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Small Income Effects: A Marshallian Theory of Consumer Surplus and Downward Sloping Demand

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We formalize the Marshallian idea that when the proportion of income spent on any commodity is small then the income effects are small. If n is the number of goods, we show, under certain assumptions on preferences and prices, that the order of magnitude of the norm of the income derivative of demand is $1/\sqrt{n}$. As a corollary we get that for the case of a single price change the percentage error in approximating the Hicksian Deadweight Loss by its Marshallian counterpart goes to zero at least at the rate $1/\sqrt{n}$ and that demand is downward sloping for n large enough.

1. INTRODUCTION

This paper considers a consumer with smooth preferences giving rise to differentiable demands and finds conditions under which the income effect (the income term in the Slutsky equation) is small. The smallness of the income effect is often postulated to justify partial equilibrium welfare analysis, consumer surplus type, "piecemeal" policy in the second best literature and to get downward sloping demand for a single consumer, to put some examples.

A common argument starts with the Slutsky relation in elasticity terms (the own price elasticity of a good equals the compensated price elasticity minus the expenditure share of the good times the income elasticity of demand) and observes that, ceteris paribus, when the proportion of income spent upon a particular commodity is small the income effect on that commodity should be small. For example, when discussing the "Law of Demand" Hicks says that even if a good is inferior, "the demand curve will still behave in an orthodox manner so long as the proportion of income spent upon the commodity is small, so that the income effect is small" (Hicks (1946), p. 35). The problem with this type of reasoning is the ceteris paribus assumption: what happens to the income elasticity of demand when the expenditure share on the good gets small? Marshall discussed in his Principles of Economics consumer surplus and downward sloping demand on the supposition that "the marginal utility of money to the individual purchaser is the same throughout" and he based this supposition "on the assumption, which underlies our whole reasoning, that his expenditure on any one thing, as, for instance, tea, is only a small part of his whole expenditure" (Marshall (1920), p. 842). Obviously a "constant" marginal utility of money means that income effects are absent. The aim of the paper is to make rigorous the Marshallian notion that when any good represents a small part of the expenditure of a consumer then income effects are negligible.

In our model we have a countable infinity of potential commodities and our consumer has well defined preferences for any finite subset of them. These preferences are representable by a sequence of utility functions which are not necessarily separable. To get small shares of expenditure for any commodity and still have our consumer consume a significant amount of every good, we increase at the same time the number of commodities (n) and income available to the consumer (we normalize income to equal the number of goods, n). In a static context we make the conceptual experiment of increasing the number of goods and income to have significant demands for any good. In an intertemporal context we may think of the consumer living for n periods, in each period he/she receives a unit of income and can borrow or lend at a zero interest rate. Adding more periods one adds more income and more goods.

We argue in the paper that provided prices are in a compact and positive interval, preferences for different goods are symmetric enough and we never have any two goods being close to perfect substitutes (in our terms, if the sequence of utility functions satisfies the Uniform Inada Property) then the consumer spreads her/his income over all the goods, and demands and the marginal utility of income are uniformly bounded above and away from zero. Furthermore, using a curvature assumption (the sequence of utility functions must have a second derivative uniformly bounded on compact sets), we show that the order of magnitude of the norm of the income derivative of demand is $1/\sqrt{n}$ and that for any n the substitution term in the Slutsky matrix is non-degenerate. That is, income effects vanish and the substitution effects remain as n gets large. Individual income derivatives of demand are at most of the order $1/\sqrt{n}$ ($1/n$ if preferences admit an additive separable representation satisfying the assumptions of our theorem). This means, in particular, that for large n the slope of the (Marshallian) demand function will be close to the slope of the Hicksian (or compensated) demand function. As a corollary we get that for large n demand will be downward sloping and that the Marshallian consumer surplus will be a good approximation of the true measure of welfare change, the Hicksian consumer surplus. In fact if only one price changes the percentage error in approximating the Hicksian consumer surplus (or deadweight loss) by its Marshallian counterpart goes to zero at least at the rate $1/\sqrt{n}$. When many prices change it is well known that the Marshallian consumer surplus depends on what particular sequence of price changes one considers. Restricting attention to simple monotonic sequences we show that for any ordering of the sequence of price changes the order of magnitude of the approximation error done in measuring the Hicksian consumer surplus by its Marshallian counterpart is at most $1/\sqrt{n}$.

