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Strategic incentives in dynamic duopoly☆

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Abstract

We compare steady states of open loop and locally stable Markov perfect equilibria (MPE) in a general symmetric differential game duopoly model with costs of adjustment. Strategic incentives at the MPE depend on whether an increase in the state variable of a firm hurts or helps the rival and on whether at the MPE there is intertemporal strategic substitutability or complementarity. A full characterization is provided in the linear-quadratic case. Then with price competition and costly production adjustment, static strategic complementarity turns into intertemporal strategic substitutability and the MPE steady-state outcome is more competitive than static Bertrand competition.

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1. Introduction

Much progress has been made in the study of dynamic interaction among firms, particularly in the study of collusive behavior. Quite a few models of strategic

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noncollusive rivalry have been developed that also consider Markov perfect equilibria as solution concept. However, owing to the inherent difficulty of analyzing fully dynamic models, two-stage models continue to be the workhorse of the analysis.

In this paper we provide a taxonomy of strategic incentives arising in duopolistic interaction over an infinite horizon in the presence of adjustment costs. Adjustment costs are important in quite a few industries as evidenced by several empirical studies at the micro level.[1] Indeed, capacity (or the production run) is costly to adjust in some industries; in others, because of menu costs, it is prices that are difficult to adjust. Adjustment cost dynamics are typically rich in that they depart from the repeated game framework, allow commitment possibilities, and make the steady-state outcomes different from the outcomes of static competition. The presence of adjustment costs, for example, must be taken into account when estimating the degree of product differentiation in a market. This is so because the standard hypothesis of static Bertrand pricing will not hold if either production or prices are costly to adjust and estimation results that take no account of this adjustment will be subject to bias. Dynamics with adjustment costs are also critical for characterizing such macro phenomena as the dynamics of aggregate investment or the effect of monetary policy on price levels and inflation.[2]

The analysis is cast in the context of a differential game of a duopoly market with differentiated products. The presence of adjustment costs will imply that a firm will have incentives to behave strategically (e.g., trying to condition rival's responses) and to depart from the naive optimization of a firm that does not try to influence future market outcomes. Strategic incentives will be characterized by comparing trajectories and steady states of open loop and Markov perfect equilibria (MPE) of the dynamic game. Our aims may be listed as follows.

- Establish an infinite-horizon differential game counterpart of the classification by [21] of strategic incentives in two-stage games.[3]
- Provide a complete characterization of the linear-quadratic case-extending previous work of [38,14], who examined the case of Cournot competition with production adjustment costs.
- Explain the role of adjustment costs in preserving, or reversing, short-run (static) strategic substitutability or complementarity in the intertemporal framework.
- Provide a dynamic equilibrium rationalization of the "Stackelberg warfare point" [46], showing that the outcomes of dynamic interaction need not lie between the

[1] See, for example, [24]. Hall [23] provides recent evidence on adjustment costs. For the empirical implementation of dynamic models with adjustment costs (and evidence of adjustment factors in the rice and coffee export markets) see [27–29]. Slade [44] provides estimates of price adjustment costs in the retail grocery sector.

[2] See, for example, [7,40].

[3] See also [9]. Lapham and Ware [31] provide a taxonomy of the strategic incentives in discrete time dynamic games for small adjustment costs. Benoit and Krishna [4] and Davidson and Deneckere [12] analyze strategic incentives in the choice of capacity followed by collusive pricing supported with repeated competition.

Cournot and Bertrand points when firms compete in prices and production is costly to adjust.

- Examine the comparative statics of steady states as well as comparative dynamics issues (e.g., whether there is increasing or decreasing dominance, or whether there is overshooting of the steady state).

We study first a general (nonlinear) symmetric duopoly model with adjustment costs and compare MPE with open-loop equilibria (OLE). The general model allows for either Cournot or Bertrand competition with production or price adjustment costs. Mixed cases, where the adjustment cost falls on a variable different from the strategic variable of the firm, are also allowed. A firm controls the rate of change of its strategic variable (price or output) at any point in time. The open-loop strategies are those in which the firms commit to a path for the game. Markov strategies depend on the payoff-relevant variables, that is, the state variables. At a MPE strategies are optimal for a firm for any state of the system given the strategy of the rival. Hence an MPE captures the strategic incentives that firms face.

The OLE provide a benchmark against which the strategic incentives at an MPE can be compared. We note first that the steady states of OLE are in one-to-one correspondence with the (interior) static Nash equilibria y^{N} of the duopoly game, provided that adjustment costs are minimized when there is no adjustment. Consider a symmetric and locally stable MPE with steady state y^* and assume that the static Nash equilibrium y^{N} is unique. We show that if an increase in the state variable of firm j hurts firm i, then the sign of $\{y^* - y^{\mathrm{N}}\}$ is positive or negative depending on whether there is intertemporal strategic substitutability or complementarity at y^*—that is, whether (respectively) an increase in the state variable of firm i decreases or increases the action of firm j.

Equipped with these results, we turn to the linear-quadratic model and provide a complete characterization of the stable symmetric linear Markov perfect equilibria (LMPE). Our contribution completes the map of possible strategic interaction in continuous-time dynamic duopoly models with one strategic variable per firm and adjustment costs.

We find that, if production (resp., price) is costly to adjust then there is intertemporal strategic substitutability (resp., complementarity) and that the steady state of the LMPE is more (resp., less) competitive than the static outcome (and this holds irrespective of whether competition is in prices or quantities). The results build on work by Reynolds [38] and Driskill and McCafferty [14] for the Cournot model with production adjustment costs, on a duality result that yields the case of Bertrand competition with price adjustment costs, and the novel analysis of this paper, which studies the "mixed" case of price competition with production adjustment costs (again, by duality, the result for quantity competition with price adjustment costs follows). Our study of the mixed cases pushes the frontier in deriving explicit results in linear-quadratic differential games by allowing the adjustment cost of a firm to depend on the controls of both firms.

The rest of the paper is devoted to the mixed case of price competition with production adjustment costs. It is assumed that firms are committed to supply whatever demand is forthcoming at the set price and that there are no inventories. In other words, we consider the classical Bertrand competition[4] where a firm faces an adjustment cost of changing production to meet the demand that is determined jointly by prices set by the firm and its rival. The production adjustment cost comes typically from the cost of altering the short-run use of capital and labor. We find (a) that the steady-state LMPE outcome with price competition and costly production adjustment is more competitive than static Bertrand competition, and (b) that LMPE price trajectories involve lower prices uniformly than the OLE trajectory. By cutting its price today, a firm will make its rival smaller (and hence less aggressive in the future) because the rival's short-run marginal cost will have increased owing to costly production adjustment. That is, in order to raise the rival's cost a firm must cut prices today. This will push the rival firm toward setting higher prices. When firms face symmetric (or not too asymmetric) production adjustment costs, this incentive to cut prices is self-defeating and the firms become locked into a price war.[5]

Price leadership can be understood as an attempt by one firm to soften the price policy of its rival (or rivals). When firms are symmetric (and thus have symmetric commitment capacities), the leadership attempt by each firm turns into Stackelberg warfare and yields a steady-state outcome that is more competitive than static Bertrand competition. The strategic complementarity of the static price game is transformed into an intertemporal strategic substitutability in the presence of costly production adjustment. As a consequence, the MPE steady state of our dynamic market provides an equilibrium story for the "Stackelberg warfare point", where each firm in a duopoly attempts to be leader in a quantity-setting game.[6]

The plan of the paper is as follows. Section 2 provides a general framework and derives the results on the steady state of OLE and the relationship between steady states of OLE and MPE. Section 3 examines a linear-quadratic specification of the game and provides a complete characterization of strategic incentives, extending the results of the literature with the results obtained in this paper. Section 4 is devoted to studying price competition with production adjustment costs. The concluding remarks of Section 5 close the paper. Most of the proofs are gathered in the Appendix.

[4] This may arise because of an (implicit) contract with customers or because of regulation. For example, common-carrier regulation in utilities (e.g., electricity, gas, and local phone) typically stipulates an obligation to serve, and firms must fulfill all the demand forthcoming at the set price [45].

[5] The basic force is present also when there is a learning curve (with no industry spillovers). Then a decrease in the price charged by a firm raises its output and lowers its rival's output, with the effect of lowering the marginal cost of the firm and increasing the marginal cost of its rival. For a strategic analysis of the learning curve, see [10,11,20]. The dynamic pricing implications of the learning curve are considered in [3], which builds on the work of [16]. Miravete [36] examines a differential game model where learning by doing reduces fixed costs of production.

[6] Stackelberg [46, pp. 194–195] thought that his leader–follower solution "is unstable, for the passive seller can take up the struggle at any time.... It is possible, of course, that the duopolists may attempt to supplant one another in the market so that 'cut-throat' competition breaks out." There are several attempts in the literature to endogenize leadership; see [13,25,32].

2. A general dynamic duopoly framework

Consider a duopoly market in which each firm, $i = 1, 2$, produces a differentiated variety. Time is continuous, indexed by $t \in [0, \infty)$, and firms compete over an infinite horizon, discounting the future at rate r. The flow of revenue (net of production costs) for each firm is given by $R_i(y_1, y_2)$, where (y_1, y_2) are state variables. For example, the variables (y_1, y_2) would be capacities or quantities in a Cournot market and prices in a Bertrand market. The action ("control") of firm j, $u_j(t) \equiv \dot{y}_j(t) \equiv \frac{dy_j}{dt}$, $j = 1, 2$, belongs to a subset of the real line U_j.[7] In a Cournot (Bertrand) market, firm i would control the rate of output (price) change. We make the convention that the state variables in Cournot competition are quantities whereas in Bertrand competition they are prices. Firm i also faces an adjustment cost of the form $F(\dot{z}_i)$, where the variable z_i is a (smooth) function of the state variables (y_1, y_2) which we denote, with some abuse of notation, by $z_i(y_1, y_2)$. The instantaneous flow of profit of firm i is then given by $\pi_i(y_1, y_2, u_1, u_2) = R_i(y_1, y_2) - F(\dot{z}_i)$ where $\dot{z}_i = \frac{\partial z_i}{\partial y_1}(y_1, y_2)u_1 + \frac{\partial z_i}{\partial y_2}(y_1, y_2)u_2$. If the variable that is costly to adjust is the same as the control, as with Cournot (Bertrand) competition with output (price) adjustment costs, then $z_i = y_i$ and $\pi_i = R_i(y_1, y_2) - F(u_i)$. Those situations we term the "pure" cases. We also consider "mixed" cases, where the adjustment cost falls on a variable different from the control of the firm—for example, there is price competition and output is costly to adjust.

The control variables u_i yield the law of motion of the system with initial values $y_i(0) = y_i^0$ for $i = 1, 2$. We assume that the duopoly game is symmetric except possibly for the initial conditions y_i^0, $i = 1, 2 : U_1 = U_2, z_1(y_1, y_2) = z_2(y_2, y_1)$, and $R_1(y_1, y_2) = R_2(y_2, y_1)$. Furthermore, we assume that R_i is smooth and concave in y_i and that F is smooth and convex in $\dot{z}_i$, with $F' > 0$, $F'' > 0$, $F'(0) = 0$ and $F(0) = 0$ for $i = 1, 2$. The first assumption (subject to standard boundary conditions) implies that there is a Nash equilibrium of the static simultaneous-move game in which firm i has payoff R_i and strategy y_i, $i = 1, 2$. The second assumption implies that π_i is concave in u_i and that adjustment costs are minimized when there is no adjustment. We assume further that π_i is concave in y_i. These assumptions are satisfied by the linear-quadratic model, the focus of our analysis in Section 3 and beyond.

Our formulation encompasses both quantity or price competition with quantity or price adjustment costs. Let us see this more explicitly. To ease notation, suppose that production costs are zero. We have Cournot competition for $R_i = P_i(x_1, x_2)x_i$, where $P_i(x_1, x_2)$ is the inverse demand of firm i and $u_i = \dot{x}_i$ is the rate of change of its output (the state variable $y_i = x_i$). We have Bertrand competition for $R_i = p_i D_i(p_1, p_2)$, where $D_i(p_1, p_2)$ is the demand of firm i and $u_i = \dot{p}_i$ is the rate of change of its price (the state variable $y_i = p_i$).[8] Adjustment costs are given by $F(\dot{x}_i)$

[7] A dotted variable represents the time derivative of the variable.

