$$

$$B_\omega(\omega_T^*, t, T) = 0, \tag{27}$$

$$B_\omega(-\infty, t, T) = 0. \tag{28}$$

Proof. See appendix. ■

Five remarks arise from Theorem 2. First, the term $\widehat{C}(\omega_t) = \frac{C_t}{K_t}$ in equation (25) is a function of only ω_t because $\frac{C_t}{K_t} = e^{-\omega_t}\left(\frac{g'(\omega_t)}{(1-\gamma)e^{\omega}}\right)^{\frac{1}{\beta(1-\gamma)-1}}$. Second, $\widehat{\Lambda}(\omega_t) = \frac{I_t}{K_t}(e^{\omega_t}+1)$ is zero in the no-investment region ($I_t = 0$). Third, the boundary condition (26) is necessary to impose that $B(\omega_t, t, T)$ is the price of a security that pays \$1 at time T. Fourth, conditions (27) and (28) are necessary to rule out arbitrage opportunities. Finally, the term structure of interest rates at time t for different maturities T, with $T \geqslant 0$, is given by $y_t(\omega_t, T)$ and the expression:

$$y_t(\omega_t, T) = -\frac{log\left(B(\omega_t, t, t+T)\right)}{T}. \tag{29}$$

3.5. Closed-form solutions for consumption, the short rate, and the term structure of interest rates

There are no known closed-form solutions to the ODE in equation (17), subject to equations (15), (16), and (22) for the constants defined in equations (18)-(21). Therefore, this ODE should be solved numerically. However, Theorem 3 shows the existence of a closed-form solution under certain restrictions.

Theorem 3. *If $\gamma = 0$, then the solution of the ODE in equation (17), subject to equations (15), (16), and (22), has the following functional form:*

$$g(\omega_t) = e^{\lambda\omega_t} \tag{30}$$

where $\lambda = -\frac{\beta_2}{\beta_1} + \sqrt{\left(\frac{\beta_2}{\beta_1}\right)^2 - 2\frac{\beta_4}{\beta_1}}$, *for* $\omega_t < \omega^*$, *and* $\omega^* = log(\frac{\lambda}{1-\lambda})$.

Proof. See appendix. ■

Using equation (14) and the results from Theorem 3, the model provides the following closed form for the optimal consumption policy $\widehat{C}_t$ which only depends on ω_t:

$$\widehat{C}_t = e^{-\omega_t} \left(\lambda e^{\lambda\omega_t - 1}\right)^{\frac{1}{\beta-1}}. \tag{31}$$

After taking logarithms, this equation becomes:

$$log\left(\frac{C_t}{K_t}\right) = \underbrace{\frac{1}{\beta-1} \cdot (log\lambda - 1)}_{\textit{Constant term}} + \underbrace{\left(\frac{\lambda}{\beta-1} - 1\right) log\left(\frac{K_t}{H_t}\right)}_{\textit{Terms that depend on } H_t/K_t}. \tag{32}$$

Assume that the collateral constraint is binding, such that the agent borrows LTV times the value of her housing stock. This gives the following linear relationship between interest rates and the H_t/K_t ratio, when applying the findings in Theorem 3 to equation (23):

$$r_t = [\mu_K + \sigma_K(\lambda - 1) - \sqrt{\sigma_K\sigma_H}\rho_{KH}(\lambda - 1)] + [(\sigma_K - \sqrt{\sigma_K\sigma_H}\rho_{KH})(\lambda - 1)LTV] \cdot \frac{H_t}{K_t}. \tag{33}$$

Therefore, the equilibrium of the model can be stated in terms of the state variable $X_t = H_t/K_t$. I find the dynamics of this state variable by applying Ito's lemma to X_t and using equations (1) and (2):

$$dX_t = d(H_t/K_t) = \mu_X(X_t)dt + \sqrt{\sigma_H X_t}dW_t^H + \sqrt{\sigma_K X_t}dW_t^K, \tag{34}$$

where $\mu_X(X_t)$ is a function of the state variable $X_t = H_t/K_t$:

$$\mu_X(X_t) = \left[\mu_H - (LTV X_t + 1)\left(\mu_K + (LTV X_t + 1)\sigma_K - \rho_{KH}\sqrt{\sigma_K \sigma_H}\right) - \widehat{C}_t(X_t) + \widehat{\Psi_t}(X_t)\right] X_t, \tag{35}$$

where $\widehat{C}(X_t) = C_t/K_t$ and $\widehat{\Psi}(X_t) = \frac{\Psi_t}{K_t}\left(\frac{1}{H_t/K_t} + 1\right)$.[14] At this point, I can linearize the function $\mu_X(X_t)$ by applying a Taylor expansion of this function around the point $X^* = H^*/K^*$ that separates the no-investment region from the investment region:

$$\mu_X(X_t) \approx \mu_X(X^*) + \beta_X(X^*) \cdot (X_t - X^*), \tag{36}$$

where

$$\beta_X(X^*) = X^* \left[LTV\left(\mu + 2(LTV X^* + 1)\sigma_K - \rho\sqrt{\sigma_K \sigma_H}\right) - \widehat{C}\,'(X^*) + \widehat{\Psi}'(X^*)\right] + \frac{\mu_X(X^*)}{X^*}.$$

When I consider the linearization of term (35) shown in equation (36), the model presents the characteristics of the family of the affine term structure (ATSM) models developed and classified in Duffie and Kan (1996). In particular, the three hypothesis related to the functional forms of ATSMs stated and developed in Piazzesi (2010) hold for this model:

1. The process for the short interest rate, r_t, is affine on $X_t = H_t/K_t$:

$$r_t = \alpha_r + \beta_r \cdot X_t \tag{37}$$

   for $\alpha_r \in \mathbb{R}$ and $\beta_r \in \mathbb{R}$. Note that $\alpha_r = \left[\mu_K + \sigma_K(\lambda - 1) - \sqrt{\sigma_K \sigma_H}\rho_{KH}(\lambda - 1)\right]$, and $\beta_r = \left[(\sigma_K - \sqrt{\sigma_K \sigma_H}\rho_{KH})(\lambda - 1)LTV\right]$ as shown in equation (33).

2. The process X_t is an affine diffusion. This means that X_t solves

$$dX_t = \mu_X(X_t)dt + \sqrt{\sigma_H X_t}dW_t^H + \sqrt{\sigma_K X_t}dW_t^K \tag{38}$$

[14] Two of the remarks that follow Theorem 2 justify the nature of the second and third terms in (35). For example, note that from equation (31), the C_t/K_t ratio is only a function of ω_t and, consequently, from X_t and the H_t/K_t ratio.

for $\mu_X(X_t) = \kappa(\bar{x} - X_t)$ and $\kappa \in \mathbb{R}$ and $\bar{x} \in \mathbb{R}$. From equation (36), I obtain $\kappa = -\beta_X(X^*)$ and $\bar{x} = X^* - \frac{\mu_X(X^*)}{\beta_X(X^*)}$.

3. The local expectations hypothesis holds.

Under these three hypothesis, the following proposition provides a functional form for the bond prices in equilibrium in terms of the state variable X_t, time t, and maturity T.

Proposition 2. *The equilibrium bond prices $B(X_t, t, T)$ that determine the term structure of interest rates in this economy are given by the following exponential affine form:*

$$B(X_t, t, T) = exp\left[A_1(T) + A_2(T) \cdot X_t\right], \tag{39}$$

where $A_1(T)$ and $A_2(T)$ solve the system of ODEs:

$$A_1(T) = -\alpha_r + \kappa\bar{x}B_1(T) \tag{40}$$

$$B_1(T) = -\beta_r - \kappa B_1(T) + \frac{1}{2}\sigma_X(B_1(T))^2 \tag{41}$$

with the starting conditions $A_1(0) = 0$ and $B_1(0) = 0$, and σ_X is the standard deviation of the process that defines the dynamics of the state variable X_t.