Recently some results about downward sloping *market* demand have been obtained by Hildenbrand (1983), Chiappori (1985) and Novshek and Sonnenschein (1979). Other results on demand aggregation are reported in Freixas and Mas-Colell (1982) and Jerison (1984). Our results concern *individual* demand. Consumer surplus has been controversial ever since its introduction by Jules Dupuit (1969). Marshall (1920) Hotelling (1969) and Hicks (1941, 1946) used it, and Samuelson (1947) was very critical. (See Chipman and Moore (1976) for a good exposition of the issue). Willig (1976) provided bounds on the percentage error of approximating the Hicksian consumer surplus by the Marshallian one. However even when this percentage error is small it may be the case that the percentage error in approximating the Hicksian deadweight loss by its Marshallian counterpart is very large as Hausman (1981) has pointed out. Usually in economic models we are interested more in the deadweight loss (surplus loss minus an appropriate transfer payment) than in the total loss in surplus to the consumer. Our results apply to the deadweight loss as well.

The plan of the Sections is as follows. In Section 2 we present the model and we discuss the assumptions needed to prove our main result. Section 3 contains the proof of the Theorem. Section 4 deals with consumer surplus and Section 5 with downward sloping demand. Concluding remarks follow.

2. THE MODEL

We suppose there is a countable infinity of potential commodities and our consumer has well defined preferences for any finite subset of them. In particular we assume that those preferences are representable by a sequence of utility functions, $\{U^n\}_{n=1}^{\infty}$, where the domain of U^n is the positive orthant of R^n. To fix ideas suppose for the moment that $U^n(\cdot)$ is a symmetric (interchanging the amounts consumed of any two goods i and j the level of utility remains constant) strictly quasiconcave and increasing utility function. Let all the prices equal p^n, $p^n > 0$. Given n goods in the market, our consumer maximizes utility U^n subject to the budget restriction with income M and prices p^n. In this completely symmetric situation the demand for good i is M/np^n, its expenditure share, $1/n$ and the income derivative of demand is $1/np^n$. By letting income equal the number of goods, $M = n$, demand for good i is $1/p^n$ and the income effect on good i (the consumption of good i times the income derivative of demand) goes to zero as n goes to infinity provided prices are bounded away from zero. The consumer spreads her/his income over all the goods, demands stay bounded and the income effect dissipates as the number of goods grows since prices are bounded away from zero.[1]

Given that there are n goods in the market with prices $p^n = (p_1^n, \ldots, p_n^n)$ the optimization program (P) of our consumer is Max $U^n(x)$ subject to $p^n x \leqq n$, $x \in R_+^n$. Notice that we normalize income to equal the number of goods n. This is the context we will use to formalize the Marshallian notion that when any good represents a small share of the expenditure of the consumer income effects go away. What can go wrong? Examples 1 and 2 below show cases of very "asymmetric" preferences where income effects do not go away by increasing the number of goods. In example 1 there is discounting and in example 2 good 1 captures all the income effects even though preferences are "nice". In both examples let $p_i^n = p_i > 0$ for all n.

Example 1. Let $U^n(x) = \sum_{i=1}^n 2^{-i} \log x_i$. Demand for good i is $n2^{-i}/(p_i \sum_{i=1}^n 2^{-i})$, its expenditure share, $2^{-i}/\sum_{i=1}^n 2^{-i}$, and the income derivative of demand is $2^{-i}/(p_i \sum_{i=1}^n 2^{-i})$. The marginal utility of income is $\sum_{i=1}^n 2^{-i}/n$. Since $\sum_{i=1}^{\infty} 2^{-i} = 1$, asymptotically the first goes to infinity, the second to 2^{-i}, the third to $2^{-i}/p_i$, and the last one to zero.

Example 2. Let $U^n(x) = x_1 + \sum_{i=2}^n \log x_i$. Suppose that $0 < p_1 < 1$, then demand for good 1 is $1 + n(1-p_1)/p_1$ and demand for good i, $i \geqq 2$, is p_1/p_i. Income derivatives of demand are $1/p_1$ and 0 respectively. Asymptotically demand for good 1 goes to infinity and its expenditure share stays constant at $1 - p_1$. The marginal utility of income stays constant at $1/p_1$.

We want the consumer to spread her/his income over all the goods, we do not want any group of goods to capture most of the income effects. To this end we will require that preferences for different goods be symmetric enough and we will limit substitutability so that demand does not concentrate on the low priced goods. This will insure that demands will be bounded above and away from zero. Bounds on the income derivatives of demand in terms of the number of goods will be derived then from uniform bounds on the second derivative of the utility functions U^n. Furthermore we will show that substitution effects are nondegenerate for large n. Before stating our results some definitions are needed.[2]

Definitions

Let $\{U^n\}_{n=1}^{\infty}$ be our sequence of utility functions $U^n: R^n_{++} \to R$.