[8] It is assumed that the firms must supply all the demand at the going prices; the products are not storable or the cost of holding inventory is infinite.

when production is costly to change and by $F(\dot{p}_i)$ when price is costly to change.[9] In the pure cases, firm i's adjustment cost depends only on its own rate of adjustment. That is, with quantity competition there are production adjustment costs $F(\dot{x}_i)$, and with price competition there are price adjustment costs $F(\dot{p}_i)$. In the mixed cases there is quantity (price) competition and price (quantity) is costly to adjust. For example, suppose that firms compete in price but that production is costly to adjust. Then for firm i we have $y_i = p_i, u_i = \dot{p}_i$, and adjustment costs given by $F(\dot{x}_i)$. Since $x_i = D_i(p_1, p_2)$ it follows that $F(\dot{x}_i) = F(\frac{\partial D_i}{\partial p_1}\dot{p}_1 + \frac{\partial D_i}{\partial p_2}\dot{p}_2)$. The mixed case with quantity competition and price adjustment costs is analogous.

We will study both the OLE and MPE of this (stationary) differential game. When a firm's strategy is a function only of time, it is called an open-loop strategy. An OLE is an open-loop strategy profile such that each firm's strategy is a best response to the other's choice. The open-loop strategy space for firm i will consist of the piecewise continuous functions of time. In general, firms' strategies can depend on past histories. Markov strategies are those that depend only on payoff-relevant state variables (in our case, the vector (y_1, y_2)). A MPE is a Markov strategy profile such that each strategy is a best response to the others for any state. Hence, a Markov perfect equilibrium is a subgame-perfect equilibrium. We will restrict attention to Markov strategies that are stationary (i.e., time-independent), continuous, and (almost everywhere) differentiable functions of the state variables. With some abuse of notation we will denote them by $u_i(y_1, y_2),\ i = 1, 2$.

At an MPE, firm i chooses $u_i(.)$ to maximize the discounted sum of profits, $\int_0^\infty \pi_i(y_1(t), y_2(t), u_i(t), u_j(y_1(t), y_2(t)))\mathrm{e}^{-rt}\,\mathrm{d}t$, given $u_j(.)$, for any possible initial condition $y_1(0) = y_1^0,\ y_2(0) = y_2^0$, where $\dot{y}_i(t) = u_i(t)$ and $\dot{y}_j(t) = u_j(y_1(t), y_2(t))$. From the necessary conditions for $u_i(y_1, y_2),\ i = 1, 2$ to form an MPE, we easily obtain under our assumptions (see Appendix A) that, at a steady state for $i,\ j = 1, 2$ with $j \neq i,\ r - \frac{\partial u_j}{\partial y_j} \neq 0$ and

$$\frac{\partial R_i}{\partial y_i} + \frac{\dfrac{\partial R_i}{\partial y_j}\dfrac{\partial u_j}{\partial y_i}}{r - \frac{\partial u_j}{\partial y_j}} = 0. \qquad (*)$$

Now, at an OLE, firms do not take into account the effect of changes of the state variables on the strategies; that is, there is no feedback from state variables and $\frac{\partial u_j}{\partial y_i} = 0$ for $i,\ j = 1, 2$. A (interior) static Nash equilibrium is characterized under our assumptions by the first-order conditions $\frac{\partial R_i}{\partial y_i} = 0, i = 1, 2$. It follows that stationary states of OLE are in one-to-one correspondence with interior Nash equilibria of the static duopoly game. The intuition for this result is based on the fact that adjustment costs are minimized when there is no adjustment. At a stationary state, the strategy of rival firm j is not to change its current action. Firm i can make the marginal cost of adjustment arbitrarily small by choosing u_i small enough. It follows that not

[9] See [42] for a model with convex costs of adjusting prices.

changing is a best response only if the net marginal revenue of a change in action (namely, $\frac{\partial R_i}{\partial y_i}$) is equal to 0. This holds only at a static interior Nash equilibrium.

At an MPE there is generally feedback from state variables, and the steady state differs from the stationary OLE or static Nash equilibrium. It is difficult to characterize MPE in differential games. However, it is possible to ascertain the effects of strategic incentives at a locally stable steady state of an MPE, at least in the symmetric version of the model (with symmetric product differentiation and symmetric adjustment costs).

Consider a symmetric MPE $u_i(y_1, y_2)$ (i.e., with $u_1(y_1, y_2) = u_2(y_2, y_1)$) and a symmetric steady state $y_1 = y_2 = y^*$ of the dynamical system $\dot{y}_j = u_j(y_1, y_2),\ j = 1, 2$. We assume that the steady state (y^*, y^*) is a regular point of u, in other words, that the Jacobian of $u = (u_1, u_2)$ at the steady state is nonsingular. In addition we assume that $\frac{\partial u_i}{\partial y_j}(y^*, y^*) \neq 0$ for $j \neq i$ and $i = 1, 2$. If the static game is symmetric and the Nash equilibrium is unique then the equilibrium will be symmetric also. At the symmetric (interior) equilibrium, $\frac{\partial R_i}{\partial y_i}(y_1, y_2) = 0$ for $i = 1, 2$. Now consider the set Y defined by

$$Y = \{(y_1, y_2) \mid y_i \geqslant 0,\ S_i(y_1, y_2) \geqslant 0,\ i = 1, 2\},$$

where $S_i(.) = P_i(.)(= D_i(.))$ in Cournot (Bertrand) competition. We assume that Y is compact. If we regard the function $v(y) = (\frac{\partial R_1}{\partial y_1}(y),\ \frac{\partial R_2}{\partial y_2}(y))$ as a vector field (defined on Y), then the static Nash equilibrium corresponds to the steady state of the dynamical system $\dot{y} = v(y)$. We assume that v points inward at all boundary points of Y.[10] The assumptions are fulfilled in the linear-quadratic model and in regular models with demand choking off at finite prices.[11] We say that a symmetric Nash equilibrium of the (symmetric) static game is regular if it is a regular point of v. Using Eq. (*), strategic incentives at a locally stable MPE can be characterized as follows (see Appendix A for a proof).

Proposition 2.1. *Suppose that there is a unique Nash equilibrium (y^N, y^N) of the static game and that it is regular. Consider a locally stable, regular, and symmetric steady state (y^*, y^*) of a given symmetric MPE of the dynamic game $u_i(y_1, y_2)$, where $\frac{\partial u_i}{\partial y_j}(y^*, y^*) \neq 0$ for $i = 1, 2$. Then* $\text{sign}\{y^* - y^{\text{N}}\} = \text{sign}\{\frac{\partial R_i}{\partial y_j}(y^*, y^*)\frac{\partial u_j}{\partial y_i}(y^*, y^*)\}$.

The proposition extends the taxonomy of strategic behavior due to Fudenberg and Tirole [21] to the differential game duopoly. Strategic incentives to under- or over-invest in a state variable at a locally stable MPE, with respect to the OLE benchmark, depend (a) on whether there is *intertemporal* strategic substitutability ($\frac{\partial u_j}{\partial y_i} < 0$) or complementarity ($\frac{\partial u_j}{\partial y_i} > 0$), and (b) on whether "investment" of a firm in its state variable makes the rival worse off ($\frac{\partial R_i}{\partial y_j} < 0$) or better off ($\frac{\partial R_i}{\partial y_j} > 0$). In the Cournot case we have that $\frac{\partial R_i}{\partial y_j} < 0$; in the Bertrand case, that $\frac{\partial R_i}{\partial y_j} > 0$. In the linear-quadratic

[10] Hence, a steady state (Nash equilibrium) must be in the interior of Y if one exists.
[11] See, for example, [19].

model we will see that intertemporal strategic substitutability (complementarity) obtains when production (price) is costly to adjust.

What determines whether intertemporal strategic complementarity or substitutability prevails? In order to answer this question, we need to characterize MPE.

Let (u_i^*, u_j^*) be a (smooth) MPE and consider the following heuristic characterization. In equilibrium, the value function for firm i, $V_i(y_1, y_2)$, is the present discounted value of profits at the MPE with initial conditions $(y_1(0), y_2(0)) = (y_1, y_2)$ and law of motion $\dot{y}_j = u_j^*$ for $j = 1, 2$. Given u_j^*, the Bellman equation for firm i $(i = 1, 2)$ is given by

$$rV_i(y_1, y_2) = \max_{u_i} H_i\left(\frac{\partial V_i}{\partial y_i}(y_1, y_2), \frac{\partial V_i}{\partial y_j}(y_1, y_2), y_1, y_2, u_i, u_j^*(y_1, y_2)\right). \quad (1)$$

The maximand on the right-hand side is the current Hamiltonian:

$$H_i = \pi_i + \frac{\partial V_i}{\partial y_i} u_i + \frac{\partial V_i}{\partial y_j} u_j^*,$$

where $\pi_i(y_1, y_2, u_1, u_2) = R_i(y_1, y_2) - F(\dot{z}_i)$, $\dot{z}_i = \frac{\partial z_i}{\partial y_1}(y_1, y_2)u_1 + \frac{\partial z_i}{\partial y_2}(y_1, y_2)u_2$ is the instantaneous profit, and $\frac{\partial V_i}{\partial y_k}$ is the shadow value of state variable y_k for firm i. Eq. (1) must hold for any state variable vector (owing to the perfection requirement). Since u_i^* is a maximizer of the current Hamiltonian, the first-order condition

$$\frac{\partial H_i}{\partial u_i} = \frac{\partial \pi_i}{\partial u_i} + \frac{\partial V_i}{\partial y_i} = 0 \quad (**)$$

must hold for $i = 1, 2$ and $j \neq i$.

In the pure cases, $\dot{z}_i = u_i$ and $\frac{\partial \pi_i}{\partial u_i} = -F'(u_i)$. Hence equation $F'(u_i) = \frac{\partial V_i(y_1, y_2)}{\partial y_i}$ holds for all (y_1, y_2). By partially differentiating with respect to y_j, we obtain $F''(u_i)\frac{\partial u_i}{\partial y_j} = \frac{\partial^2 V_i}{\partial y_i \partial y_j}$. Since $F'' > 0$, it follows that sign $\frac{\partial u_i}{\partial y_j}$ = sign $\frac{\partial^2 V_i}{\partial y_i \partial y_j}$. When $\frac{\partial^2 V_j}{\partial y_j \partial y_i} <$ (resp., >0) 0 we can say that contemporaneous strategic substitutability (complementarity) prevails. We thus have that intertemporal strategic substitutability (complementarity) prevails if and only if contemporaneous strategic substitutability (complementarity) does. The mixed cases are not so simple and in fact in the next section we shall see that, for the linear-quadratic model, the result is actually reversed.

3. The linear-quadratic model: overview of results

In the remainder of this paper we examine the linear-quadratic specification of the model. Let (net) revenues for firm i be given by $R_i = (a - by_i + cy_j)y_i$ with $b > |c| \geqslant 0$. Then the unique (and symmetric) Nash equilibrium of the static game is given by $y^{\mathrm{N}} = a/(2b - c)$. Adjustment costs are quadratic: $F(\dot{z}_i) = \lambda(\dot{z}_i)^2/2$ for $\lambda > 0$.[12] If the adjustment costs are borne by the strategic variable of the firm (e.g., production in a

[12] Nothing substantive would change in the analysis by including a linear term in the adjustment.

Cournot model or price in a Bertrand model), then $\dot{z}_i = \dot{y}_i = u_i$. In the mixed case, the adjustment cost borne by firm i depends also on the control of the rival firm j (but in the linear case it is independent of the state variables): $z_i = -bu_i + cu_j$. For example, with price competition $(y_i = p_i, u_i = \dot{p}_i, R_i = (a - bp_i + cp_j)p_i)$ and production adjustment costs, $F(\dot{x}_i) = \lambda(\dot{x}_i)^2/2$ with $\dot{x}_i = -b\dot{p}_i + c\dot{p}_j$. It is worth noting that, for quantity competition $(y_i = x_i, u_i = \dot{x}_i, R_i = (a - bx_i + cx_j)x_i)$, the case of homogenous product and increasing marginal cost of production can be accommodated. Indeed, let $c<0$ and note that $R_i = (a - bx_i + cx_j)x_i = (a + c(x_i + x_j))x_i - (b + c)(x_i)^2$. The slope of marginal cost is $2(b + c)$.

We shall investigate LMPE; namely, equilibria in which the strategies are linear (or affine, to be precise) functions of the state variables.[13] We provide in Proposition 3.1 a complete characterization of strategic incentives in LMPE of the linear-quadratic model. This characterization is based on the following building blocks:

- The study by [14,38,39] of Cournot dynamic duopoly games with homogenous product and production adjustment costs.
- An extension of their results covering the case of Bertrand competition with differentiated products and price adjustment costs.
- The results for mixed cases based on the analysis developed in Section 4 on the study of price competition with production adjustment costs.