Proof. See appendix. ■

Given equations (29) and (39), the term structure of interest rates at time t for different maturities T, $y_t(T)$ has the following functional form:

$$y_t(T) = -\frac{A_1(T)}{T} - \frac{A_2(T)}{T} \cdot X_t. \tag{42}$$

Finally, note that the interest rates in the model are in real terms (see Evans, Keef, and Okunev, 1994, and Dahlquist, 1996, for further details on modeling real interest rates). As in Boudoukh (1993), Campbell and Viceira (2002), and Wachter (2006), I assume that inflation follows an exogenous process in order to model nominal bonds. Let π_t denote log realized inflation and z_t denote log expected inflation. I assume that π_t follows an AR(1) process:

$$\pi_{t+1} = z_t + \nu_{\pi,t+1}, \tag{43}$$

$$z_{t+1} = (1 - \phi_z)\mu_z + \phi_z z_t + \nu_{z,t+1}, \tag{44}$$

where ϕ_z is a constant parameter. I use the estimation of the parameters for the inflation process in Campbell and Viceira (2002): $\phi_z = 0.992$ for the period 1962-1983 and $\phi_z = 0.8674$ after 1983.

4. Numerical and empirical results

4.1. Data

This section presents details about the data and the intuition behind the dynamics of the term structure that the model provides. I use data on: (i) fixed assets (i.e., stocks of capital); (ii) consumption and investments (i.e., flows of capital); and (iii) short-term and long-term interest rates. I consider data from 1962 to 2016.

The dataset for fixed assets consists of the net stock of fixed assets from the fixed asset tables (FAT) of the U.S. National Economic Accounts, which are provided by the Bureau of Economic Analysis (BEA). I consider both private (FAT, Tables 2.1 and 2.2) and government assets (FAT, Tables 7.1.A, 7.1.B, 7.2.A, and 7.2.B). Residential assets are treated as housing investments. Non-residential assets are separated into equipment and software (non-housing) and structures (housing). These data cover the stock of the non-housing capital account, K_t, and the housing capital account, H_t, in the model.

For consumption and investments, I use data from the National Income and Product Accounts (NIPA) on real consumption of nondurable goods. The data come from personal consumption expenditures (NIPA, Tables 2.3.4 and 2.3.5), real gross private domestic fixed investment (NIPA, Tables 5.3.4 and 5.3.5), and real government consumption expenditures and gross investments (NIPA, Tables 3.9.4 and 3.9.5) provided by BEA. The NIPA data on government consumption and investment are provided as an aggregate of consumption expenditures, gross investments in structures, and gross investments in equipment and software.[15]

For the short- and long-term interest rates, I use data for the term structure of interest rates from the Federal Reserve Board. I use the three-month rate as the short-term rate r_t and the five-year rate as the long-term rate R_t. Note that the five-year rate $y(\omega_t, t, 5)$ is equivalent to the long-term rate R_t, according to equations (29) and (42).

Moreover, I need to deflate the variables when working on real (not nominal) terms. Deflated data on the NIPA accounts can be obtained directly from the BEA. I use two main alternatives to estimate the real interest rates. First, I estimate real interest rates as deflated nominal interest rates.[16] Second, I dene an exogenous process for inflation and infer the real rates from the nominal rates. I use data on the consumer price index (CPI) for the nominal inflation.

The summary statistics for the main variables for the period 1962 through 2016 are shown in Table II. The mean of the real short-term rate (1.38%) is lower than the mean of the real long-term rate (2.42%) for this period. Meanwhile, the standard deviation of both variables is similar. The nominal real short-term (long-term) rate averages 5.28% (6.32%) with a standard deviation of 3.35% (2.83%.) The slope of the term structure measured as the difference between the rates of the 10-year and 5-year T-bonds has a mean of 1.04% with a maximum of 3.12% and a minimum of -1.22%. Consumption grows at an average of 2.85% per year in the sample period. The average ratio of consumption to non-housing

[15] I consider gross investments in structures as housing investments and gross investments in equipment and software as non-housing investments. I include government consumption expenditures as non-housing investments. Accordingly, I use the data on real consumption of nondurable goods as C_t, data on housing investments as I_t, and data on non-housing investments estimated as the portion of the account K_t that has neither been invested in housing nor consumed.

[16] Series covering an extended period of time are needed to estimate the model, but the U.S. Treasury started issuing Treasury Ination-Protected Securities (TIPS) in 1997. The differences between TIPS rates and the real rates calculated as deflated nominal rates are minor. Therefore, I can assume that the deflated nominal interest rate is a good proxy for the real interest rate. The differences between TIPS and real rates calculated as deflated nominal rates were more remarkable in the early years of the TIPS markets. This may be due to the significant illiquidity in the early years of the TIPS market.

assets is 0.29. The mean of the H_t/K_t ratio is 0.47. This ratio increases by an average of 0.21% per year during the period of analysis.

[TABLE II HERE]

The correlation matrix for the main variables is shown in Table III. The correlation between the real short-term and long-term rates is high (0.89). Similarly, the correlation between the nominal short-term and long-term rates is high (0.95). However, the correlation between the real and nominal short-term rates is lower (0.57), as is the correlation between the real and nominal long-term rates (0.40).

[TABLE III HERE]

4.2. Model estimation

This subsection presents the implementation of Duffie and Singleton's (1993) SMM to estimate the model's parameters. The model's set of parameters is $\Psi = \{\gamma, \rho, \beta, \mu_K, \sigma_K, \mu_H, \sigma_H, \rho_{KH}, LTV\}$. I must find the parameters that minimize the weighted distance between a set of model unconditional moments, $F_Z(\widetilde{\psi})$, and their moment conditions from the empirical data F_T.

Let f_t denote the vector of the time series of the following variables: (i) the real short-term interest rate, (ii) the real long-term interest rate, and (iii) the ratio of consumption to non-housing assets. Let F_T denote the set of unconditional moments of these variables in the data: $F_T = \frac{1}{T}\sum_{t=1}^{T} f_t$. I fix the subset of parameters $\Psi = \{\gamma, \rho, \beta, \mu_K, \rho_{KH}, LTV\}$ and estimate the set of parameters $\widetilde{\Psi} = \{\sigma_K, \mu_H, \sigma_H\} \subset \Psi$. As I am mostly interested in estimating the characteristics of housing and non-housing capital, I focus on the estimation of the subset of parameters $\widetilde{\Psi}$. The rest of the parameters are set to reasonable values according to existing studies and observed data (see Campbell and Cocco (2003), Cocco (2004), Yao and Zhang (2004), and Corradin, Fillat and Vergara-Alert (2013)).

More specifically, I fix the time preference parameter, ρ, at 2.5%, which is consistent with an expected long term inflation between 2% and 3%. The housing flow services parameter,

$1-\beta$, measures the degree to which households value housing consumption relative to non-housing consumption. I set $1-\beta$ at 0.35. This figure is consistent with the share of household housing expenditures in the US estimated using expenditure data from the U.S. Bureau of Labor Statistics. The value of the drift of the non-housing process, μ_K, is set at 6.3%, which corresponds to the annualized return on the S&P 500 index in the focal period 1962-2016. The correlation between the non-housing and housing processes, ρ_{KH}, is set at 0.745 given that the correlation between these two processes using US data for the period 1962-2016 is 0.745. I set the loan-to-value, LTV, at 0.420 because the average loan-to-value of across all homes in the US is 42.0%.[17] Table IV reports the model's fixed and estimated parameters.