Definition 1. We say that U^n is *differentiably strictly monotone* if U^n is continuously differentiable and $DU^n(x) \gg 0$ for all $x \in R^n_{++}$.

Definition 2. We say that U^n is *differentiably strictly concave* if U^n is twice-continuously differentiable and its Hessian, $D^2U^n(x)$, is negative definite for all $x \in R^n_{++}$.

Definition 3. The sequence $\{U^n\}$ satisfies the *Uniform Inada Property* (U.I.P.) if there exist non-increasing functions $\underline{\phi}$ and $\bar{\phi}$ from R_{++} to R_{++}, $\underline{\phi}(z) \leqq \bar{\phi}(z)$ for all $z \in R_{++}$, with $\underline{\phi}(z)$ approaching infinity when z approaches zero and with $\bar{\phi}(z)$ approaching zero when z approaches infinity, such that for any n

$$\underline{\phi}(x_i) \leqq \partial_i U^n(x) \leqq \bar{\phi}(x_i) \quad \text{for all } x \in R^n_{++}, \text{for all } 1 \leqq i \leqq n.$$

That is to say, the sequence $\{U^n\}$ satisfies the U.I.P. if we can find non-increasing functions $\underline{\phi}$ and $\bar{\phi}$ which bound the marginal utility of any good, which depend only on the quantity of the good considered (in particular they are independent of n) and which satisfy the Inada requirement for marginal utility (as the consumption of the good goes from zero to infinity, marginal utility goes from infinity to zero). A sequence $\{U^n\}$ satisfying the U.I.P. formalizes the idea that preferences are not "very asymmetric" and that we never have two goods being close to perfect substitutes. In the monopolistic competition literature we find assumptions similar in spirit to our Uniform Inada Property. Examples of this are the symmetry assumptions in the representative consumer models of Spence (1976) and Dixit and Stiglitz (1977) or the "no neighbouring goods" assumption in Hart's Chamberlinian model (Hart (1985)). (See Jones (1984) for a general competitive model of commodity differentiation.) The U.I.P. assumption is a very strong one. It means that no good has a "neighbour" no matter how many goods are in the market. We could think in terms of goods being drawn from an unbounded space of characteristics. Otherwise, with a compact space of characteristics, most of the goods would eventually become very close substitutes as n goes to infinity. The U.I.P. implies that demands will be uniformly bounded above and away from zero provided that prices are not of different orders of magnitude.

Definition 4. The sequence $\{U^n\}$ has a *second derivative uniformly bounded on compact sets* (B.S.D. for short) if the absolute values of the eigenvalues of $D^2U^n(\cdot)$ are bounded above and away from zero uniformly in n provided the consumption of each good lies in a compact set bounded away from zero.

What is the economic interpretation of this condition? Lemma 1 below gives a sufficient condition for $\{U^n\}$ to have a second derivative uniformly bounded on compact sets.

Lemma 1. *Suppose that $\{U^n\}$ is a sequence of twice-continuously differentiable utility functions such that $|\partial_{ii}U^n(\cdot)|$ is bounded above and $\partial_{ii}U^n(\cdot)+\sum_{j\neq i}|\partial_{ij}U^n(\cdot)|$ is negative and bounded away from zero for all $1\leqq i\leqq n$ and for all n provided that the consumption of each good lies in a compact interval bounded away from zero. Then each U^n in the sequence is differentiably strictly concave and $\{U^n\}$ has a second derivative uniformly bounded on compact sets.*

Proof. See Appendix. ||

Lemma 1 says that a sufficient condition for $\{U^n\}$ to have the B.S.D. property is that $D^2U^n(\cdot)$ be strongly dominant diagonal and that its diagonal be bounded. This means that the aggregation of the interaction terms $\sum_{j\neq i}|\partial_{ij}U^n(\cdot)|$ cannot overwhelm the own effect $\partial_{ii}U^n(\cdot)$. One possibility is for the interaction of good i and j, $|\partial_{ij}U^n(\cdot)|$, to vanish as $|i-j|$ gets larger.

Let $f(p,m)$ solve Max $U(z)$ subject to $pz=m$, $z\in R^n_{++}$ and let $S(p,m)$ be the associated Slutsky matrix

$$S(p,m)=D_pf(p,m)+f(p,m)^\tau D_mf(p,m)$$

where τ indicates transpose.

It is well known that $S(p,m)$ is negative semidefinite and negative definite on $T_p=\{v\in R^n\colon v^\tau p=0\}$. Correspondingly let $f^n(p^n,n)$ solve Max $U^n(z)$ subject to $p^nz=n$, $z\in R^n_+$ and let $S^n(p^n,n)$ be the associated Slutsky matrix.