We will say that a steady state is "more competitive" when it involves a lower (higher) price (quantity) in Bertrand (Cournot) competition.

Proposition 3.1. *In the linear-quadratic model there is a unique (globally) stable symmetric LMPE. The strategies are given by* $u_i = \alpha + \beta y_i + \gamma y_j$ *for* $i,\ j = 1,2$ *and* $j \neq i$, *where* $\beta<0$ *and* $|\beta|>|\gamma|>0$. *The steady state is symmetric and is given by* $y^* = a/(2b - c(1 - \gamma(\beta - r)^{-1}))$. *When production (price) is costly to adjust,* $\gamma<0$ $(\gamma>0)$ *and* y^* *is more (less) competitive than the static Nash equilibrium* $a/(2b - c)$.

Proof. First of all, given MPE strategies $u_i = \alpha + \beta y_i + \gamma y_j (i,j = 1,2 \text{ and } j \neq i)$ with $\beta<0$ and $\beta^2 - \gamma^2>0$, the steady state is symmetric and is given by $y^* = a/(2b - c(1 - \gamma(\beta - r)^{-1}))$. This follows from $(^*)$ by first setting $\frac{\partial R_i}{\partial y_j}(y, y) = cy, \frac{\partial u_j}{\partial y_i} = \gamma, \frac{\partial u_j}{\partial y_j} = \beta$, and $\frac{\partial R_i}{\partial y_i}(y, y) = a - (2b - c)y$ and then obtaining $a - (2b - c)y + (c\gamma y/(r - \beta)) = 0$. Obviously, the equilibrium parameters β and γ depend on the exogenous parameters of the model (b, c, λ and r; a is a scale parameter and does not affect β or γ). We also have from Proposition 2.1 that $\text{sign}\{y^* - y^{\text{N}}\} = \text{sign}\{c\gamma\}$. Let us consider in turn the cases of (i) Cournot competition and production adjustment costs, (ii) Bertrand competition and price adjustment costs, and finally (iii) the mixed cases.

[13] Equilibria in linear strategies can be rationalized as the limit (as the horizon lengthens) of the strategies used in finite-horizon games. In a linear-quadratic differential finite-horizon dynamic game, the linear solution is unique in the class of strategies that are analytic functions of the state variables [37]. We do not explore potential nonlinear equilibria in our model.

(i) Refs. [14,38,39] have characterized the case of Cournot competition with homogenous product, quadratic production costs ($c<0$), and production adjustment costs. (As we have seen, this is equivalent to our linear-quadratic model.) Those authors show the existence of a unique stable symmetric LMPE.[14] They find that $\gamma<0$; indeed, the steady-state output is larger than the Cournot static output $a/(2b-c)$, according to Proposition 2.1, because $c\gamma>0$.

(ii) Using the duality between price and quantity competition in the duopoly model with product differentiation (see [43]), we can characterize price competition with price adjustment costs. In fact, this case is formally identical to case (i) but with $c>0$. The effect of this is that now, at the unique stable LMPE, the rate of change of prices of each firm is increasing in the price of the rival ($\gamma>0$) and $c\gamma>0$. This makes the steady-state price larger than the Bertrand static price.

In (i), the result $\gamma<0$ follows because the static strategic substitutability of the Cournot model ($\frac{\partial^2 \pi_j}{\partial y_j \partial y_i}<0$) translates into contemporaneous strategic substitutability in the dynamic game (with $\frac{\partial^2 V_j}{\partial y_j \partial y_i}<0$). Similarly, in (ii) the result $\gamma>0$ follows because the static strategic complementarity of the Bertrand model ($\frac{\partial^2 \pi_j}{\partial y_j \partial y_i}>0$) translates into contemporaneous strategic complementarity in the dynamic game (with $\frac{\partial^2 V_j}{\partial y_j \partial y_i}>0$).

(iii) In Section 4 we consider the "mixed" case in which there is price competition ($c>0$) and production is costly to adjust. This case introduces further complexity in the analysis because now the action of a firm affects the adjustment cost of the rival. We show that there is a unique stable LMPE. At this equilibrium the rate of change of price of each firm is decreasing in the price of the rival ($\gamma<0$); see Proposition 4.2. Now the steady-state price is smaller than the Bertrand static price because $c\gamma<0$. A duality argument gives us the results for Cournot competition ($c<0$) with costly price adjustment. Then, at the unique stable LMPE, the rate of change of production of each firm is increasing in the output of the rival ($\gamma>0$) and $c\gamma<0$.[15] □

For the mixed cases we easily obtain, from (∗∗) and using the equation $\dot{z}_i = -bu_i + cu_j$, that $u_i(y_1, y_2) = \frac{1}{\lambda b(b^2-c^2)}(b\frac{\partial V_i}{\partial y_i} + c\frac{\partial V_j}{\partial y_j})$ and $\frac{\partial u_i}{\partial y_j} = \frac{1}{\lambda b(b^2-c^2)}(b\frac{\partial^2 V_i}{\partial y_i \partial y_j} + c\frac{\partial^2 V_j}{(\partial y_j)^2})$. Stability of the equilibrium requires that $-\frac{\partial^2 V_j}{(\partial y_j)^2}>|\frac{\partial^2 V_i}{\partial y_i \partial y_j}|$ (because $-\beta>|\gamma|$). In Section 4 we show that the own effect $c\frac{\partial^2 V_j}{(\partial y_j)^2}$ always dominates the cross effect $b|\frac{\partial^2 V_i}{\partial y_i \partial y_j}|$; hence when $c>0$ (price competition), even though there is contemporaneous

[14] Reynolds [38] needed some restrictions on parameter values in order to prove existence and uniqueness of a stable LMPE. Driskill and McCafferty [14] introduced a graphical apparatus that enabled an analysis without parameter restrictions. We follow a similar method in Section 4.

[15] The duality can be seen from Figs. 1 and 2 in the Appendix (Proof of Proposition 4.2). As the sign of C changes the graphs change symmetrically with respect to the β-axis. The same is true for part (ii).

strategic complementarity $(\frac{\partial^2 V_j}{\partial y_j \partial y_i} > 0)$, intertemporally strategic substitutability prevails and $\frac{\partial u_i}{\partial y_j} < 0$.

We thus see that what determines the competitiveness of a market is whether the (symmetric) adjustment costs are supported by prices or rather by quantities. With price adjustment costs, intertemporal strategic complementarity $(\gamma > 0)$ prevails and this pushes prices up. With production adjustment costs, intertemporal strategic substitutability $(\gamma < 0)$ prevails and this pushes prices down.

Consider the case of production adjustment costs. With Cournot competition, a larger output by firm i today leads the firm to be more aggressive tomorrow. With symmetric adjustment costs, both firms are in the same situation and quantities are pushed beyond the Cournot level. Perhaps more interestingly, as we will see in the next section, with price competition intertemporal strategic substitutability also prevails. A firm must cut its price in order to induce the rival to price softly in the future and since both firms do this they end up with prices that are lower than the static Bertrand level. A cut in prices makes the rival softer in the future because it makes the rival smaller, and thus facing a higher marginal adjustment cost to increase output. The softening effect on the rival happens even though the original price-cutting firm has become more aggressive (because it is larger and hence its marginal production adjustment cost is smaller). The increase in the rival's marginal costs dominates the indirect effect through the decrease in marginal costs of the original firm. In contrast, with price adjustment costs intertemporal strategic complementarity prevails. If there is price competition, then a firm that prices high today will elicit high prices from the rival tomorrow. The cost of adjusting the price lends credibility to this strategy.

Our results for a standard model show that the outcomes of dynamic competition, even when firms condition only on payoff-relevant variables, need not be bounded between the Cournot and Bertrand static long-run outcomes. Indeed, with price competition and production adjustment costs we have seen how the steady-state LMPE price is actually below the static Bertrand price. This result may come as a surprise if one supposes that static strategic complementarity or substitutability will translate, respectively, into intertemporal strategic complementarity or substitutability. This happens only in the "pure" cases of Cournot competition with production adjustment costs and of Bertrand competition with price adjustment costs, as we have just seen. The results in those pure cases have a parallel in the literature. Fershtman and Kamien [17] consider a quantity-setting game with slowly adjusting prices in which the steady-state price is below the Cournot price. In their model, there is intertemporal strategic substitutability (a higher output of firm 1 today leads to lower prices and lower output from firm 2 tomorrow). Similarly, in the alternating-move quantity-setting duopoly game of [33], the (MPE) dynamic reaction functions of the firms are monotone decreasing and there is intertemporal strategic substitutability. In the price competition models with switching costs, Markovian equilibria yield steady-state prices above the one-shot level [2]. The reason is that intertemporal strategic complementarity holds because a higher price today increases the rival's market share and hence makes the rival price more softly tomorrow.[16] Our mixed case provides an example where static strategic complementarity (substitutability) is turned into intertemporal strategic substitutability (complementarity).

An interesting comparative statics result is the following.

Proposition 3.2. *As the adjustment cost λ tends to zero, $y^* - y^N$ does not tend to zero; and as the discount rate r tends to infinity, y^* tends to y^N.*

The comparative statics result of y^* with respect to r is intuitive. The discrepancy between y^* and y^N is governed by $|\gamma|/(r-\beta)$. When the discount rate is low, the future matters more and the strategic incentive increases. Thus it is not surprising that, when r grows unboundedly and the future does not matter, the steady state converges to the static Nash level. The other result (that when λ tends to zero, y^* does not tend to y^N) needs more explanation. One effect is that when λ is low, the strategic incentive (as measured by $|\gamma|$) should be larger. This follows because for low λ it will be less costly for the rival firm to change its action and then firm i has more incentive to change its own state variable to influence the rival's behavior. However, a low λ should also increase the response to the own state variable, $|\beta|$. In fact, as λ tends to zero both $|\beta|$ and $|\gamma|$ tend to infinity whereas $|\gamma|/|\beta|$, and so $|\gamma|/(r-\beta)$, tend to a number between 0 and 1.[17]

Finally, we explore briefly the effect of asymmetric adjustment costs. In the pure cases, with Cournot (Bertrand) competition and production (price) adjustment costs, whether those costs are symmetric or asymmetric does not make a difference for strategic incentives. In the Cournot (Bertrand) case there is intertemporal strategic substitutability (resp. complementarity) independently of whether adjustment costs are symmetric or asymmetric. In both cases, furthermore, if the adjustment cost of firm 2 is very small then the LMPE steady state is close to the Stackelberg outcome with firm 1 as leader. These results show the emergence of the Stackelberg equilibrium (the first with quantity leadership and the second with price leadership) as the steady state of dynamic competition, where the leader is the firm that faces an adjustment cost and hence can commit.[18] However, for the mixed cases the strategic incentives with symmetric or asymmetric adjustment costs may differ. In Section 4.2 it is shown that, with production adjustment costs and price competition, if the adjustments costs are sufficiently asymmetric intertemporal strategic complementarity can be restored.

[16] However, in [34] alternating-move Markov price game with homogenous product of different types of equilibria can be supported because the (equilibrium) dynamic reaction function of a firm is not monotonic. See also [15] for results with product differentiation.

[17] When $\lambda = 0$ there does not exist a LMPE but there does exist a nonlinear MPE yielding y^N (firm i jumps to y^N if its state variable is not at the Nash level and stays put otherwise). There is thus a discontinuity of LMPE as the friction in the market disappears (this has been found also in [14,17,38]).

[18] The result is based on simulations; [26] shows it for the Cournot case and we have done it for the Bertrand case. In both cases, at the steady-state firm 2 will necessarily be very close to its static best response function, since the firm faces almost no adjustment cost (and has almost no commitment power). Firm 1 will optimize accordingly and hence will be close to its Stackelberg level.

4. Price competition with costly production adjustment

The rest of the paper concentrates on the mixed case of the linear-quadratic model with Bertrand competition and production adjustment costs.