[TABLE IV HERE]

I simulate the economy described by the model for a particular set of parameters $\widetilde{\Psi}$ using Theorem 3. This economy is uniquely determined by its state variable, ω_t, and endogenously determined by the optimal consumption and investment strategy of the agent. Hence, I first have to solve the optimal control problem developed in Section 3. I estimate the implied density function of the state variable, $h(\omega;\widetilde{\Psi})$, and I calculate the implied moments of the model: $F_Z(\widetilde{\Psi}) = E\left[f(\omega;\widetilde{\Psi})\right] \approx \int f(\omega;\widetilde{\Psi})h(\omega;\widetilde{\Psi})d\omega$.

Finally, SMM requires solving the problem:

$$\widetilde{\Psi}^* = \underset{\psi\in\widetilde{\Psi}}{\arg\min}\,[F_Z(\psi) - F_T]'\,W_T\,[F_Z(\psi) - F_T]\,, \tag{45}$$

where W_T is the weighting matrix. I assume $W_T = V^{-1}$, where $V$is the asymptotic unbiased estimate covariance matrix of the sample averages F_T. Table IV shows the subset of estimated parameters $\widetilde{\Psi} = \{\sigma_K, \mu_H, \sigma_H\} \subset \Psi$. The model provides estimations of the standard deviations for the non-housing process, σ_K, and the housing process, σ_H, of 10.2% and

[17] The average loan-to-value of across all homes in the US is estimated as the value of all mortgages divided by the value of all US homes using data from FRED, Federal Reserve Bank of St. Louis: *Owner-Occupied Real Estate including Vacant Land and Mobile Homes at Market Value* (code: HOOREVLMHMV) and *Households and Nonprofit Organizations; Home Mortgages; Liability* (code: HHMSDODNS)).

17.8%, respectively. These estimates correspond to variances in the non-housing and housing processes of 1.04% and 3.17%, respectively. The estimation of the drift of the housing process, μ_H, is 1.6%. Note that the standard deviation of the non-housing process is lower than the standard deviation of the S&P 500 (16.6%) but higher than the standard deviation for the growth process, dK/K, that I find in the data (3.6%). Nevertheless, the parameters of the housing process are similar to those found in Campbell and Cocco (2003), who report μ_H of 1.6% and σ_H of 16.2%.

Table V reports the first and second moments provided by the SMM estimation, and compares them to the moments in the data. The model slightly overvalues the first moments of the real short-term rate (1.55%), the real long-term rate (2.53%), and the ratio of consumption to housing assets (0.61), while it undervalues their second moments (0.21%, 0.16%, and 0.01, respectively). As I focus on the first moments of these three variables, only three parameters can be independently estimated from the data.

[TABLE V HERE]

Finally, I perform sensitivity analyses with respect to the loan-to-value ratio, LTV, the correlation between housing and non-housing, ρ_{HK}, and the standard deviation of the housing process, σ_H, using the parameters described in Table IV. Figure 3 and 4 exhibit these analyses for the short-term interest rates and consumption, respectively. The top two graphics in Figure 3 show the real short-term interest rate as a function of the K_t/H_t ratio for two values of the parameter ρ_{KH}. The bottom two graphics display the real short-term interest rate as a function of the K_t/H_t ratio for two values of the parameter σ_H. The model captures the trend of decreasing interest rates as the H_t/K_t ratio increases.

The top two graphics in Figure 4 show consumption in terms of housing capital, C_t/K_t, as a function of the K_t/H_t ratio for two values of the parameter ρ_{KH}. The bottom two graphics display the C_t/K_t ratio as a function of the K_t/H_t ratio for two values of the parameter σ_H. The model captures the trend of increasing K_t/H_t values as the K_t/H_t ratio increases as well as the convexity of the relationship between the C_t/H_t and K_t/H_t ratios.

[FIGURES 3 AND 4 HERE]

Moreover, Figures 3 and 4 highlight four interesting facts. First, interest rates decrease with the correlation between housing and non-housing capital. Second, the ratio of consumption to housing capital increases with this correlation. The economic intuition behind these two facts is as follows. Assume that the economy experiences a sequence of negative shocks in non-housing capital. This will lead to an economy with “too much” housing capital that cannot be transferred to non-housing capital because of the irreversibility constraint. As a result, consumption growth will become more volatile and the interest rate will fall. However, a higher correlation between housing and non-housing capital will decrease the effect of the irreversibility constraint, which will lead to a lower interest rate and a higher consumption-to-non-housing ratio. Third, interest rates decrease as the LTV ratio rises that is, lower LTV ratios lead to higher interest rates. Fourth, the ratio of consumption to non-housing capital increases as the LTV ratio rises. The intuition behind these two facts is as follows. Assume that the economy experiences a sequence of negative shocks in non-housing capital, which leads to an economy with “too much housing capital. If the LTV ratio is low, then the agent cannot borrow as much as she would find optimal in an economy with no collateral constraints. Consequently, interest rates are high. At the same time, the higher the LTV ratio, the more attractive are investments in housing capital. Therefore, the agent is willing to lower the ratio of consumption to housing capital.

4.3. Empirical tests

In this subsection, I present the results of the empirical tests. To validate the model, I use the closed-form solutions for consumption, the short-term interest rate, and the long-term interest rate shown in (32), (33), and (42), respectively. I show the tests for both the real and nominal interest rates.

The first empirical analysis, which focuses on the dynamics of consumption, tests the power of the closed-form solutions for consumption in equation (32). Table VI shows the results of this exercise. The negative sign for $log(H_t/K_t)$ dominates the positive sign for H_t/K_t for the range $[0.42, 0.55]$ of the ratio H_t/K_t that it is observed in the data (see Table II). In other words, an increase in the H_t/K_t ratio leads to a decrease in the C_t/H_t ratio. These results are statistically significant and consistent with the economic intuition behind the model: the lower level of available (liquid) non-housing capital K_t, the less the agent is able to consume, C_t, for a fixed amount of capital locked into housing investments, H_t. Columns [1] and [2] show the results of the OLS regressions for the logarithm of the consumption-to-housing ratio using the estimated end-of-the-year real rate. Column [3] shows the results of the regressions using the lagged H_t/K_t ratio as the instrumental variable.

[TABLE VI HERE]

The second empirical analysis focuses on the dynamics of the real and nominal interest rates. Tables VII and VIII show the tests of the power of the closed-form solutions for the short-term and long-term rates, respectively. In both tables, columns [1], [2], [4], and [5] show the results of the OLS regressions for the real and nominal rates in equation (33) using the estimated end-of-the-year real rate. Columns [3] and [6] show the results of the regressions for the real and nominal rates using the lagged H_t/K_t ratio as the instrumental variable.

[TABLES VII AND VIII HERE]

Table VII shows that a 0.10% higher value in the H_t/K_t ratio relates to a 0.57% lower value in the real short-term interest rate and a 0.86% lower value in the nominal short-term interest rate in the analysis period. Table VIII shows that a 0.10% higher value in the H_t/K_t ratio relates to a 0.60% lower value in the real long-term interest rate and a 0.89% lower value in the nominal long-term interest rate in the analysis period. Overall, these empirical results are consistent with the economic intuition provided by the mode's equilibrium results.

5. Conclusions

This paper differentiates between housing and non-housing capital, and applies this distinction to the CIR model to study the dynamics of the term structure of interest rates. The proposed model contains two risky production technologies: one for producing non-housing assets and another for housing assets. A representative agent decides how much capital to invest in the non-housing and housing technologies, how much to consume, and how much to borrow using housing as collateral. The capital invested in the non-housing technology is liquid, while the capital allocated to the housing technology cannot be disinvested whenever capital is needed for consumption. The illiquidity that characterizes housing production makes consumption smoothing harder, which leads the agent to increase her precautionary savings. Therefore, the short interest rate needs to be low in order to clear markets.