Definition 5. We say that the sequence of Slutsky matrices $\{S^n(p^n,n)\}$ is *non-degenerate* if the absolute values of the eigenvalues of $S^n(p^n,n)$ restricted to $T^n_p=\{v\in R^n\colon v^\tau p^n=0\}$ are bounded above and away from zero for all n.

We are now ready to state our main result.

Theorem. *Let $\{U^n\}$ be a sequence of utility functions such that U^n is differentiably strictly monotone and strictly concave for all n. Suppose that $\{U^n\}$ satisfies the Uniform Inada Property and has a second derivative uniformly bounded on compact sets. The consumer solves Program P, where the prices are in a compact and positive interval. Then*:

- (a) *The demand for any good and the marginal utility of income are uniformly bounded above and away from zero.*
- (b) *The order of magnitude of the* (*Euclidean*) *norm of the income derivative of demand is $1/\sqrt{n}$ and*
- (c) *The associated Slutsky matrix is non-degenerate.*

Corollary. *If preferences are representable by a sequence $\{U^n\}$ of additive separable, $(U^n(x)=\sum_{i=1}^n u_{in}(x_i))$, or homothetic utility functions which satisfy the assumptions of the Theorem then the order of magnitude of the income derivative of demand of any good is $1/n$.*

A rigorous proof of the Theorem and Corollary is in Section 3. We sketch here the argument. The U.I.P. guarantees that (a) is true. If demand for any good were unbounded then the U.I.P. would imply that demand for all goods is unbounded which is a contradiction since from the budget constraint average demand has to be bounded. Similarly one shows that demands are bounded away from zero. Once demands are in a compact set bounds on the norm of the income derivative of demand in terms of the number of goods are easily derived from the bounds on the Hessian of the utility function (B.S.D. assumption) through the effect of the expansion of the price vector. (c) follows also from the B.S.D. assumption. The norm of the income derivative of demand is seen to be of the order $1/\sqrt{n}$ which means that the order of magnitude of *individual* income derivatives of demand is at most $1/\sqrt{n}$. When preferences are representable by additive separable utility functions with strictly concave components then all goods are normal and from the bounds on second derivatives it follows that all individual income derivatives are of the same order of magnitude which must be $1/n$ because of the budget constraint.[3] If preferences are homothetic then uniform bounds on demand (given by the U.I.P.) imply the result since with income m the demand for good i is given by $f_i(p, m) = mg_i(p)$, where g_i is an appropriate positive function, and therefore $\partial_m f_i = g_i(p)$ will be of the order of x_i/m which is $1/n$ since $m = n$ according to our normalization.[4] Two examples of sequences $\{U^n\}$ satisfying the assumption of the Theorem follow.

Example 3. Let

$$U^n(x) = \sum_{i=1}^n \alpha_i^n \log x_i + \sum_{k=1}^{n-1} \frac{1}{2^{k+1}} \sum_{i=1}^{n-k} \frac{x_i}{1+x_i} \frac{x_{i+k}}{1+x_{i+k}}$$

where $\bar{\alpha} \geqq \alpha_i^n \geqq 1$ for all i and n. It is easily seen that $D^2U^n(\cdot)$ satisfies the hypothesis of Lemma 1 and therefore U^n is differentiably strictly concave and $\{U^n\}$ satisfies the B.S.D. property. Furthermore it satisfies the U.I.P. The functions which bound marginal utility are

$$\bar{\phi}(z) = \frac{\bar{\alpha}}{z} + \frac{1}{(1+z)^2} \quad \text{and} \quad \underline{\phi}(z) = \frac{1}{z}.$$

Note that the weights given to the interaction between good i and j decrease as the goods get farther apart, that is as $|i-j|$ increases.

Example 4. Let $U^n(x) = \sum_{i=1}^n \alpha_i^n x_i^{\rho_i^n}$ with $\underline{\alpha} \leqq \alpha_i^n \leqq \bar{\alpha}$ and $\underline{\rho} \leqq \rho_i^n \leqq \bar{\rho}$ for all i and n, where $\underline{\alpha}$ and $\underline{\rho}$ are positive and $\bar{\rho} < 1$.

Remark 1. The theorem applies to preferences which are *representable* by sequence of utility functions $\{U^n\}$ satisfying the stated assumptions. Obviously the boundedness result on the marginal utility of income applies only to the sequence $\{U^n\}$. For example, the theorem applies to the preferences given by utility functions of the type $U^n(x) = \prod_{i=1}^n x_i$ since $V^n(x) = \sum_{i=1}^n \log x_i$ is an increasing transformation of $U^n(\cdot)$ and the sequence $\{V^n\}$ satisfies the assumptions.