Let us set the notation for this model. The instantaneous profit of firm i is given by $\pi_i = p_i D_i(p_1, p_2) - F(\dot{x}_i)$, for $i = 1, 2$ where $D_i(p_1, p_2) = a - bp_i + cp_j$ with $b > c \geqslant 0$ and where $F(\dot{x}_i) = \lambda(\dot{x}_i)^2/2$ for $\lambda > 0$ with $\dot{x}_i = -b\dot{p}_i + c\dot{p}_j$. The state variables are prices (p_1, p_2). We require that the initial state $(p_1(0), p_2(0)) = (p_1^0, p_2^0)$ belong to the region in price space P for which the demand for both firms is nonnegative.[19] In this model firm i controls the rate of change of its price $u_i = \dot{p}_i$, and the output of the firm must adjust to clear the market because the firm must fulfill the forthcoming demand at the set prices. The magnitude of the output adjustment depends also on the rival's rate of price change, $\dot{p}_j$: $\dot{x}_i = -b\dot{p}_i + c\dot{p}_j$. That is, the rate of change of a firm's output is only partially controlled by the firm itself. In any case this output adjustment is costly, $F(\dot{x}_i) = \lambda(\dot{x}_i)^2/2$, and the cost is incurred by firm i.

Demands can be derived from the maximization problem of a representative consumer who has a quadratic and symmetric utility function for the differentiated goods (and utility is linear in money): $U(x_1, x_2) = A(x_1 + x_2) - (B(x_1^2 + x_2^2) + 2Cx_1x_2)/2$. This yields both inverse demands $P_i(x_1, x_2) = A - Bx_i - Cx_j$ and demands $D_i(p_1, p_2) = a - bp_i + cp_j$. Then $a = A/(B + C)$, $b = B/(B^2 - C^2)$, and $c = C/(B^2 - C^2)$ with $B > |C| \geqslant 0$. When $B = C$ the two products are homogeneous from the consumer's view point, when $C = 0$ the products are independent, and when $C < 0$ they are complements.

We deal first with OLE and then with MPE.

4.1. Open-loop equilibria

Let $u_i \equiv \dot{p}_i$. We know from Section 2 that the Bertrand equilibrium $p_i = p^{\mathrm{B}} \equiv a/(2b - c)$, $i = 1, 2$, is the unique stationary state of OLE. We claim now that there is a unique OLE that yields a stable trajectory. The following proposition states the result (the proof is standard and is omitted).[20]

Proposition 4.1. *There is a unique pair of OLE strategies that yield stable price trajectories. These strategies are given by, for* $i = 1, 2$,

$$u_i(t) = ((p_i^0 + p_j^0)/2 - p^{\mathrm{B}})\phi_1 e^{\phi_1 t} + ((p_i^0 - p_j^0)/2)\phi_2 e^{\phi_2 t},$$

where $\phi_1 = \frac{1}{2}\left\{r - \sqrt{r^2 + \frac{4(2b-c)}{\lambda b(b-c)}}\right\}$ *and* $\phi_2 = \frac{1}{2}\left\{r - \sqrt{r^2 + \frac{4(2b+c)}{\lambda b(b+c)}}\right\}$. *We have that* $\phi_1 < \phi_2 < 0$. *Whenever* (p_1^0, p_2^0) *is in* P, *it follows that* $(p_1(t), p_2(t))$ *is also in* P *for all* t.

[19] The region P is just region Y (as defined in Section 2, when $S_i(.) = D_i(.)$) and is given by the intersection of a cone-shaped region with vertex $(a/(b-c), a/(b-c))$ and the nonnegative orthant: $a - bp_i + cp_j \geqslant 0$ with $i \neq j$, $i = 1, 2$, and $p_i \geqslant 0$.

[20] A similar result can be derived when adjustment costs are asymmetric. See [18] for related results on stability of the OLE of Cournot-type models.

4.2. Markov perfect equilibria

Markov strategies depend only on payoff-relevant variables, the level of prices in our case. We restrict attention to strategies that are stationary (i.e., time-independent), continuous, and (almost everywhere) differentiable functions $u_i(p_1, p_2),\ i = 1, 2$, of the prices.

Let (u_i^*, u_j^*) be an MPE and consider the associated value function for firm i, $V_i(p_1, p_2)$. The first-order condition (✻✻) from the Bellman equation in Section 2 is given for $i = 1, 2$ and $j \neq i$ by

$$\frac{\partial \pi_i}{\partial u_i} + \frac{\partial V_i}{\partial y_i} = \lambda b(-bu_i^* + cu_j^*) + \frac{\partial V_i}{\partial p_i} = 0.$$

This is immediate from $\dot{x}_i = -b\dot{p}_i + c\dot{p}_j = -bu_i + cu_j$ and $\pi_i = p_i D_i(p_1, p_2) - \lambda(-bu_i + cu_j)^2/2$. This first-order condition defines i's instantaneous best response (and is also sufficient for a maximum given the concavity of the objective function with respect to u_i). We can therefore derive the equilibrium of the instantaneous game given p_1 and p_2, $i = 1, 2$:

$$u_i^*(p_1, p_2) = \frac{1}{\lambda b(b^2 - c^2)}\left(b\frac{\partial V_i}{\partial p_i}(p_1, p_2) + c\frac{\partial V_j}{\partial p_j}(p_1, p_2)\right). \tag{2}$$

We will characterize stable LMPE by following a standard approach and finding a quadratic value function for the optimization problem that firms face.[21] The key steps of the characterization are as follows.

(i) Posit a quadratic value function for firm i (and a symmetric function for firm j):

$$V_i(p_1, p_2) = z + vp_i + wp_j + \frac{m}{2}p_i^2 + np_ip_j + \frac{s}{2}p_j^2. \tag{3}$$

(ii) Obtain a system of partial differential equations substituting the instantaneous equilibrium u_i^* and u_j^* in the necessary conditions for equilibrium (1):

$$\begin{aligned} rV_i = {} & p_iD_i - \frac{1}{2\lambda b^2}\left(\frac{\partial V_i}{\partial p_i}\right)^2 + \frac{1}{\lambda b(b^2 - c^2)}\left(b\frac{\partial V_i}{\partial p_i} + c\frac{\partial V_j}{\partial p_j}\right)\frac{\partial V_i}{\partial p_i} \\ & + \frac{1}{\lambda b(b^2 - c^2)}\left(b\frac{\partial V_j}{\partial p_j} + c\frac{\partial V_i}{\partial p_i}\right)\frac{\partial V_i}{\partial p_j}. \end{aligned} \tag{4}$$

Both the left- and the right-hand sides are functions of (p_1, p_2). If V_i is a quadratic function, then the right-hand side of (4) will also be a quadratic function. Furthermore, the equation must hold for any pair (p_1, p_2) according to the definition of MPE.

(iii) Find the coefficients of the value function by taking derivatives of V_i and V_j with respect to p_i and p_j, substituting the result in the right-hand side of (4), and comparing the coefficients using (3). Candidate LMPE strategies follow from (2). If

[21] For examples of computation and characterization of LMPE see [14,22, Chapter 13; 38], [47].

we write

$$u_i^* = \alpha + \beta p_i + \gamma p$$

then we obtain $\alpha = v/(\lambda b(b-c))$, $\beta = (bm+cn)/(\lambda b(b^2-c^2))$, and $\gamma = (cm+bn)/(\lambda b(b^2-c^2))$. In terms of the utility parameters A, B, C and normalizing so that $B=1$, we have that $\alpha = \lambda^{-1}(1-C^2)(1+C)v, \beta = \lambda^{-1}(1-C^2)(m+Cn)$, and $\gamma = \lambda^{-1}(1-C^2)(Cm+n)$. The exercise, after tedious computations summarized in Appendix B.1 yields *six solutions in closed form* (listed in Table 2) that correspond to the six candidate value functions. It can be checked that v, and therefore α, is linear in A.

(iv) Identify a unique stable solution. This necessarily is a MPE because the stable solution to the partial differential equation system (4) fulfills the transversality condition. In Appendix B.2 we show (using a method similar to [14]) that only one of the solutions generates a stable stationary state. The unique LMPE that generates a stable solution has $\alpha > 0$ and $\beta < \gamma < 0$.

(v) Check that when the initial state belongs to P, the LMPE path stays in P. The LMPE path can be obtained by solving the differential equations $u_i^* = \alpha + \beta p_i + \gamma p_j$ for $i, j = 1, 2$ and $i \neq j$. It is given by

$$p_i^{\mathrm{MP}}(t) = p^* + \left(\frac{p_i^0 + p_j^0}{2} - p^*\right)\mathrm{e}^{(\beta+\gamma)t} + \left(\frac{p_i^0 - p_j^0}{2}\right)\mathrm{e}^{(\beta-\gamma)t}, \tag{5}$$

where p^* is the stationary state of the stable LMPE path, $p^* = -\alpha/(\beta+\gamma) = a/(2b - c(1-\gamma(\beta-r)^{-1}))$, the last equality following from Proposition 3.1. That $(p_1(t), p_2(t))$ belongs to P can be shown as follows: $(p_1(t), p_2(t))$ is contained in the rectangle formed by the lines $p_1(t) + p_2(t) = p_1^0 + p_2^0, p_1(t) + p_2(t) = 2p^*, p_1(t) - p_2(t) = p_1^0 - p_2^0$, and $p_1(t) - p_2(t) = 0$. Since the rectangular region lies inside P when the initial state belongs to P, it follows that $(p_1(t), p_2(t))$ belongs to P whenever (p_1^0, p_2^0) belongs to P.

Proposition 4.2 summarizes the characterization of the stable LMPE. A stable LMPE generates a stable stationary state starting from any initial condition in P. The proof is provided in Appendix B.

Proposition 4.2. *There exists a unique symmetric stable LMPE*:

$$u_i^* = \alpha + \beta p_i + \gamma p_j$$

with $\beta < \gamma < 0$ *and* $\alpha > 0$. *It corresponds either to solution* #3 *or solution* #5 *in Table* 2. *The steady state is symmetric, with prices equal to* $p^* = a/(2b - c(1-\gamma(\beta-r)^{-1})) < p^{\mathrm{B}} = a/(2b-c)$.[22]

Remark. Table 2 in Appendix B provides closed-form expressions for the LMPE parameters α, β and γ as functions of the underlying parameters of the model. The

[22] In terms of the utility parameters we have $p^* = A(B-C)/(2B - C(1-\gamma(\beta-r)^{-1})) < p^{\mathrm{B}} = A(B-C)/(2B-C)$.

stable LMPE switches between "labels" of the solution depending on parameters (see Result B.1 in Appendix B).

Remark. The described LMPE remains an equilibrium when the law of motion of the system is subject to additive shocks with mean zero. Suppose, for example, that the demand intercept is time-dependent according to a shock that follows a Brownian motion, $a(t) = a + \sigma w(t)$, where $\mathrm{d}w(t)$ is normally distributed with mean zero and variance $\mathrm{d}t$. If we consider outputs as the state variables, then the law of motion is given by $\mathrm{d}x_i(t) = (-bu_i(t) + cu_j(t))\,\mathrm{d}t + \sigma\mathrm{d}w(t),\ i = 1,2,$ where $\sigma > 0$. Under these conditions our deterministic LMPE is also an equilibrium for the stochastic game (and is independent of σ).[23]

Remark. It is possible to compare the OLE and the LMPE price paths using numerical simulations. We find that prices at the stable LMPE trajectory are (strictly) lower than prices at the OLE trajectory for all initial states that belong to P. We have verified the property in the fine parameter grid considered, that is, in a parameter set that covers an extensive range and has a fine grid.[24]

Firms are more aggressive at the LMPE than at the OLE owing to a strategic incentive. Firm i would like rival firm j to price softly, but the rate of price change u_j depends negatively on p_i (since, in equilibrium, $\gamma < 0$) and hence there is an incentive to cut prices. This happens to both firms, which become trapped in trying to elicit soft behavior from the rival by cutting prices. To reiterate: if firm i cuts prices today then its rival (j) is made softer tomorrow because j is smaller, which raises j's (short-run) marginal cost of increasing output (because of the production adjustment cost). This happens even though firm i has become more aggressive, because the firm is larger and therefore its (short-run) marginal cost is lower. The first effect (the increase in firm j's marginal costs) dominates the indirect effect through the decrease in marginal costs of firm i.

If the initial state is the static Bertrand equilibrium then both firms have an incentive to decrease prices. This did not happen at the OLE because in that case j would not react to i's price cut over an interval (and then such a move would not have a first-order effect on profit). At the LMPE, a price cut represents a commitment because of costly production adjustment. Once a firm has deviated from the Bertrand equilibrium, it is costly to recover that equilibrium. A firm then has the incentive to cut prices to make the rival softer tomorrow, transforming the strategic complementarity of static price competition into intertemporal strategic substitutability. Given that both firms have symmetric commitment capacities (adjustment costs), this leadership attempt is self-defeating and the outcome is price

[23] See, for example, [1, Section 6.5].