Under certain general conditions, the model is an affine term-structure model (ATSM) as defined in Duffie and Kan (1996). Its dynamics are driven by one state variablethe ratio of aggregate housing to non-housing capital invested in the economy, $X_t = H_t/K_t$. In contrast to standard ATSMs, this state variable is not imposed on the market price of risk in the economy, but found in equilibrium from a structural model. By endogenizing the market price of risk, the model provides rich economic intuition on how the housing and non-housing investment channels and the consumption channel jointly determine the dynamics of the term structure of interest rates.

The model provides functional forms for consumption, short-term interest rates, and long-term interest rates that can be used to empirically test the model. I estimate the parameters of the model using SMM and US data from 1962 to 2016. I find that the model captures the dynamics of consumption as well as the short-term and long-term interest rates. The empirical analysis shows that a 0.10% higher value in the ratio of housing to non-housing assets in the economy relates to a 0.57% (0.86%) lower value in the real (nominal) short-term interest rate and a 0.60% (0.89%) lower value in the real (nominal) long-term interest rate. These results confirm the importance of separating housing and non-housing investments when modeling the term structure of interest rates. This distinction has not been explored in the classic term-structure models and highlights an area for future research with high relevance for policy makers and bond investors.

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Appendix

A.1. Proofs of Propositions and Theorems

Proof of Proposition 1

Apply Ito's Lemma to equation (23) and rearrange terms to obtain the following expression:

$$dr_t = [\alpha_1 + \alpha_2 r_t]\, dt + \left[\alpha_1^K + \alpha_2^K r_t\right] dW_t^K + \left[\alpha_1^H + \alpha_2^H r_t\right] dW_t^H,$$

where

$$\alpha_1 = \alpha_1(\widehat{\Psi_t}, \omega_t, t) = x_t\sqrt{\sigma_K}\Upsilon_1(\widehat{\Psi_t}, \omega_t, t)\Upsilon_2(\omega_t, t) + \Gamma_{KH}\Upsilon_3(\omega_t, t)$$

$$\alpha_2 = \alpha_2(\widehat{\Psi_t}, \omega_t, t) = x_t\sqrt{\sigma_K}\Upsilon_1(\widehat{\Psi_t}, \omega_t, t)\Upsilon_2(\omega_t, t) + \Gamma_{KH}\left(\frac{g''(\omega_t)}{g'(\omega_t)}\right)^2$$

$$\alpha_1^K = \alpha_1^K(\omega_t, t) = x_t\sqrt{\sigma_K}\Upsilon_2(\omega_t, t)$$

$$\alpha_2^K = \alpha_2^K(\omega_t, t) = -x_t\sqrt{\sigma_K}\frac{g''(\omega_t)}{g'(\omega_t)}$$

$$\alpha_1^H = \alpha_1^H(\omega_t, t) = \sqrt{\sigma_H}\Upsilon_2(\omega_t, t)$$

$$\alpha_2^H = \alpha_2^H(\omega_t, t) = -\sqrt{\sigma_H}\frac{g''(\omega_t)}{g'(\omega_t)}$$

and where $\Upsilon_1(\widehat{\Psi_t}, \omega_t, t)$, $\Upsilon_2(\omega_t, t)$, $\Upsilon_3(\omega_t, t)$, and Γ_{KH} are defined as follows:

$$\Upsilon_1(\widehat{\Psi_t}, \omega_t, t) = (x_t\mu_K + (1-x_t)r_t - \mu_H) - 0.5(x_t^2\sigma_K - \sigma_H) - e^{-\omega_t}\left(\frac{g'(\omega_t)}{(1-\gamma)e^{\omega_t}}\right)^{\frac{\beta(1-\gamma)}{\beta(1-\gamma)-1}} + \Lambda_t(\widehat{\Psi_t}, \omega_t)$$

$$\Upsilon_2(\omega_t, t) = (x_t\mu_k + (1-x_t)r_t)\frac{g''(\omega_t)}{g'(\omega_t)} + x_t^2\sigma_K\frac{g'''(\omega_t) - g''(\omega_t)}{g'(\omega_t)} + x_t\sqrt{\sigma_K\sigma_H}\rho_{KH}\frac{(1-\gamma)g'''(\omega_t) - g''(\omega_t)}{g'(\omega_t)}$$

$$\Upsilon_3(\omega_t, t) = -x_t\mu_K\left(\frac{g''(\omega_t)}{g'(\omega_t)}\right)^2 + 0.5\left(\frac{g^{(iv)}(\omega_t)}{g'(\omega_t)} - 3\frac{g'''(\omega_t)g''(\omega_t)}{(g'(\omega_t))^2} + 2\left(\frac{g''(\omega_t)}{g'(\omega_t)}\right)^2\right)x_t\sqrt{\sigma_K} +$$

$$+0.5\left((1-\gamma)\left(\frac{g^{(iv)}(\omega_t)}{g'(\omega_t)} - 3\frac{g'''(\omega_t)g''(\omega_t)}{(g'(\omega_t))^2}\right) - 2\left(\frac{g''(\omega_t)}{g'(\omega_t)}\right)^2\right)x_t\sqrt{\sigma_K\sigma_H}\rho_{KH}$$

$$\Gamma_{KH} = x_t^2\sigma_K + \sigma_H - 2x_t\sqrt{\sigma_K\sigma_H}\rho_{KH}.$$

Q.E.D.

Proof of Proposition 2

See the proof in Duffie and Kan(1996) and take into consideration the following facts: (i) the drift $\mu_X(X_t)$ of the process for the state variable X_t in equation (34) is affine in X_t; (ii) the coefficients involved in the volatility term σ_K and σ_H are affine (e.g. in fact, they are constant as in Cox, Ingersoll and Ross (1985b)); and (iii) the process for the interest rate r_t is affine as shown in (33).

Q.E.D.

Proof of Theorem 1

The ordinary differential equation (ODE) shown in (17) comes from the left part of the maximization function in equation (7) evaluated at the optimal consumption shown in (14). I know from the HJB equation in (7) that the following ODE holds in the no-investment region:

$$\sup_{C_t, x_t, I_t} \left\{ E_0 \left[\widehat{dJ^*} + e^{-\rho t} U(C_t, H_t) dt \right] \right\} = 0$$

where $\widehat{dJ^*}$ is represented by the following expression:

$$\widehat{dJ^*} = J_t + \left[x_t K_t \mu_K + (1 - x_t) K_t r_t - C_t \right] J_{K_t} + \frac{1}{2} \left[x_t^2 K_t^2 \sigma_K \right] J_{K_t K_t} +$$

$$+ \left[H_t \mu_H \right] J_H + + \frac{1}{2} \left[H_t^2 \sigma_H \right] J_{H_t H_t} + \left[x_t \sqrt{\sigma_K \sigma_H} \rho_{KH} K_t H_t \right] J_{KH}$$

and J_K, J_{KK}, J_H, J_{HH}, and J_{KH} are the first and second order partial derivatives of the value function $J(K_t, H_t, t)$ with respect to K_t and H_t. When taking into account Lemma 1 and the fact that in the no-investment region $I_t = 0$, we obtain the following ODE:

$$0 = e^{-\rho t} U_{C_t}(C_t, H_t) + J_t + \left[x_t K_t \mu_K + (1 - x_t) K_t r_t - C_t \right] J_{K_t} + \frac{1}{2} \left[x_t^2 K_t^2 \sigma_K \right] J_{K_t K_t} +$$

$$+ \left[H_t \mu_H \right] J_H + \frac{1}{2} \left[H_t^2 \sigma_H \right] J_{H_t H_t} + \left[x_t \sqrt{\sigma_K \sigma_H} \rho_{KH} K_t H_t \right] J_{KH}.$$