Remark 2. Since $\{S^n\}$ is non-degenerate it follows that the diagonal elements of the Slutsky matrix S^n are bounded above and away from zero for all n. That is, the slopes of the Hicksian demand functions do not degenerate to zero or infinity as n grows.

Remark 3. The marginal utility of income, λ^n, is bounded above and away from zero for the sequence $\{U^n\}$ in the Theorem. Obviously this does not mean that λ^n converges. For example, if preferences correspond to the linear expenditure system, $U^n(x) = \sum_{i=1}^n \log(x_i - \beta)$, with $0 < \beta < 1$ and prices are in a compact subset of $(0, 1)$ then $\lambda^n = 1 - \beta\bar{p}^n$ where $\bar{p}^n$ is the average price. Clearly the average price need not converge. Suppose now that U^n is additively separable $U^n(x) = \sum_{i=1}^n u_i(x_i)$ and that we have given sequences $\{u_i\}_{i=1}^\infty$ and $\{p_i\}_{i=1}^\infty$. Assume furthermore that there is a well defined limit demand for any good. That is, the demand for good i, x_i^n, tends to x_i as n goes to infinity. This means that λ^n tends to $\hat{\lambda}=u_i'(x_i)/p_i$ since $u_i'(x_i^n)=\lambda^n p_i$ and that for n large the solution to Program P, x^n, can be approximated by the solution to $\text{Max}\sum_i u_i(z_i)-\hat{\lambda}p^n z^n$, $z^n \in R_+^n$. This "marginal utility of money constant" demand functions (which are called Frisch demands by Browning *et al.* (1985)) show no income effects. For n large the consumer acts almost as if she/he had a fixed constant marginal utility for income, $\hat{\lambda}$.[5]

3. PROOF OF THE THEOREM

Fact. *Let $\underline{\phi}$ and $\bar{\phi}$ be functions as in Definition* 3 *and let z^n be a sequence of positive numbers. We have,* (i) *if $\underline{\phi}(z^n)\to_n 0$ then $z^n\to_n\infty$ and* (ii) *if $\bar{\phi}(z^n)\to_n\infty$ then $z^n\to_n 0$.*

Proof. To show (i) suppose z^n does not go to infinity, then there exists a subsequence z^{n_j} bounded above, say by $k>0$. We have thus $z^{n_j}\leqq k$ and $\underline{\phi}(z^{n_j})\geqq\underline{\phi}(k)>0$ for all j since $\underline{\phi}$ is non-increasing and always positive, but this contradicts the fact that $\underline{\phi}(z^n)\to_n 0$. Therefore $z^n\to_n\infty$. One gets (ii) similarly. ||

Lemma 2. *Under the assumptions of the Theorem the utility function U^n satisfies the following boundary condition: Let $z^k\in R_{++}^n$ and $z^k\to_k z$, where z belongs to the boundary of R_+^n and $z\neq 0$, then $z_i=0$ if and only if $\partial_i U^n(z^k)\to_k\infty$.*

Proof. From the U.I.P. we know that $\underline{\phi}(z_i)\leqq\partial_i U^n(z)\leqq\bar{\phi}(z_i)$ for all $z\in R_{++}^n$. If $z^k\to_k z$ with $z_i=0$ (for some i) then $\underline{\phi}(z_i^k)\to_k\infty$ and $\partial_i U^n(z^k)\to_k\infty$. Conversely, if $\partial_i U^n(z^k)\to_k\infty$ then $\bar{\phi}(z_i^k)\to_k\infty$ and $z_i=0$ from part (ii) of the Fact. ||

Fix n and let $x^n=f^n(p^n,n)$ solve Program P. From the boundary condition we know that $x_i^n>0$ for $i=1,\dots,n$ and the budget restriction will be satisfied with equality since U^n is strictly monotone, accordingly there exists a positive λ^n which satisfies

$$\partial_i U^n(x^n)=\lambda^n p_i^n \qquad (i=1,\dots,n).$$

Furthermore, since U^n is differentiably strictly concave, x^n and λ^n are continuously differentiable functions of prices and income (see Debreu (1972)). Suppose that prices lie in $[\underline{p},\bar{p}]$ where $\bar{p}\geqq\underline{p}>0$.

Proof of (*a*). We show that (α) $\max\{x_i^n: 1\leqq i\leqq n\}$ is bounded above, (β) $\min\{x_i^n: 1\leqq i\leqq n\}$ is bounded away from zero and (γ) λ^n bounded above and away from zero.