[24] The parameter range and grid considered are the following: $B = 1$, C is between 0.001 and 1, and λr^2 is between 0 and 1000 with step size 0.1 for C and 10 for λr^2. See Appendix C for details of the numerical simulation results.

warfare. In this sense the steady state represents the "Stackelberg warfare point" in which the leadership attempt of both firms turns into highly aggressive behavior.[25]

4.3. Comparative dynamics

4.3.1. Comparative statics of the steady state

A number of comparative statics and comparative dynamics results can be obtained. Recall that $p^* < p^{\mathrm{B}}$. The following proposition is proved in Appendix C.

Proposition 4.3. *As $\psi \equiv \lambda r^2$ tends to infinity, $p^{\mathrm{B}} - p^*$ tends to zero; and as ψ tends to zero, $p^{\mathrm{B}} - p^*$ tends to a strictly positive number.*

If the adjustment cost is large (or if the future does not matter much), the strategic incentive of a firm is small because it is costly for the rival to change prices. We have in particular that the more costly it is to adjust output, the closer we are in steady state to the Bertrand equilibrium.[26] The explanation for the result obtained when ψ tends to zero was provided in Section 3.

Remark. Using numerical methods, we can check that the ratio $(p^{\mathrm{B}} - p^*)/p^{\mathrm{B}}$ is decreasing in $\psi \equiv \lambda r^2$ and increasing in the degree C of product substitutability. We have verified the property in a parameter set that covers an extensive range and has a fine grid.[27]

Products that are closer substitutes yield larger (relative) strategic incentives, because a cut in prices is more effective in inducing softer behavior of the rival (as C increases, $|\gamma|/|\beta|$ also increases). The ratio $(p^{\mathrm{B}} - p^*)/p^{\mathrm{B}}$ can be as large as 35% when ψ is close to zero and C is close to unity. Table 1 gives $(p^{\mathrm{B}} - p^*)/p^{\mathrm{B}}$ for different values of C and λr^2. For example, even with high adjustment costs ($\lambda = 100$) and

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[25] With sufficiently asymmetric commitment capacities intertemporal strategic complementarity may be preserved and soft pricing induced. Consider a two-period version $(t = 0, 1)$ of the game with production adjustment costs equal to $F_i(x_i^t - x_i^{t-1}) = \lambda_i(x_i^t - x_i^{t-1})^2/2$ for $\lambda_i \geqslant 0$. Suppose that $\lambda_1 > 0$ and $\lambda_2 = 0$. At the last period (period 1), firm 2 will price according to its static Bertrand best reply function, since neither firm can manipulate the costs of firm 2. However, an increase in the price of firm 1 in period 0 induces a decrease in its output and therefore an increase in its marginal cost in period 1. This moves the best response function of firm 1 outward in period 1. The outcome is higher prices for both firms, giving a strategic incentive for firm 1 to raise its price in period 0. The described incentives will be the same whenever λ_2 is close to zero, in which case the period-1 best reply of firm 2 will also be affected by changing prices in period 0.

[26] This seems to provide a counterpoint to the idea that "quantity precommitment and price competition yields Cournot outcomes" [30] since the source of the precommitment value of quantity is that quantity is more costly to adjust than price. However, in our model firms have only one strategic variable (the rate of price change) whereas firms in [30] make both quantity/capacity and price choices.

[27] The parameter range and grid considered are the following: $B = 1$, C is between 0.001 and 1, and λr^2 is between 0 and 1000 with step size 0.1 for C and 10 for λr^2.

Table 1
Values of $(p^{\mathrm{B}} - p^{*})/p^{\mathrm{B}}$ in percentage

	$C = 0.1$	$C = 0.3$	$C = 0.5$	$C = 0.7$	$C = 0.9$	$C = 0.95$
$\Psi = 0$	0.13	1.38	4.73	12.11	28.42	35.58
$\Psi = 0.01$	0.13	1.33	4.55	11.65	27.33	34.21
$\Psi = 0.1$	0.12	1.23	4.18	10.65	24.81	30.89
$\Psi = 1$	0.09	0.91	3.07	7.63	16.98	20.6
$\Psi = 10$	0.03	0.34	1.10	2.58	5.26	6.21

$r = 10\%$ (yielding $\psi = 1$), we have that $(p^{\mathrm{B}} - p^{*})/p^{\mathrm{B}} \approx 21\%$ if $C = 0.95$; if $C = 0.7$, the ratio drops to about 7.6%.[28]

4.3.2. Price dynamics

Price changes at the OLE (rewriting the strategies in terms of state variables[29]) are given by

$$\begin{aligned} u_i(t) &= \phi_1(((p_1(t) + p_2(t))/2) - p^{\mathrm{B}}) + \phi_2((p_i(t) - p_j(t))/2) \\ &= \frac{\phi_1 + \phi_2}{2}(p_i(t) - p^{\mathrm{B}}) + \frac{\phi_1 - \phi_2}{2}(p_j(t) - p^{\mathrm{B}}) \end{aligned}$$

and at the LMPE by

$$\begin{aligned} u_i^*(t) &= \beta(p_i(t) - p^*) + \gamma(p_j(t) - p^*) \\ &= (\beta + \gamma)(((p_1(t) + p_2(t))/2) - p^*) + (\beta - \gamma)((p_i(t) - p_j(t))/2). \end{aligned}$$

The OLE and LMPE trajectories have the following four properties:

(i) A higher adjustment cost or discount rate slows down convergence to the steady state. Indeed, a price change toward the steady state today increases adjustment costs today but decreases them in the future and, when r increases, the future is discounted more ($-\phi_1$ and $-\phi_2$ as well as $-(\beta + \gamma)$ and $-(\beta - \gamma)$ decrease with λ and r).[30]

(ii) There is *decreasing dominance*. Starting from an asymmetric initial position, the system converges to the symmetric steady state. This happens because the larger firm is softer in pricing: $u_i - u_j = \phi_2(p_i - p_j)$ and $u_i^* - u_j^* = (\beta - \gamma)(p_i - p_j)$ are positive when $p_i - p_j < 0$.[31]

[28] Note that, a fortiori, p^* is decreasing in C (equaling the monopoly price $A/2B$ when products are independent, $C = 0$, and the competitive price 0 when products are perfect substitutes, $C = B$).

[29] However, this yields the open-loop price changes only along the equilibrium price trajectories.

[30] The results for the LMPE equilibrium parameters are checked numerically in the following parameter range: $B = 1$ and C is between 0.02 and 0.99 for the graphs of $-\sqrt{\lambda}(\beta + \gamma)$ and $-\sqrt{\lambda}(\beta - \gamma)$ as functions of λr^2 (with domain ranging from 0 to 10,000).

[31] Cabral and Riordan [10] find conditions for increasing dominance in a learning-by-doing model. See also [8,41].

(iii) Convergence to the steady state is slower in the LMPE than in the OLE case. Indeed, we have that $-(\beta+\gamma) < -\phi_1$ and $-(\beta-\gamma) < -\phi_2$.[32]

(iv) There are trajectories for which there is *overshooting* with respect to the steady-state prices. The initially larger firm is the one that overshoots the steady-state level. For example, this happens for firm 2 both at the OLE and the LMPE when: $A = B = 1$, $C = 0.5$, $r = 0.05$, $\lambda = 0.1$, $p_1(0) = 0.7$, and $p_2(0) = 0.4$. We have then $p^{\mathrm{B}} = 1/3$ and $p^* = 0.3177$. Firm 2 features a price that starts higher than the steady-state value, decreases below it after a while (beneath 0.314), and then increases again toward the steady state level.[33]

5. Concluding remarks

We have shown that what drives the competitiveness of a market in relation to the static benchmark is whether production or prices are costly to adjust, not the character of competition (Cournot or Bertrand). Indeed, when output (price) is costly to adjust, the MPE steady state is more (less) competitive than the static Nash equilibrium. In particular, the static strategic complementarity characterizing price competition is turned into intertemporal strategic substitutability whenever firms face similar adjustment production costs. The outcome is fierce competition and a steady state that is below the static Bertrand benchmark.

The consideration of adjustment costs has implications for empirical work. The importance of taking into account the dynamic structure of the market when estimating product differentiation models cannot be underestimated. For example, in the work of [5,6] it is assumed that firms compete according to a static Bertrand model. From this assumption sophisticated estimates of patterns of elasticities and cross-elasticities of substitution among products are derived, building on discrete choice theory. An obvious problem is that if a dynamic structure exists in the industry then there will be biases estimating the degree of product differentiation. For example, if the true model of an industry corresponds to our case of price competition with production adjustment costs, then the estimates based on static Bertrand competition would systematically overstate the degree of substitutability of the products. The lesson to draw is that, even when the modeler is reasonably certain that industry collusion is not an issue, neglecting the dynamic structure is dangerous because it can lead to biases in estimation.

Appendix A

Let $H_i = R_i(y_1, y_2) - F(\dot{z}_i) + \mu_{ii}u_i + \mu_{ij}u_j$ be the (current-value) Hamiltonian of firm i, where $\mu_i = (\mu_{ii}, \mu_{ij})$ is the vector of costate variables associated to the

[32] See Appendix C.

[33] This contrasts with "switching costs" models, where convergence is monotone [2].

maximization problem of firm i at an MPE where $\dot{z}_i = \frac{\partial z_i}{\partial y_1}(y_1, y_2)u_1 + \frac{\partial z_i}{\partial y_2}(y_1, y_2)u_2$. The following are necessary conditions ($i = 1, 2$) for (u_1, u_2) to form an MPE pair:

$$\frac{\partial H_i}{\partial u_i} = -\frac{\partial F}{\partial u_i} + \mu_{ii} = 0, \tag{A.1}$$

$$\dot{\mu}_{ii} = r\mu_{ii} - \frac{\partial H_i}{\partial y_i} - \frac{\partial H_i}{\partial u_j}\frac{\partial u_j}{\partial y_i} = r\mu_{ii} - \frac{\partial R_i}{\partial y_i} + \frac{\partial F}{\partial y_i} + \left(\frac{\partial F}{\partial u_j} - \mu_{ij}\right)\frac{\partial u_j}{\partial y_i},$$
$$\dot{\mu}_{ij} = r\mu_{ij} - \frac{\partial H_i}{\partial y_j} - \frac{\partial H_i}{\partial u_j}\frac{\partial u_j}{\partial y_j} = r\mu_{ij} - \frac{\partial R_i}{\partial y_j} + \frac{\partial F}{\partial y_j} + \left(\frac{\partial F}{\partial u_j} - \mu_{ij}\right)\frac{\partial u_j}{\partial y_j}. \tag{A.2}$$

At a steady state we have that $u_i = \dot{\mu}_{ij} = 0$ for $i,j = 1, 2$ and so $\dot{z}_i = 0$ and $\mu_{ii} = 0$ (because $F'(0) = 0$ and $\frac{\partial F}{\partial u_j} = -F'(0)\frac{\partial z_i}{\partial y_j}(y_1, y_2) = 0$ for $i,j = 1, 2$). We have also that $\frac{\partial F}{\partial y_j} = F'(\dot{z}_i)(\frac{\partial^2 z_i}{\partial y_1 \partial y_j}u_1 + \frac{\partial^2 z_i}{\partial y_2 \partial y_j}u_2) = 0$ $(i,j = 1, 2)$. Hence, at a steady state (A.2) may be simplified to

$$\frac{\partial R_i}{\partial y_i} + \mu_{ij}\frac{\partial u_j}{\partial y_i} = 0,$$
$$r\mu_{ij} - \frac{\partial R_i}{\partial y_j} - \mu_{ij}\frac{\partial u_j}{\partial y_j} = 0. \tag{A.2$'$}$$

Note that $r - \frac{\partial u_j}{\partial y_j} = 0$ would be inconsistent with (A.2′) at the steady state whenever $\frac{\partial R_i}{\partial y_j} \neq 0$. Solving for μ_{ij} in (A.2′), we obtain

$$\frac{\partial R_i}{\partial y_i} + \frac{\frac{\partial R_i}{\partial y_j}\frac{\partial u_j}{\partial y_i}}{r - \frac{\partial u_j}{\partial y_j}} = 0 \tag{$*$}$$

for $i,\ j = 1, 2$ and $j \neq i$.