Consider the utility function $U(C_t, H_t) = \frac{\left(C_t^{\beta} H_t^{(1-\beta)} \right)^{1-\gamma}}{1-\gamma}$ and the functional form of the value function defined by $J(K_t, H_t, t) = \frac{{H_t}^{1-\gamma}}{1-\gamma} g(\omega_t)$. When we plug the corresponding derivatives of $U(C_t, H_t)$ and J in the two dimensional ODE, I obtain the following one dimensional form of the ODE:

$$0 = 0.5 \hat{\beta}_1 g''(\omega) + \hat{\beta}_2 g'(\omega) + \beta_3 \left(\frac{g'(\omega)}{e^{\omega}} \right)^{\frac{\beta(1-\gamma)}{\beta(1-\gamma)-1}} + \beta_4 g(\omega),$$

where:

$$\hat{\beta}_1 = x_t^2\sigma_K + \sigma_H - 2x_t\sqrt{\sigma_K\sigma_H}\rho_{KH},$$

$$\hat{\beta}_2 = \left(x_t\mu_K + (1-x_t)r_t - 0.5x_t^2\sigma_K\right) - (\mu_H + 0.5(2-\gamma)\sigma_H) + (1-\gamma)x_t\sqrt{\sigma_K\sigma_H}\rho_{KH},$$

$$\beta_3 = \gamma\,(1-\gamma)^{\frac{-\beta(1-\gamma)}{\beta(1-\gamma)-1}},$$

$$\beta_4 = -\rho + (1-\gamma)(\mu_H - 0.5\gamma\sigma_H).$$

Note that $\hat{\beta}_2$ depends on r_t and x_t. Hence, we plug equation (23) into $\hat{\beta}_2$ and assume a constant $x_t = \bar{x}$. Note that Lemma 1 accounts for the particular case $\bar{x} = 1$, in which the markets clear or the agent does not borrow. Then, the ODE above becomes:

$$0 = 0.5\beta_1 g''(\omega) + \beta_2 g'(\omega) + \beta_3\left(\frac{g'(\omega)}{e^{\omega}}\right)^{\frac{\beta(1-\gamma)}{\beta(1-\gamma)-1}} + \beta_4 g(\omega),$$

where:

$$\beta_1 = (2-\bar{x})\bar{x}\sigma_K + \sigma_H - (1-0.5\bar{x})\bar{x}\sqrt{\sigma_K\sigma_H}\rho_{KH},$$

$$\beta_2 = \mu_K - (1-0.5\bar{x})\sigma_K\bar{x} - (\mu_H + 0.5(2-\gamma)\sigma_H) + (1-\gamma)(2-\bar{x})\bar{x}\sqrt{\sigma_K\sigma_H}\rho_{KH},$$

$$\beta_3 = \gamma\,(1-\gamma)^{\frac{-\beta(1-\gamma)}{\beta(1-\gamma)-1}},$$

$$\beta_4 = -\rho + (1-\gamma)(\mu_H - 0.5\gamma\sigma_H).$$

This ODE must hold under the boundary conditions shown in equations (15) and (16). Besides, a third boundary condition is needed in order to account for the states in which ω becomes very small. I must impose that when the amount of short-term capital K_t becomes very small, then the value function is zero, because $C_t = 0$ when H_t goes to zero by the non-negativity constraint for the illiquid investment I_t. This final boundary condition is equivalent to impose that:

$$\lim_{\omega\to-\infty} g(\omega) = +\infty.$$

Q.E.D.

Proof of Theorem 2

The price at time t of a zero-coupon bond paying one unit at time T, $T > t$ is:

$$B(t,T) = E_t\left[e^{-\int_t^T r_s ds}|\Im_t\right]$$

with $0 \leqslant t \leqslant T$ in the probability space $(\Omega, \Im, \widetilde{P})$. Because this problem is Markov (the processes dK_t and dH_t are Markov processes) and the equilibrium interest rate r_t given by (23) is a function of the factor ω_t, there must be a function $f(t,\omega_t)$ such that $B(t,T) = f(t,\omega_t)$.

The price of the risk-free money-market account or discount factor D_t follows the process $\frac{dD_t}{D_t} = -r_t dt$. Iterated conditioning implies that the discounted bond price $D_tB(t,T)$ is a martingale under the probability measure $\widetilde{P}$. I calculate the differential of $D_tB(t,T)$ by using the Ito′s lemma:

$$\begin{aligned}
d(D_t, B(t,T)) &= d(D_t, f(t,\omega_t)) \\
&= -r_t D_t f(t,\omega_t)dt + D_t df(t,\omega_t) \\
&= D_t \left[-r_t f dt + f_t dt + f_\omega d\omega_t + 0.5 f_{\omega\omega} d[\omega_t,\omega_t]\right] \\
&= D_t \left[-r_t f dt + f_t dt + f_\omega d\omega_t + 0.5 f_{\omega\omega} [\sigma_K dt + \sigma_H dt + \rho_{KH}\sqrt{\sigma_K \sigma_H} dt]\right]
\end{aligned}$$

where ω_t is given by equation (13).

Because $D_t B(t,T)$ is a martingale under the probability measure $\widetilde{P}$, the dt term of $D_t B(t,T)$ must be zero. If I set the dt term equal to zero, and I take into account that $f = B(t,T)$, then I obtain the following PDE:

$$\begin{aligned}
0 = B_t - r_t B + &\left[(\bar{x}\mu_K + (1-\bar{x})r_t - \mu_H) - 0.5(\bar{x}^2\sigma_K - \sigma_H) - \widehat{C}(\omega_t) - \widehat{\Lambda}(\omega_t)\right] B_\omega + \\
&+ 0.5\left[\bar{x}^2\sigma_K + \sigma_H + \bar{x}\rho_{KH}\sqrt{\sigma_K\sigma_H}\right] B_{\omega\omega}
\end{aligned}$$

Q.E.D.

Proof of Theorem 3

First, let us consider a function and, then, let us check that it is actually the solution of the ODE in (17), subject to (15), (16), and (22). Assume that

$$g(\omega_t) = e^{\lambda\omega_t}$$

where $\lambda = -\frac{\beta_2}{\beta_1} + \sqrt{\left(\frac{\beta_2}{\beta_1}\right)^2 - 2\frac{\beta_4}{\beta_1}}$ is the solution of this problem for any $\omega_t < \omega^*$. Note that the first derivative of $g(\omega_t)$ is

$$g'(\omega_t) = \lambda e^{\lambda\omega_t}$$

and its second derivative is

$$g''(\omega_t) = \lambda^2 e^{\lambda\omega_t}.$$

Now, consider the case $\gamma = 0$, in which the equation (17) gets simplified to

$$0 = 0.5\beta_1 g''(\omega) + \beta_2 g'(\omega) + \beta_4 g(\omega)$$

and plug these functions $g(\omega_t)$, $g'(\omega_t)$ and $g''(\omega_t)$ into it. I plug $g(\omega_t)$ into the smooth pasting condition in (15) or the super-contact condition in (16) in order to find ω^*. As a result, $\omega^* = ln(\frac{\lambda}{1-\lambda})$.

Q.E.D.

A.2. Adjustment Costs in Housing Markets

Adjustment costs are the expenses associated with moving to a different house. Quigley (2002) documents six different types of costs for housing purchases: search cost, legal cost, administrative cost, adjustment cost, financial cost and uncertainty cost. There are four type of costs that are quantitatively relevant. First, search costs are the monetary expenses and the value of time spent on looking for a property (buyer) or a buyer for your property (seller). Real estate agent fees may capture part of the search costs. Second, legal costs are fees paid to lawyers or consultants in the preparation of the sales and purchase agreement. Lawyers are typically asked to ensure that there are no liens on the property. Third, administrative costs are mainly registration costs. They are the taxes and fees incurred in the process of registering the property with the competent registry (e.g., registration fees, stamp duties, and notary fees). Four, sales and transfer taxes on the sale and purchase of real estate must be paid to local and national governments. They include deed taxes, transfer Taxes and value-added taxes (VAT). In general, most of the adjustment costs are paid by the buyers.