(α) Suppose it is not. Then without loss of generality we may assume that $\max_i x_i^n\to_n\infty$ (take a subsequence if necessary). We have

$$\min_i\partial_i U^n(x^n)\leqq\min_i\bar{\phi}(x_i^n)=\bar{\phi}(\max_i x_i^n)$$

by the U.I.P. and since $\bar{\phi}$ is non-increasing. As $\max_i x_i^n\to_n\infty$, $\bar{\phi}(\max_i x_i^n)\to_n 0$ and $\min_i\partial_i U^n(x^n)\to_n 0$. From the first order conditions $\min_i\partial_i U^n(x^n)=\lambda^n\min_i p_i^n\geqq\lambda^n\underline{p}$, therefore $\lambda^n\to_n 0$. Also $\max_i\partial_i U^n(x^n)=\lambda^n\max_i p_i^n\leqq\lambda^n\bar{p}$ and $\max_i\partial_i U^n(x^n)\to_n 0$. Furthermore

$$\max_i\partial_i U^n(x^n)\geqq\max_i\underline{\phi}(x_i^n)=\underline{\phi}(\min_i x_i)$$

so that $\phi(\min_i x_i) \to_n 0$ and $\min_i x_i^n \to_n \infty$ according to part (i) of the Fact. On the other hand $\sum_{i=1}^n p_i^n x_i^n = n$, therefore $\sum_i \underline{p} x_i^n \leqq \sum_i p_i^n x_i^n = n$ or $1/n \sum_i x_i^n \leqq 1/\underline{p}$ for all n. Which is a contradiction since $\min_i x_i^n \leqq 1/n \sum_i x_i^n$.

(β) Similarly.

(γ) From the first order conditions one gets $\lambda^n = 1/n \sum_{i=1}^n \partial_i U^n(x^n) x_i^n$ and therefore $\underline{\phi}(\bar{x})\underline{x} \leqq \lambda^n \leqq \bar{\phi}(\underline{x})\bar{x}$, where $x_i^n \in [\underline{x}, \bar{x}]$ for all $1 \leqq i \leqq n$ and for all n. ($\bar{x} \geqq \underline{x} > 0$). ||

Proof of (*b*). We show that the order of magnitude of $\|D_m f^n(p^n, n)\|$ is $1/\sqrt{n}$.

Fix n. Differentiating with respect to income the first order conditions we get $H_n D_m f^n = p^n(p^n H_n^{-1} p^n)^{-1}$ where $H_n = D^2 U^n(x^n)$ and f^n is evaluated at (p^n, n). Therefore $D_m f^n = H_n^{-1} p^n (p^n H_n^{-1} p^n)^{-1}$ (recall that H_n is negative definite and therefore invertible) and $\|D_m f^n\| = \|H_n^{-1} p^n\| |p^n H_n^{-1} p^n|^{-1}$. Now, since demands lie in a compact set we know that the (absolute value of the) eigenvalues of H_n will lie in a compact set of the type $[\underline{\mu}, \bar{\mu}]$, where $\bar{\mu} \geqq \underline{\mu} > 0$. We have then

$$\|p^n\|^2 \bar{\mu}^{-1} \leqq |p^n H_n^{-1} p^n| \leqq \|p^n\|^2 \underline{\mu}^{-1}$$

and

$$\|p^n\| \bar{\mu}^{-1} \leqq \|H_n^{-1} p^n\| \leqq \|p^n\| \underline{\mu}^{-1}.$$

We conclude that

$$\frac{\underline{\mu}}{\bar{\mu}\bar{p}} \frac{1}{\sqrt{n}} \leqq \|D_m f^n\| \leqq \frac{\bar{\mu}}{\underline{\mu}\underline{p}} \frac{1}{\sqrt{n}}. \qquad \|$$

Proof of the Corollary. We show that if $U^n(x) = \sum_{i=1}^n u_{in}(x_i)$ then the order of magnitude of $\partial_m f_i^n(p^n, n)$ is $1/n$. First note that since each u_{in} is strictly concave all goods will be normal. Differentiating with respect to income the first order conditions one gets that $\partial_m f_i^n / \partial_m f_j^n = u''_{jn}(x_j^n) p_i^n / u''_{in}(x_i^n) p_j^n$ and therefore

$$\frac{\underline{\mu}}{\bar{\mu}} \frac{\underline{p}}{\bar{p}} \leqq \frac{\partial_m f_i^n}{\partial_m f_j^n} \leqq \frac{\bar{\mu}}{\underline{\mu}} \frac{\bar{p}}{\underline{p}}$$

for all i, j and for all n. We conclude that all $\partial_m f_i^n$ are of the same order of magnitude, which has to be $1/n$ since from the budget constraint we know that $\sum_{i=1}^n p_i^n \partial_m f_i^n = 1$. ||

Proof of (*c*). We show that the eigenvalues of $S^n(p^n, n)$ restricted to $T_p^n = \{v \in R^n : v^\tau p^n = 0\}$, $S^n | T_p^n$, are bounded above and away from zero for all n.