Proof of Proposition 2.1. Let $\gamma \equiv \frac{\partial u_i}{\partial y_j}(y^*, y^*)$ and $\beta \equiv \frac{\partial u_i}{\partial y_i}(y^*, y^*)$. If the steady state (y^*, y^*) is regular, then $\det\begin{pmatrix} \beta & \gamma \\ \gamma & \beta \end{pmatrix} \neq 0$. This ensures, in particular, that (y^*, y^*) is an isolated rest point of the dynamical system. It follows then from Poincaré's linearization result that if (y^*, y^*) is locally stable, then necessarily the matrix $\begin{pmatrix} \beta & \gamma \\ \gamma & \beta \end{pmatrix}$ has negative trace $2\beta < 0$ and positive determinant $\beta^2 - \gamma^2 > 0$.[34] Rewriting the

[34] Indeed, if the nonlinear dynamical system $\dot{y}_i = u_i(y_1, y_2),\ i = 1, 2$, has a locally stable steady state (y^*, y^*) then the trace (determinant) of the matrix of the linearized system is negative (positive) unless the roots of the matrix are pure imaginary or the roots are real and equal (see, e.g. [48, pp. 405–411]). In our case the roots of $\begin{pmatrix} \beta & \gamma \\ \gamma & \beta \end{pmatrix}$ are real and different: $\beta + \gamma$ and $\beta - \gamma$, with $\gamma \neq 0$. (Note also that the stated conditions on the matrix are necessary and sufficient for a linear system to be stable.) This means that we have two possible situations: either $\beta < \gamma < 0$ or $|\beta| > \gamma > 0$.

necessary conditions (*) for the steady state of an MPE we obtain

$$\frac{\partial R_i}{\partial y_i}(y^*, y^*) = -\frac{\frac{\partial R_i}{\partial y_j}(y^*, y^*)\gamma}{(r-\beta)}.$$

We show that $\text{sign}\{y^* - y^{\text{N}}\} = \text{sign}\{\frac{\partial R_i}{\partial y_j}\gamma\}$. Since there is a unique interior and regular equilibrium that is symmetric, it is necessary (with a direct application of the Poincaré–Hopf index theorem for the vector field $-v$ defined on Y; see [35, p. 35]) that $(\frac{\partial^2 R_i}{(\partial y_i)^2})^2 - (\frac{\partial^2 R_i}{\partial y_j \partial y_i})^2 > 0$ at the equilibrium. Because $\frac{\partial^2 R_i}{(\partial y_i)^2} < 0$ this means that $\frac{\partial^2 R_i}{(\partial y_i)^2} + \frac{\partial^2 R_i}{\partial y_j \partial y_i} < 0$ at the equilibrium. Hence, if we define $\phi(z) \equiv \frac{\partial R_i}{\partial y_i}(z, z)$, then $\phi'(y^{\text{N}}) = \frac{\partial^2 R_i}{(\partial y_i)^2}(y^{\text{N}}, y^{\text{N}}) + \frac{\partial^2 R_i}{\partial y_j \partial y_i}(y^{\text{N}}, y^{\text{N}}) < 0$. Since y^{N} is the unique solution to the equation $\phi(z) = 0$, we have $\text{sign}\{y^* - y^{\text{N}}\} = \text{sign}\{-\frac{\partial R_i}{\partial y_i}(y^*, y^*)\}$. From this and the fact that $\beta < 0$ and so $r - \beta > 0$, the conclusion follows. □

Appendix B

B.1. Derivation of the LMPE

By substituting for the derivatives in (4) using (3) and then comparing the coefficients (in terms of $A, B = 1$, and C), we obtain the following equations:

$$0 = 2\lambda + \lambda r(1 - C^2)m - (1 - C^2)^2\{(1 + C^2)m^2 + 4Cmn + 2n^2\}, \tag{B.1}$$

$$0 = \lambda C - \lambda r(1 - C^2)n + 1(1 - C^2)^2 \times \{(Cm + n)s + Cm^2 + (2 + C^2)mn + 2Cn^2\}, \tag{B.2}$$

$$0 = \lambda rs - (1 - C^2)\{2(m + Cn)s + 2Cmn + (1 + C^2)n^2\}, \tag{B.3}$$

$$0 = \lambda A - (1 + C)(\lambda rv - (1 - C^2) \times \{(1 + C + C^2)mv + (1 + 2C)nv + (Cm + n)w\}), \tag{B.4}$$

$$0 = \lambda rw - (1 - C^2)\{Cmv + (1 + C + C^2)nv + (1 + C)sv + (m + Cn)w\}, \tag{B.5}$$

$$0 = 2\lambda rz - (1 - C^2)(1 + C)\{(1 + C)v^2 + 2vw\}. \tag{B.6}$$

In order to solve for linear strategies we need to solve for m, n, and v. Using (B.3), one can substitute s out in (B.2) and (B.5). Also, using (B.5), one can substitute w out in (B.4). Then transform the resulting equations into equations in α, β, and γ, using $\alpha = \lambda^{-1}(1 - C^2)(1 + C)v, \beta = \lambda^{-1}(1 - C^2)(m + Cn)$, and $\gamma = \lambda^{-1}(1 - C^2)(Cm + n)$.

The following system of equations emerges:

$$0 = 2(1 - C^2) + \lambda r(\beta - C\gamma) - \lambda\beta^2 + 2\lambda C\beta\gamma - \lambda(2 - C^2)\gamma^2, \tag{B.7}$$

$$\begin{aligned} 0 = {} & rC(1 - C^2) - \{2(1 - C^2) - \lambda r^2\}C\beta - \lambda r^2\gamma - 3\lambda rC\beta^2 + \lambda r(4 - C^2)\beta\gamma \\ & + 2\lambda C\beta^3 - \lambda(4 - C^2)\beta^2\gamma + \lambda\gamma^3, \end{aligned} \tag{B.8}$$

$$\begin{aligned} 0 = {} & A(1 - C^2)(1 + C)(r^2 - 3r\beta + 2\beta^2) - \lambda\alpha\{(1 - C)(r^3 - 4r^2\beta + 5r\beta^2 - 2\beta^3) \\ & - r^2(1 - C^2)\gamma + r(3 - 2C^2)\beta\gamma - (2 - C^2)\beta^2\gamma - r\gamma^2 + 2\beta\gamma^2 - \gamma^3\}. \end{aligned} \tag{B.9}$$

Eq. (B.9) determines α from β and γ.[35] From (B.7) and (B.8) we can derive the following equations for β and γ

$$\begin{aligned} 0 = {} & -\frac{-\{8(1 - C^2) + \lambda r^2\}C^2}{\lambda^3(9 - C^2)^2} + \frac{8(8 - 11C^2 + 5C^4) + 4\lambda r^2(4 - 3C^2) + \lambda^2 r^4}{\lambda^2(9 - C^2)^2}\gamma^2 \\ & - \frac{8(18 - 13C^2 + 3C^4) + 2\lambda r^2(9 + C^2)}{\lambda(9 - C^2)^2}\gamma^4 + \gamma^6, \end{aligned} \tag{B.10}$$

$$\beta = \frac{r}{2} + \frac{(8 - 10C^2 + \lambda r^2)\gamma - \lambda(9 - 5C^2)\gamma^3}{2C(1 - 6\lambda\gamma^2)}. \tag{B.11}$$

We can solve (B.10) for explicit values of γ because it is a cubic equation in γ^2. Next we sketch how one obtains explicit solutions for a cubic equation when there are three real roots.

Suppose a cubic equation is given in the form

$$a_0 + a_1 z + a_2 z^2 + z^3 = 0. \tag{B.12}$$

Define $\Gamma = \frac{3a_1 - a_2^2}{9}$, $\Lambda = \frac{a_0}{2} - \frac{a_1 a_2}{6} + \frac{2a_2^3}{27}$, and $\Delta = \Gamma^3 + \Lambda^2$; here Δ is the discriminant of Eq. (B.12). If $\Delta < 0$, then this cubic equation has three real roots. The three roots are given by

$$z_k = -\frac{a_2}{3} + 2\sqrt{-\Gamma}\cos\frac{\theta + 2k\pi}{3}(k = 0, 1, 2), \tag{B.13}$$

where $\theta = \arctan(\frac{\sqrt{-\Delta}}{-\Lambda}) \in [0, \pi)$. Notice that $-\Gamma > 0$ if $\Delta < 0$. In our model Δ is a complicated function of λ, r, and C, which is given in Table 2. One can check analytically that Δ is negative in our model (unless $39 - 55C^2 + 6\lambda r^2 = 0$). The solutions to (B.10) for γ^2 can be found according to formula (B.13); then the solutions for γ are obtained by taking positive and negative square roots. The six solutions are summarized in Table 2.

[35] In order to check that v (and thus α) is linear in A, note that from (B.5) we can see that w is linear in v. After substituting w out in (B.4), v can be checked to be a linear function of A.

Table 2

Solution #	γ	Solution #	γ
1	$\gamma_1 = -\gamma_2$	2	$\gamma_2 = \sqrt{-\frac{a_2}{3} + 2\sqrt{-\Gamma}\cos\frac{\theta}{3}}$
3	$\gamma_3 = -\gamma_4$	4	$\gamma_4 = \sqrt{-\frac{a_2}{3} + 2\sqrt{-\Gamma}\cos\frac{\theta+2\pi}{3}}$
5	$\gamma_5 = -\gamma_6$	6	$\gamma_6 = \sqrt{-\frac{a_2}{3} + 2\sqrt{-\Gamma}\cos\frac{\theta+4\pi}{3}}$

$$\beta = \frac{rC + (8 - 10C^2 + \lambda r^2)\gamma - 6\lambda rC\gamma^2 - \lambda(9 - 5C^2)\gamma^3}{2C(1 - 6\lambda\gamma^2)}$$

and

$$\alpha = -\frac{A(B-C)(\beta+\gamma)}{2B - C(1 - \frac{\gamma}{\beta - r})},$$

where

$$a_2 = -\frac{8(18 - 13C^2 + 3C^4) + 2\lambda r^2(9 + C^2)}{\lambda(9 - C^2)^2}, \theta = \arctan\left(\sqrt{-\Delta}/(-\Lambda)\right) \in [0, \pi),$$

$$\begin{aligned}\Gamma = &-\{8(648 - 639C^2 + 383C^4 - 321C^6 + 57C^8)\\ &+ 4\lambda r^2(324 + 171C^2 + 215C^4 - 95C^6 + 33C^8) + \lambda^2 r^4(81 + 126C^2 + C^4)\}\{9\lambda^2(9 - C^2)^4\}^{-1},\end{aligned}$$

$$\begin{aligned}\Delta = &-C^2(39 - 55C^2 + 6\lambda r^2)^2\{64(1 - C^2)(256 - 139C^2 - 15C^4 + 31C^6 - 5C^8)\\ &+ 16\lambda r^2(512 - 539C^2 + 169C^4 - 17C^6 + 3C^8) + \lambda^2 r^4(1536 - 971C^2 + 214C^4 - 11C^6)\\ &+ 4\lambda^3 r^6(32 - 9C^2 + C^4) + 4\lambda^4 r^8\}\{108\lambda^6(9 - C^2)^8\}^{-1},\end{aligned}$$

and

$$\begin{aligned}\Lambda = &\{8(93312 - 507627C^2 + 615087C^4 - 197854C^6 - 9810C^8 + 17433C^{10} - 2349C^{12})\\ &+ 3\lambda r^2(93312 - 428409C^2 + 308772C^4 - 78166C^6 + 17316C^8 - 17433C^{10})\\ &+ 48\lambda^2 r^4(729 - 2916C^2 + 945C^4 - 218C^6 - 12C^8)\\ &+ 2\lambda^3 r^6(9 + C^2)(9 - 18C + C^2)(9 + 18C + C^2)\}\{54\lambda^3(9 - C^2)^6\}^{-1}.\end{aligned}$$

B.1.1. Bounds on γ^2

We want to compare the sizes of γ^2 for the six solutions. We will use this information later. First, γ^2 can be rewritten as

$$\gamma^2 = -\frac{a_2}{3}\left(1 + t\cos\frac{\theta + 2k\pi}{3}\right) \text{ for } k = 0, 1, 2, \quad \text{(B.14)}$$

where $t = 6\sqrt{-\Gamma/a_2^2}$ and a_2 is as given in Table 2. The parameter θ is the angle associated to the complex number $-\Lambda + \sqrt{-\Delta}i$. We have that $0 \leqslant \theta \leqslant \pi$ because

$\sqrt{-\Delta} \geqslant 0$. Hence,

$$-1 < \cos\frac{\theta+2\pi}{3} \leqslant -0.5 \leqslant \cos\frac{\theta+4\pi}{3} \leqslant 0.5 \leqslant \cos\frac{\theta}{3} \leqslant 1. \tag{B.15}$$

Since $-a_2 > 0$ and $t > 0$, we know from (B.15) that solutions #3 and #4 are the smallest in absolute value and that #1 and #2 are the largest.