Empirical studies on the economic size of adjustment costs provide a median range around 10% of the property value. However, the dispersion of adjustment costs across countries is large. The European Monetary Federation (EMF) published the "Study on the Cost of Housing in Europe" in 2005. In the study, transaction costs were found to be the highest in Belgium at 17.1% of the property value and lowest in the UK, at 1.9% of the property value. Belot and Everdeen (2006) estimated housing adjustment costs for 22 OECD countries. They found that adjustment costs are highest in Italy, Belgium, and Portugal with values of 19%, 18%, and 15.5% of the property value, respectively. On the other hand, New Zealand, UK and Australia present adjustment costs of 3.5%, 4.0%, and 4.5% of the property value, respectively. In the US housing markets, Linneman (1986) and Cunningham and Hendershott (1984) find that transaction costs are at least 12% of the house value. Rosenthal (1988) and Corradin, Fillat, and Vergara-Alert (2013) document transaction costs of 7% and 10% of the house value, respectively. Zillow and Global Property Guide report average transaction costs of 10% and 9.82% of the house value in the US, respectively.

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$$dr_t = [\alpha_1 + \alpha_2 r_t + \alpha_3 r_t log(r_t)]\, dt + \left[\alpha_1^K + \alpha_2^K r_t\right]^\theta dW_t^K + \left[\alpha_1^H + \alpha_2^H r_t\right]^\theta dW_t^H$$

	α_1	α_2	α_3	α_1^K	α_2^K	α_1^H	α_2^H	θ
Merton (1973)	x	—	—	x	—	—	—	1.0
Vasicek (1977)	x	x	—	x	—	—	—	1.0
Brennan and Schwartz (1979)	x	x	—	—	x	—	—	1.0
Cox, Ingersoll and Ross (1985b), 1-factor	x	x	—	—	x	—	—	0.5
Cox, Ingersoll and Ross (1985b), 2-factor	x	x	—	—	x	—	x	0.5
Pearson and Sun (1990)	x	x	—	x	x	—	—	0.5
Black, Dearman and Toy (1990)	—	x	x	—	x	—	—	1.0
Affine Term Structure Models, 2-factor	x	x	—	x	x	x	x	0.5
This paper	x	x	—	x	x	x	x	0.5

Table I: Comparison to classic term-structure models. This table compares the functional forms of the processes for the short-term interest rate, dr_t, for the main classic term-structure models in the literature. "**x**" indicates that the coefficient is not zero.

	Mean	Median	Std. dev	Max.	Min.
Real short rate (1 year)	1.38%	1.65%	2.27%	6.59%	-2.98%
Real long rate (10 year)	2.42%	2.17%	2.29%	8.14%	-3.48%
Nominal short rate (1 year)	5.28%	5.20%	3.35%	14.80%	0.12%
Nominal long rate (10 year)	6.32%	6.16%	2.83%	13.92%	1.80%
Slope (10 year - 1 year)	1.04%	1.20%	1.08%	3.12%	-1.22%
Consumption growth	2.85%	2.99%	2.22%	7.04%	-3.11%
Ratio C_t/K_t	0.29	0.29	0.02	0.33	0.25
Ratio H_t/K_t	0.47	0.46	0.03	0.55	0.42
Ratio H_t/K_t growth	0.21%	0.41%	2.61%	6.56%	-7.36%

Table II: Summary statistics. Summary statistics. This table reports the statistics of the main variables used in the paper. Aggregate US data for the period 1962-2016.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
(1) Real short rate (1 year)	1.00								
(2) Real long rate (10 year)	0.89	1.00							
(3) Nominal short rate (1 year)	0.55	0.26	1.00						
(4) Nominal long rate (10 year)	0.57	0.40	0.95	1.00					
(5) Slope (10 year - 1 year)	-0.23	0.25	-0.61	-0.34	1.00				
(6) Consumption growth	0.46	0.46	0.00	0.01	0.01	1.00			
(7) Ratio C_t/K_t	-0.07	0.12	-0.67	-0.65	0.38	0.11	1.00		
(8) Ratio H_t/K_t	-0.15	-0.09	-0.30	-0.31	0.12	-0.14	0.54	1.00	
(9) Ratio H_t/K_t growth	-0.03	0.09	-0.21	-0.15	0.27	0.44	0.23	0.24	1.00

Table III: Correlations among variables. This table reports the correlations among the main variables used in the paper. Aggregate US data for the period 1962-2016.

Description	Symbol	Value	
Time preference	ρ	2.5%	Fixed
Housing flow services	$1-\beta$	0.35	Fixed
Drift of the non-housing process	μ_K	6.3%	Fixed
Std. dev. of the non-housing process	σ_K	10.2%	Estimated
Drift of the housing process	μ_H	2.0%	Estimated
Std. dev. of the housing process	σ_H	59.6%	Estimated
Correlation between non-housing and housing	ρ_{KH}	0.745	Fixed
Loan-to-value ratio	LTV	0.420	Fixed

Table IV: Model parameters. This table displays the parameters estimated using SMM and the first moments of the real short-term rate, the real long-term rate, and the C/K ratio for the US from 1962 to 2016. I estimate the following three parameters: σ_K, μ_H, and σ_H. The rest of the parameters are fixed.

Variable	Moment	Model	Data
Real short rate (1-year)	Mean	1.55%	1.38%
Real long rate (10-year)	Mean	2.58%	2.42%
Ratio C_t/K_t	Mean	0.13	0.29
Real short rate	Std. Dev.	0.07%	2.27%
Real long rate	Std. Dev.	0.24%	2.29%
Ratio C_t/K_t	Std. Dev.	0.07	0.02

Table V: SMM estimation. The first and second moments of the real short-term and long-term interest rates, and the ratio of consumption to housing, C_t/K_t, are obtained using SMM and the real data. In the estimation using SMM, I minimize the distance between the first moment (i.e., the mean) in the model and the first moment in the data for the real short-term and long-term interest rates and the C_t/K_t ratio, and report the squared root of the second moment (i.e., the standard deviation.)

$$log(C_t/K_t) = \alpha_c + \beta_{c,1} \cdot H_t/K_t + \beta_{c,2} \cdot log(H_t/K_t) + X_t + \epsilon_{c,t}$$

	[1] OLS	[2] OLS	[3] IV
α_c	-6,101***	-6,553***	-4,943*
	(2,167)	(2,446)	(2,634)
H/K	4,852**	5,29**	4,497*
	(2,164)	(2,447)	(2,635)
log(H/K)	-5.167**	-5,61**	-5,353*
	(2.415)	(2,718)	(2,923)
Controls, X_t	no	yes	yes
F-test	7.59	2.32	
χ^2-test			20.09

Table VI: Empirical results for consumption. This table reports the coefficients for the empirical analysis of consumption based on OLS and regressions using instrumental variables (IVs). Standard errors are shown in parentheses. ***, **, and * denote statistical significance at the 1%, 5%, and 10% levels, respectively. Controls include real GDP, growth in exports, and growth in imports. Newey-West standard errors are shown in parentheses. The error structure is assumed to be heteroskedastic and possibly autocorrelated up to a lag. US data from 1962 to 2016.

$$r_t = \alpha_r + \beta_r \cdot (H_t/K_t) + X_t + \epsilon_{r,t}$$

	[1] Real OLS	[2] Real OLS	[3] Real IV	[4] Nominal OLS	[5] Nominal OLS	[6] Nominal IV
α_r	0.110***	0.082**	0.066*	0.163***	0.166***	0.148***
	(0.035)	(0.036)	(0.036)	(0.034)	(0.053)	(0.054)
H/K	-0.089***	-0.072**	-0.057*	-0.100***	-0.102**	-0.086*
	(0.033)	(0.031)	(0.033)	(0.034)	(0.048)	(0.049)
Controls, X_t	no	yes	yes	no	yes	yes
F-test	7.24	5.82		8.81	2.28	
χ^2-test			11.93			3.27