If S is the Slutsky matrix of a differentiably strictly concave and strictly monotone utility function with Hessian H it is easily seen that $S = \lambda A$ where λ is the marginal utility of income and

$$A = H^{-1} - H^{-1} p (H^{-1} p)^\tau (p^\tau H^{-1} p)^{-1},$$

furthermore, $AHA = A$.

Fix n, we have that $S^n = \lambda^n A^n$. We know that λ^n is bounded above and away from zero. We show that the eigenvalues of $A^n | T_p^n$ are (in absolute value) in the interval $[\bar{\mu}^{-1}, \underline{\mu}^{-1}]$ for all n. For $v \in T_p^n$, $v^\tau A^n v = v^\tau A^n H^n A^n v = w^\tau H^n w$ where $w = A^n v$. Since $w \neq 0$ ($v \in T_p^n$ and therefore v is orthogonal to p^n) we know that

$$\bar{\mu} \geqq \frac{|w^\tau H^n w|}{w^\tau w} \geqq \underline{\mu}$$

for all n. Furthermore if $\underline{\alpha}$ and $\bar{\alpha}$ are, respectively, the absolute value of the minimum and of the maximum eigenvalues of $A | T_p$ then

$$\max_{\|v\|=1} \frac{|v^T A v|}{v^T A A v} = \frac{1}{\underline{\alpha}} \quad \text{and} \quad \min_{\|v\|=1} \frac{|v^T A v|}{v^T A A v} = \frac{1}{\bar{\alpha}}.$$

Therefore, noting that

$$\frac{|w^T H^n w|}{w^T w} = \frac{|v^T A^n v|}{v^T A^n A^n v}$$

we conclude that the eigenvalues of $A^n | T_p^n$ lie (in absolute value) in $[\bar{\mu}^{-1}, \underline{\mu}^{-1}]$ for all n. ||

4. CONSUMER SURPLUS

Consider a consumer with utility function $U(\cdot)$ on R_+^n and income m facing prices p. Let $f(p, m)$ denote the Marshallian demand and $h(p, u)$ the Hicksian demand. Let $e(p, u)$ and $V(p, m)$ denote respectively the expenditure and the indirect utility function. Suppose that our consumer faces a price change from p^0 to p^1. According to Hicks the compensating variation of a price change, C.V., is the amount of income the consumer must receive at prices p^1 to leave utility unaffected by the price change:

$$\text{C.V.}\,(p^0, p^1, m) = e(p^1, u^0) - m, \qquad u^0 = V(p^0, m).$$

The equivalent variation of a price change, E.V., is the amount of income that one would have to take away from the consumer at prices p^0 to make him/her as well off as with prices p^1:

$$\text{E.V.}\,(p^0, p^1, m) = m - e(p^0, u^1), \qquad u^1 = V(p^1, m).$$

It is easily seen that the E.V. is an acceptable measure of welfare change in the sense that it can be used to rank the consumer's level of well being under various sets of prices. (See Chipman and Moore (1980) for example). That is for all p^0, p^1, p^2 and m, $\text{E.V.}\,(p^0, p^1, m) \gtreqless \text{E.V.}\,(p^0, p^2, m)$ if and only if $V(p^1, m) \lesseqgtr V(p^2, m)$. This is not the case, in general, for the C.V.

We define the Hicksian Consumer Surplus corresponding to the price change, HCS, to be the Equivalent Variation. More often however in economics we are interested in the Deadweight Loss, DL, of a price change. For example, in estimating the excess burden of a tax system it is not the entire loss to the consumer we are interested in but rather the loss in excess of the revenue collected by the government. That is from our consumer surplus measure we have to subtract a transfer payment. We may define (see Auerbach (forthcoming) and Mohring (1971)) the Hicksian Deadweight Loss, HDL, from a tax system (price change from p^0 to p^1) as the difference between the Equivalent Variation and the tax revenue collected when prices are p^1:

$$\text{HDL}\,(p^0, p^1, m) = \text{E.V.}\,(p^0, p^1, m) - (p^1 - p^0) \cdot f(p^1, m).$$

Or, more generally, we may define the HDL as the difference between the E.V. and an appropriate transfer payment T:

$$\text{HDL}\,(p^0, p^1, m) = \text{E.V.}\,(p^0, p^1, m) - T(p^0, p^1, m).$$

A single price change

Suppose that only the price of good i changes, then the Marshallian Consumer Surplus, MCS, is given by the integral

$$\text{MCS}(p^0, p^1, m) = \int_{p_i^0}^{p_i^1} f_i(p, m)\,dp_i$$

while the Hicksian Consumer Surplus is given by

$$\text{HCS}(p^0, p^1, m) = \int_{p_i^0}^{p_i^1} h_i(p, V(p^1, m))\,dp_i.$$

The Marshallian Deadweight Loss, MDL, equals MCS $-\, T$.