B.2. There is a unique stable LMPE

We show that there is a unique solution to (B.7) and (B.8) that satisfies the stability condition. First, (B.7) can be rewritten as

$$\gamma = \frac{C(\beta - r/2)}{2 - C^2} \pm \frac{\sqrt{(2-C^2)\{8(1-C^2)+\lambda r^2\} - 8\lambda(1-C^2)(\beta - r/2)^2}}{2\sqrt{\lambda}(2-C^2)}. \tag{B.7$'$}$$

Using (B.7), we can derive the following from (B.8):

$$\gamma = \frac{2C(3-C^2)(\beta - r/2)}{18 - 7C^2 + C^4} + \frac{F_1(\beta - r/2)}{F_2(18 - 7C^2 + C^4)}, \tag{B.8$'$}$$

where

$$F_1 = 8C\{24 - 76C^2 + 46C^4 - 11C^6 + C^8 + (12 - 7C^2 + C^4)\lambda r^2\},$$

$$F_2 = (2 - C^2)\{8 + (1 - C^2)\lambda r^2\} - 4\lambda(18 - 7C^2 + C^4)(\beta - r/2)^2.$$

The graphs of (B.7′) and (B.8′), including the asymptotes, are drawn in Fig. 1 for the case $F_1 > 0$. The two graphs have six intersections corresponding to the six solutions listed in Table 2. The graphs are symmetric with respect to the point $(r/2, 0)$ in the $\beta - \gamma$ plane. The derivative of the second term on the RHS of (B.8′) with respect to β is either positive for all values of β or negative for all values of β. The sign of the slope is the same as the sign of F_1 in (B.8′). In Fig. 1 the slope is positive. We deal with the two cases separately. In both cases, all we need to show is that the two graphs representing (B.7′) and (B.8′) intersect each other only once in the region defined by the stability condition: namely, $\gamma > \beta$ and $\gamma < -\beta$.

Case 1: $F_1 \geqslant 0$. The proof can be most clearly explained graphically, so the reader should refer to Fig. 2 in the subsequent argument. Point A is the intersection of the graph of (B.7′) and that of $\gamma = -\beta$. Point B is the point on the graph of (B.8′) that has the same β-coordinate as A. Point C is the intersection of the horizontal asymptote of (B.8′) (the straight line represented by the first term on the RHS of (B.8′)) and the graph of (B.7′). Point D is the intersection of the graph of (B.7′) and the graph of $\gamma = \beta$. Finally, point S is the intersection of the graphs of (B.7′) and (B.8′) with the smallest β-coordinate. We claim that point S is the unique stable solution. The claim is proved by checking that:

(1) point A lies to the left of the smaller vertical asymptote;
(2) point B lies below point A; and
(3) point C lies above the line $\gamma = \beta$.

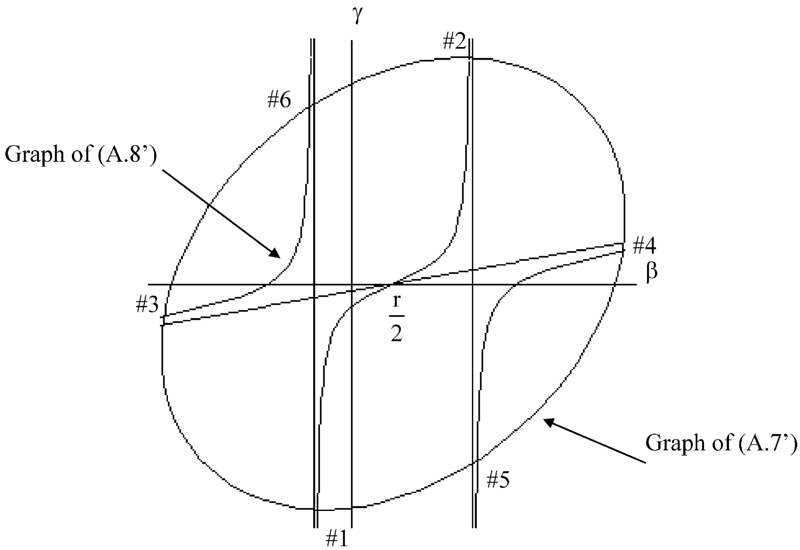

Fig. 1.

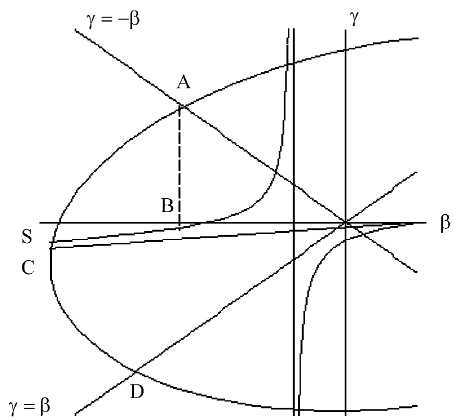

Fig. 2.

We omit the details. When point A lies to the left of the smaller vertical asymptote, so does point D. Since the stable solution must lie on arc AD and since the graph of (B.8′) intersects arc AD only once, uniqueness follows.

Case 2: $F_1 < 0$. Again the proof can be most clearly explained graphically; see Fig. 3. Points A, C, D, and S are defined as before. Points E and F are the points on the graph of (B.8′) that have the same β-coordinate as D and A, respectively. Again

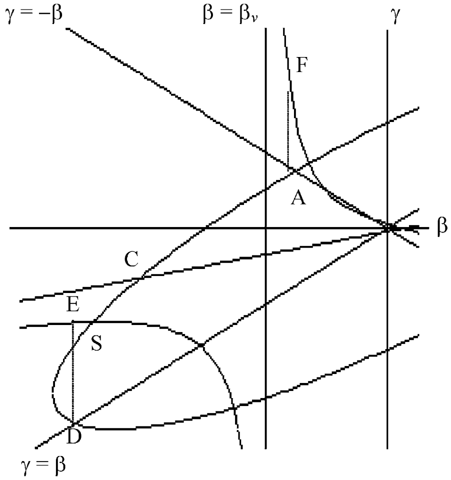

Fig. 3.

we prove that point S is the unique stable solution. The proof is done by showing that:

(1) point D lies to the left of the smaller vertical asymptote;
(2) point E lies above point D; and
(3) if point A lies to the right of the smaller vertical asymptote, then point F lies above point A.

Again, we skip the details. Claims (1)–(3) imply that point S is the only solution on arc AD, which proves the uniqueness of the stable solution.

B.3. Result B.1 (Identifying the stable solution)

Let $B = 1$. If $C > \sqrt{39/55}$ and $\lambda r^2 < (-39 + 55C^2)/6$, then solution #5 in Table 2 is the stable LMPE. Otherwise, solution #3 is the stable LMPE.

Derivation: We can check analytically that #3 is the stable solution when $F_1 \geqslant 0$. We sketch the proof (refer to Fig. 4). Point C is defined in the same way as in Fig. 2. Points G and H are, respectively, the intersections of the graphs of (B.7′) and (B.8′) with the β-axis on the left of the smaller vertical asymptote. Point I is the point on the graph of (B.7′) that has the same β coordinate as point H. Points K and L are the intersections of the vertical asymptotes of (B.8′) and the graph of (B.7′). First one can show that the β-coordinate of point H is larger than that of point G. This proves that the γ-coordinate of S is negative, since the slope of the graph of (B.8′) is positive. Second, one can check that the γ-coordinate of point I is larger than the absolute value of the γ-coordinate of point C. Third, the γ-coordinate of point L is larger than that of point K, because (B.8′) is the sum of an ellipse with center at $(r/2, 0)$ and a

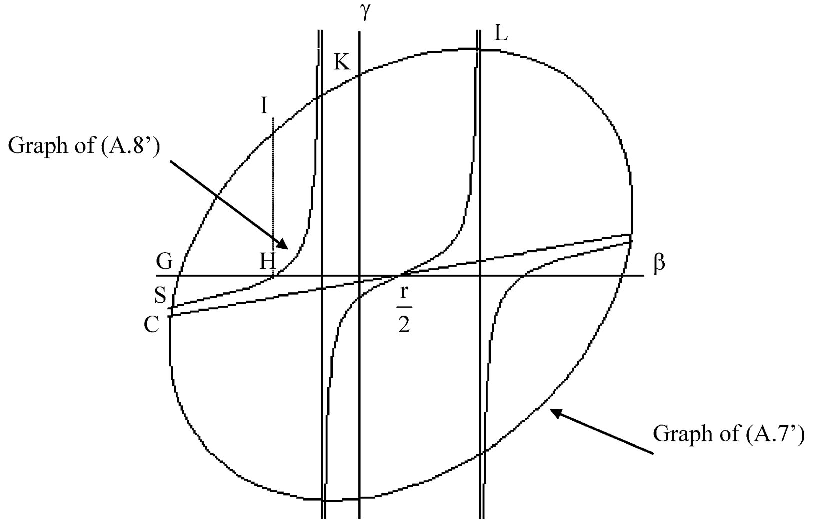

Fig. 4.

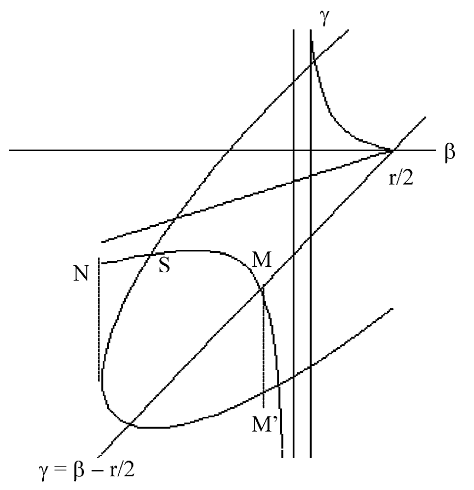

Fig. 5.

straight line through $(r/2, 0)$ with positive slope. This proves that the stable solution is the smallest in absolute value with negative γ coordinate, namely #3.

We can also show analytically that the stable solution is either #3 or #5 when $F_1 < 0$. It is clear from Fig. 3 that S has $\gamma < 0$ (i.e., S is either #1 or #3 or #5 in Table 2). We can show also that the γ-coordinate of S cannot be the largest in absolute value. We sketch the proof (refer to Fig. 5). Point M is the intersection of the graph of (B.8′) and the graph of $\gamma = \beta - r/2$ on the left of the smaller vertical

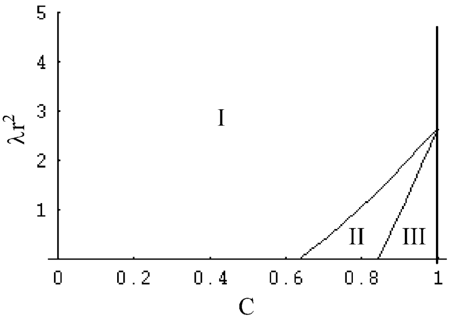

Fig. 6. Region I: $F_1 \geqslant 0$, stable solution is #3. Region II: $F_1 < 0$, stable solution is #3. Region III: $F_1 < 0$, stable solution is #5.

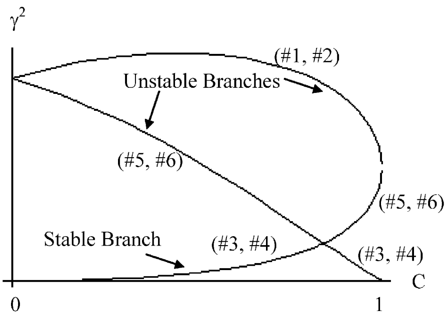

Fig. 7. Graphs of γ^2.

asymptote of (B.8′). Point M' is the point on the lower part of (B.7′) that has the same β-coordinate as point M. Point N is the point on the graph of (B.8′) whose β-coordinate is the minimum of the points on the graph of (B.7′). One can check that the γ-coordinate of point N is larger than that of point M (which in turn is larger than that of point M') and that the slope of (B.8′) at M is negative. These facts, together with the fact that (B.8′) is concave on the left of the smaller vertical asymptote, proves that the γ-coordinate of S is not the largest in absolute value.