Table VII: Empirical results for short-term interest rates. The real short-term rate is the deflated end-of-year one-year Treasury constant maturity rate from the St. Louis Fed's FRED (Federal Reserve Economic Data). Standard errors are shown in parentheses. ***, **, and * denote statistical significance at the 1%, 5%, and 10% levels, respectively. Controls include real GDP, growth in exports, and growth in imports. Newey-West standard errors are shown in parentheses. The error structure is assumed to be heteroskedastic and possibly autocorrelated up to a lag. US data from 1962 to 2016.

$$R_t = \alpha_r + \beta_r \cdot (H_t/K_t) + X_t + \epsilon_{R,t}$$

	[1] Real OLS	[2] Real OLS	[3] Real IV	[4] Nominal OLS	[5] Nominal OLS	[6] Nominal IV
α_r	0.110***	0.089***	0.081**	0.163***	0.089**	0.164***
	(0.027)	(0.044)	(0.039)	(0.023)	(0.038)	(0.045)
H/K	-0.080***	-0.068***	-0.060*	-0.091***	-0.068*	-0.089**
	(0.025)	(0.024)	(0.036)	(0.022)	(0.035)	(0.041)
Controls, X_t	no	yes	yes	no	yes	yes
F-test	10.61	5.91		17.91	4.16	
χ^2-test			7.58			4.71

Table VIII: Empirical results for long-term interest rates. The real long-term rate is the deflated end-of-year 10-year Treasury constant maturity rate from the St. Louis Fed's FRED (Federal Reserve Economic Data). Standard errors are shown in parentheses. ***, **, and * denote statistical significance at the 1%, 5%, and 10% levels, respectively. Controls include real GDP, growth in exports, and growth in imports. Newey-West standard errors are shown in parentheses. The error structure is assumed to be heteroskedastic and possibly autocorrelated up to a lag. US data from 1962 to 2016.

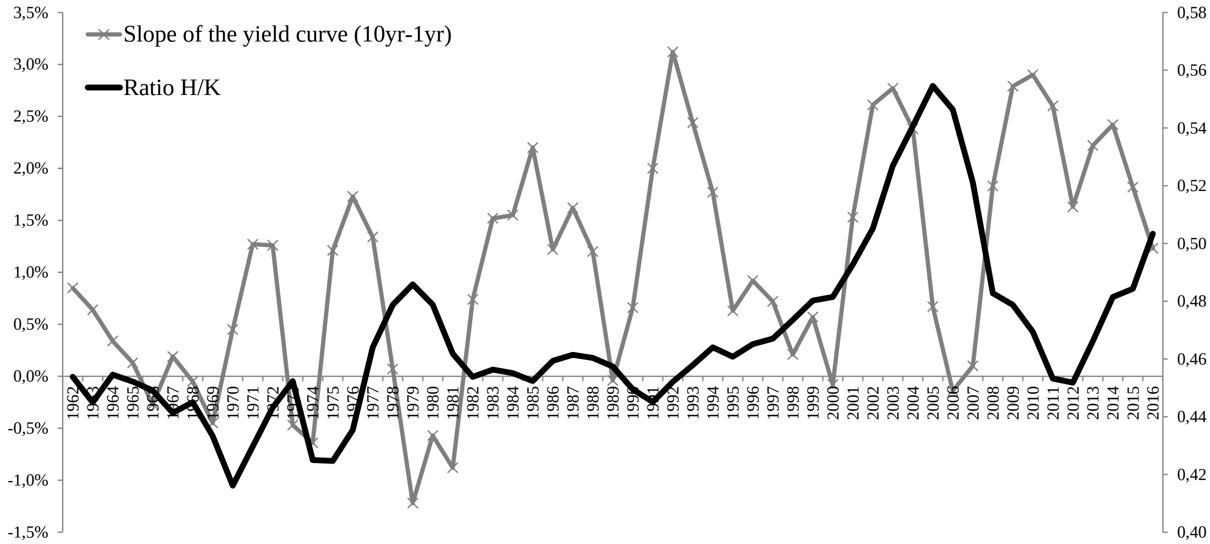


Figure 1: Dynamics of the slope of the term structure of interest rates and the ratio of housing to non-housing assets, H/K, in the US economy.

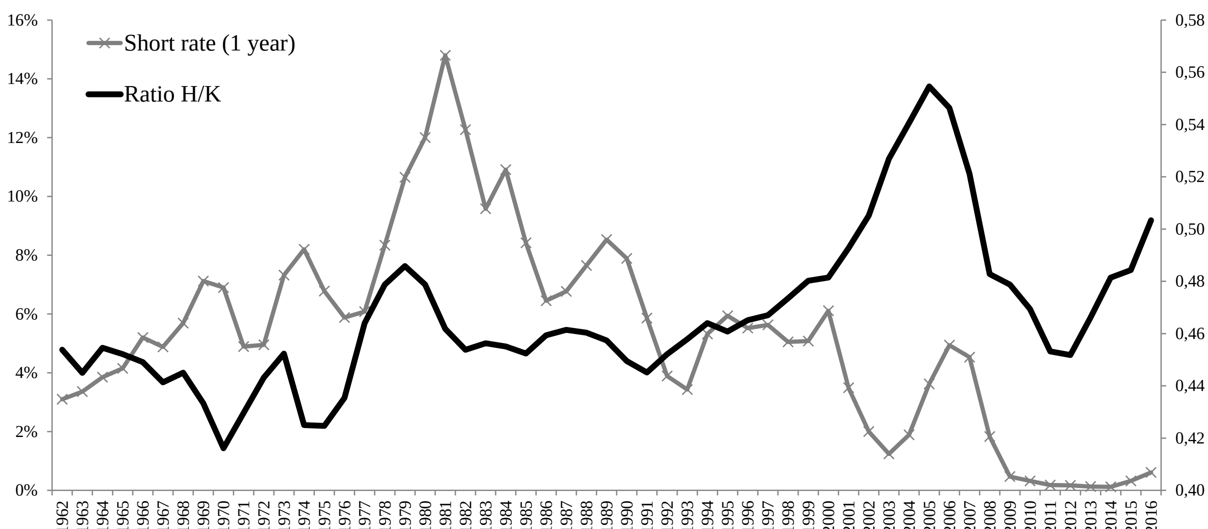
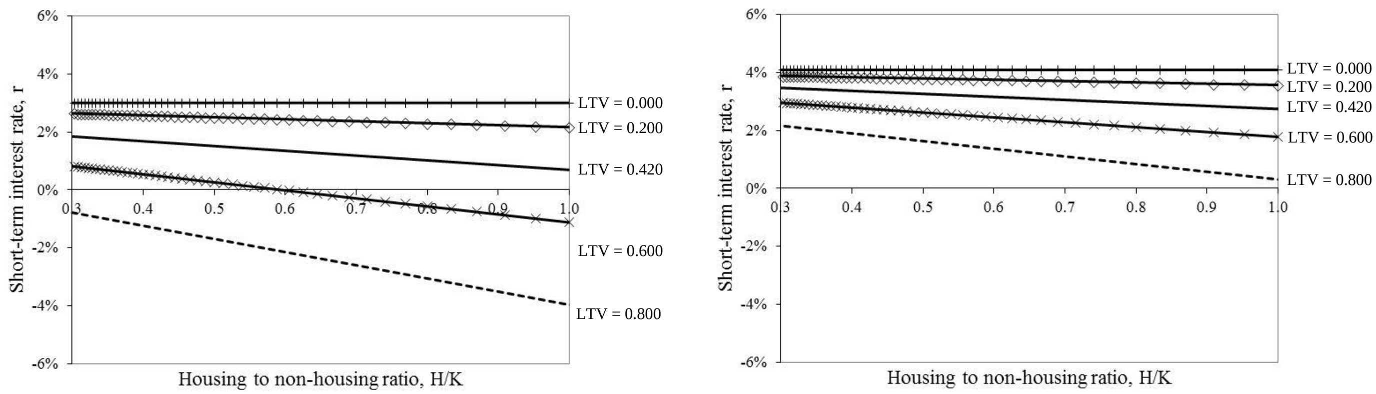
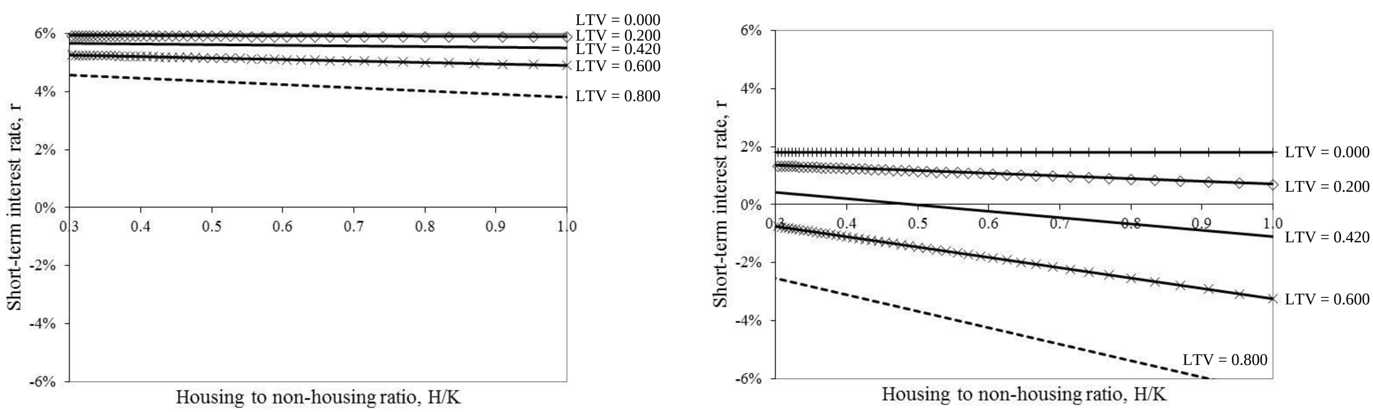