Numerical analysis shows that #3 is the stable solution if $\lambda r^2 > (55C^2 - 39)/6$ and that #5 is the stable solution otherwise. Fig. 6 summarizes the results. For $C < \sqrt{39/55} \approx 0.842075$, solution #3 is the coordinate of S. For larger values of C and for $\lambda r^2 < (-39 + 55C^2)/6$ (region III), #5 is the coordinate of S. Fig. 7 depicts three solutions of (B.10), a cubic equation in γ^2, as functions of C for $\lambda r^2 = 1/100$ and $\lambda = 1$. The stable branch starts out the smallest when C is small. As C increases the stable branch increases and crosses one of the unstable branches, to become the second largest. The crossing occurs when $\theta = 0$, that is, when $\lambda r^2 = (-39 + 55C^2)/6$.

Appendix C

C.1. Simulation results

Prices at the stable LMPE trajectory are (strictly) lower than the prices at the OLE trajectory for all initial states that belong to P for $B = 1$, where C is between 0.001 and 1 and λr^2 is between 0 and 1000, with step size 0.1 for C and 10 for λr^2.

We want to check that $p_i^{\mathrm{OL}}(t) - p_i^{\mathrm{MP}}(t) > 0 (t \geqslant 0, i = 1, 2)$ for prices in the region P. From Proposition 4.1 we have

$$p_i^{\mathrm{OL}}(t) = p^{\mathrm{B}} + \left(\frac{p_i^0 + p_j^0}{2} - p^{\mathrm{B}}\right)\mathrm{e}^{\phi_1 t} + \left(\frac{p_i^0 - p_j^0}{2}\right)\mathrm{e}^{\phi_2 t}. \tag{C.1}$$

We obtain $p_i^{\mathrm{OL}}(t) - p_i^{\mathrm{MP}}(t)$ by subtracting (5) from (C.1). First define a new set P_t for a given t to be the set of prices (p_1^0, p_2^0) for which $p_i^{\mathrm{OL}}(t) - p_i^{\mathrm{MP}}(t) > 0,\ i = 1, 2$. We want to show that P is included in P_t for all $t \geqslant 0$, which is illustrated in Fig. 8. In order to prove the inclusion it is enough to show that (1) the coordinates of the vertex N are larger than those of M and (2) the horizontal coordinate of point L is larger than that of K.

The horizontal (and vertical) coordinate of N is $\frac{(1-\exp[\phi_1 t])p^{\mathrm{B}} - (1-\exp[(\beta+\gamma)t])p*}{\exp[(\beta+\gamma)t] - \exp[\phi_1 t]}$, whereas the horizontal (and vertical) coordinate of M is A. We want to show that $(1 - \exp[\phi_1 t])(p^{\mathrm{B}}/A) - (1 - \exp[(\beta+\gamma)t])(p^*/A) > \exp[(\beta+\gamma)t] - \exp[\phi_1 t]$. Define the function $f(t)$ by

$$f(t) = (1 - \exp[\phi_1 t])(p^{\mathrm{B}}/A) - (1 - \exp[(\beta+\gamma)t])(p^*/A) - (\exp[(\beta+\gamma)t] - \exp[\phi_1 t]).$$

We want to show that $f(t) > 0$. First, $f(0) = 0$. Observe that $f'(t)$ and hence $f(t)$ would be positive if $(\beta+\gamma)(1 - p^*/A) < \phi_1(1 - p^{\mathrm{B}}/A)$ and $\phi_1 < \beta + \gamma$. Since $p^* = -\alpha/(\beta+\gamma)$ and, normalizing $B = 1, p^{\mathrm{B}} = A(1 - C)/(2 - C)$, the first inequality is

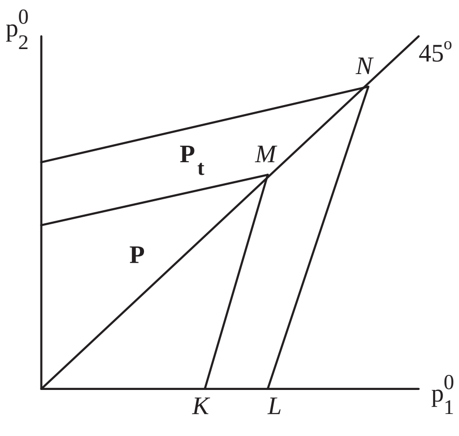

Fig. 8.

equivalent to

$$\phi_1 - (2 - C)(\alpha/A + \beta + \gamma) > 0.$$

We check this inequality numerically.

The horizontal coordinate of L is $\frac{2\{(1-\exp[\phi_1 t])p^{\mathrm{B}} - (1-\exp[(\beta+\gamma)t])p*\}}{\exp[(\beta+\gamma)t] - \exp[\phi_1 t] + \exp[(\beta-\gamma)t] - \exp[\phi_2 t]}$, whereas the coordinate of K is $A(1-C)$ after normalizing $B = 1$. The inequalities $\phi_1 < \beta + \gamma$ and $\phi_2 < \beta - \gamma$ can be checked analytically.[36] We then want to show that $(1-\exp[\phi_1 t])(p^{\mathrm{B}}/A) - (1-\exp[(\beta+\gamma)t])(p^*/A) + ((1-C)/2)\{\exp[\phi_1 t] - \exp[(\beta+\gamma)t] + \exp[\phi_2 t] - \exp[(\beta-\gamma)t]\} > 0$.

In summary, we check that

(i) (numerically) $\phi_1 - (2-C)(\alpha/A + \beta + \gamma) > 0$,
(ii) (analytically) $\phi_1 < \beta + \gamma$ and $\phi_2 < \beta - \gamma$, and
(iii) (numerically) $(1-\exp[\phi_1 t])(p^{\mathrm{B}}/A) - (1-\exp[(\beta+\gamma)t])(p^*/A) + ((1-C)/2)$ $\{\exp[\phi_1 t] - \exp[(\beta+\gamma)t] + \exp[\phi_2 t] - \exp[(\beta-\gamma)t]\} > 0$.

We describe the numerical method first. Note that p^{B}/A is a function of C only, while p^*/A, $\sqrt{\lambda\phi_1}$, $\sqrt{\lambda\phi_2}$, $\sqrt{\lambda}\alpha$, $\sqrt{\lambda\beta}$, and $\sqrt{\lambda\gamma}$ are functions of C and λr^2. For (i) we check the graphs of $\sqrt{\lambda}\{\phi_1 - (2-C)(\alpha/A + \beta + \gamma)\}$, $\sqrt{\lambda}\{\phi_1 - (\beta+\gamma)\}$, and $\sqrt{\lambda}\{\phi_2 - (\beta-\gamma)\}$ as functions of C (or λr^2) fixing the value of the other variable. Here λr^2 ranges from 0 to 1000 and C ranges from 0.001 to 1. For (iii) we substitute $t = \sqrt{\lambda}\tan\omega$ (so that, e.g., $\exp[\phi_1 t] = \exp[\sqrt{\lambda\phi_1}\tan\omega]$) and check the graph of the above expression as a function of ω for various values of C and λr^2, where $0.001 \leqslant \omega \leqslant (\pi/2) - 0.0001$ (which approximately corresponds to $0.001/\sqrt{\lambda} \leqslant t \leqslant 10{,}000/\sqrt{\lambda}$), and $0 \leqslant \lambda r^2 \leqslant 1000$. All the graphs are either monotone or single-peaked and satisfy the required sign conditions.

Proof of Proposition 4.3. As λr^2 tends to infinity $p^{\mathrm{B}} - p^*$ tends to zero; as λr^2 tends to zero $p^{\mathrm{B}} - p^*$ tends to a strictly positive number.

It is enough to show that (i) as λr^2 tends to infinity both β and γ tend to 0 and (ii) as λ or r tend to $0, \gamma/(\beta - r)$ converges to a number strictly between 0 and 1. We now prove each of these statements separately.

(i) We show that, as λr^2 tends to infinity, the graphs of (B.7′) and (B.8′) have the following properties in the limit (see Fig. 1):

(a) the vertical asymptotes have positive β-coordinates;
(b) F_1 in (B.8′) is positive. Hence the graph of (B.8′) is increasing in β;
(c) the graphs of both (B.7′) and (B.8′) go through $(0,0)$;
(d) the slope of (B.7′) is larger than that of (B.8′) at $(0,0)$.

These properties imply that among the six solutions five have positive β-coordinates and one is $(0,0)$. Since we know that $\beta < \gamma < 0$ for the stable solution (Proposition 4.2), it must be that $(0,0)$ is the limit of the stable solution.

[36] Using the graphs of (B.7′) and (B.8′) in Figs. 1 and 2.

The β-coordinates of the vertical asymptotes are the values of β that make F_2 of (B.8′) equal to 0. As λr^2 tends to infinity these asymptotes converge to $(r/2)\left[1 \pm \sqrt{\frac{(1-C^2)(2-C^2)r^2}{18-7C^2+C^4}}\right]$, which are positive. (b) This is trivial to check. (c) It is easy to see that $(0,0)$ lies on the positive part of the graph of (B.7′) and on the graph of (B.8′) in the limit. (d) The slope of (B.7′) at $(0,0)$ is $1/C$, which is larger than that of (B.8′), which is $\frac{C(3-C^2)}{4-C^2}$ in the limit.

(ii) First, define $\hat{\gamma} = \sqrt{\lambda}\gamma$ and $\hat{\beta} = \sqrt{\lambda}\beta$; then rewrite (B.7′) as

$$\hat{\gamma} = \frac{C(\hat{\beta} - \sqrt{\lambda}r/2)}{2 - C^2} \pm \frac{\sqrt{(2 - C^2)\{8(1 - C^2) + \lambda r^2\} - 8(1 - C^2)(\hat{\beta} - \sqrt{\lambda}r/2)^2}}{2(2 - C^2)}. \qquad \text{(B.7″)}$$

Eq. (B.8′) can be rewritten as

$$\hat{\gamma} = \frac{2C(3 - C^2)(\hat{\beta} - \sqrt{\lambda}r/2)}{18 - 7C^2 + C^4} + \frac{F_1(\hat{\beta} - \sqrt{\lambda}r/2)}{F_2(18 - 7C^2 + C^4)}, \qquad \text{(B.8″)}$$

where

$$F_1 = 8C\{24 - 76C^2 + 46C^4 - 11C^6 + C^8 + (12 - 7C^2 + C^4)\lambda r^2\},$$

$$F_2 = (2 - C^2)\{8 + (1 - C^2)\lambda r^2\} - 4(18 - 7C^2 + C^4)(\hat{\beta} - \sqrt{\lambda}r/2)^2.$$

Comparing (B.7″) and (B.8″) with (B.7′) and (B.8′), one can see that $\hat{\gamma}$ and $\hat{\beta}$ are the solutions to Eqs. (B.7′) and (B.8′) when $\lambda = 1$ and r is replaced by $r\sqrt{\lambda}$. Hence all the properties we show for γ and β apply also to $\hat{\gamma}$ and $\hat{\beta}$, especially when $\hat{\gamma}$ and $\hat{\beta}$ are nonpositive in the limit. In the limit as λ tends to 0, (B.7″) becomes

$$\hat{\gamma} = \frac{C\hat{\beta}}{2 - C^2} \pm \frac{\sqrt{8(1 - C^2)\{(2 - C^2) - \hat{\beta}^2\}}}{2(2 - C^2)}$$

and (B.8″) becomes

$$\hat{\gamma} = \frac{2C(3 - C^2)\hat{\beta}}{18 - 7C^2 + C^4} + \frac{2C(24 - 76C^2 + 46C^4 - 11C^6 + C^8)\hat{\beta}}{(18 - 7C^2 + C^4)\{2(2 - C^2) - (18 - 7C^2 + C^4)\hat{\beta}^2\}}.$$

The same argument given in Appendix B applies here to show that $\hat{\gamma}$ and $\hat{\beta}$ have finite negative values such that $\hat{\beta} < \hat{\gamma} < 0$ for the stable solution. Hence, $\lim_{\lambda r^2 \to 0} \gamma/\beta = \lim_{\lambda r^2 \to 0} \hat{\gamma}/\hat{\beta}$ belongs to the open interval $(0, 1)$. Since $\hat{\gamma} = \sqrt{\lambda}\gamma$ and $\hat{\beta} = \sqrt{\lambda}\beta$ converge to finite negative numbers as λ tends to 0, it follows that γ and β must tend to $-\infty$. Hence $\lim_{\lambda \to 0} \gamma/(\beta - r) = \lim_{\lambda \to 0} \gamma/\beta$, and it is also clear that $\lim_{r \to 0} \gamma/(\beta - r) = \lim_{r \to 0} \hat{\gamma}/(\hat{\beta} - \sqrt{\lambda}r) = \lim_{r \to 0} \hat{\gamma}/\hat{\beta}$.

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