Figure 2: Dynamics of short-term interest rates and the ratio of housing to non-housing assets, H/K, in the US economy.

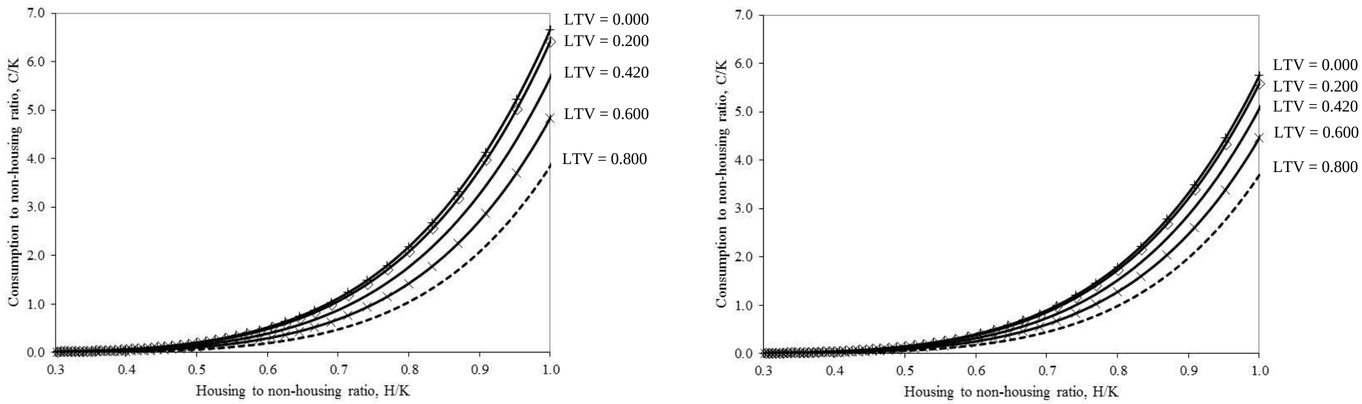


(a) Short interest rate with $\rho_{KH} = 0.745$ (left) and $\rho_{KH} = 0.600$ (right)

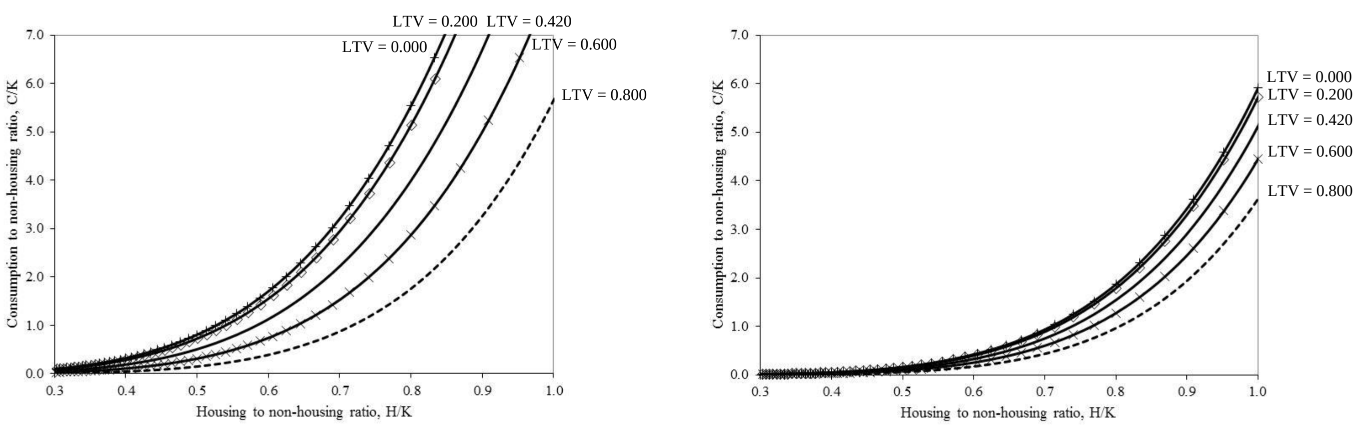


(b) Short interest rate with $\sigma_H = 30\%$ (left) and $\sigma_H = 70\%$ (right)

Figure 3: Sensitivity of short-term interest rates to the LTV ratio, ρ_{KH}, and σ_H. Sensitivity to the LTV ratio and ρ_{KH} (top figures) and sensitivity to the LTV ratio and σ_H (bottom figures). All figures show the short-term interest rate, r_t, versus the ratio of housing to non-housing capital, H_t/K_t. Results are shown for the baseline parameters detailed in Table VI. Note that the model captures the negative relation between r_t and the H_t/K_t ratio.



(a) Consumption with $\rho_{KH} = 0.745$ (left) and $\rho_{KH} = 0.600$ (right)



(b) Consumption with $\sigma_H = 30\%$ (left) and $\sigma_H = 70\%$ (right)

Figure 4: Sensitivity of consumption to the LTV ratio, ρ_{KH}, and σ_H. Sensitivity to the LTV ratio and ρ_{KH} (top figures) and sensitivity to the LTV ratio and σ_H (bottom figures). All figures show the consumption to non-housing ratio, C_t/K_t, versus the ratio of housing to non-housing capital, H_t/K_t. Results are shown for the baseline parameters detailed in Table VI. Note that the model captures the positive relation between C_t/K_t ratio and the H_t/K_t ratio.