Willig (1976) provides bounds on the percentage error one makes when approximating the true welfare indicator HCS by the MCS. This bounds depend on the income elasticity of demand and on the expenditure share of the good in question. However even when the percentage error in approximating HCS by MCS is small it may be the case that the percentage error in approximating the Hicksian Deadweight Loss by its Marshallian counterpart is very large. Hausman (1981) provides an example where the first is 3% and the second 30%. For instance, consider the imposition of a commodity tax t on good i. Its price goes from p_i^0 to $p_i^0 + t$ and the amount demanded on good i from x_i^0 to x_i^1. In Figure 1 (where good i is assumed normal), $\text{MCS} = T + B + C$ and $\text{HCS} = T + B$. $\text{MDL} = B + C$ and $\text{HDL} = B$ since T is the tax revenue collected by the government. Notice that $|\text{MCS} - \text{HCS}| = |\text{MDL} - \text{HDL}|$ and denote it by Δ. Although $\Delta/|\text{HCS}|$ may be small, $\Delta/|\text{HDL}|$ may be large. In our case, $\Delta = C$, $C/(T+B)$ may be small while C/B may be large. To get the percentage error of approximation small we need that the slopes of the Hicksian and Marshallian demand curves be close. This is an immediate consequence of our theorem (provided n is large). From the Slutsky equation and with n goods

$$\frac{\partial h_i^n}{\partial p_i} - \frac{\partial f_i^n}{\partial p_i} = x_i^n \frac{\partial f_i^n}{\partial m}$$

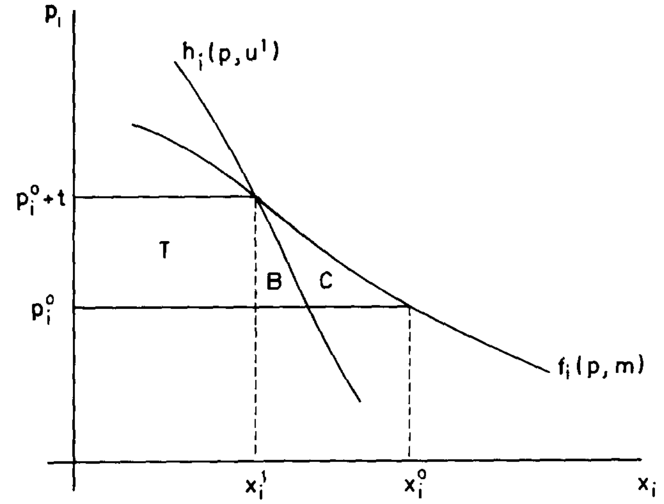

FIGURE 1

and the Theorem provides conditions under which x_i^n stays bounded and $\partial f_i^n//\partial m$ gets small as n increases while $\partial h_i^n/\partial p_i$ does not degenerate to zero or infinity. In fact the order of magnitude of $|\partial f_i^n/\partial m|$ can be at most $1/\sqrt{n}$ since this is the order of magnitude of $\|D_m f^n\|$. When preferences are representable by nice additive separable utility functions we know that the order of magnitude of $\partial f_i^n/\partial m$ is $1/n$. The upper bound on the income effect in terms of the number of goods translates immediately, by integrating twice and using the Slutsky equation, into an upper bound of the type $d(\Delta p_i)^2/2\sqrt{n}$, where d is an appropriate constant and $\Delta p_i = |p_i^1 - p_i^0|$, on the difference between the Marshallian and the Hicksian measure with n goods, Δ^n. On the other hand the Hicksian Deadweight Loss is bounded below by $(\Delta p_i)^2 b/2$ where b is a lower bound on the slope of the Hicksian demand function. We conclude that the order of magnitude of $\Delta^n/|\text{HDL}^n|$ is at most $1/\sqrt{n}$. Obviously this applies to $\Delta^n/|\text{HCS}^n|$ also since $|\text{HCS}^n| \geqq |\text{HDL}^n|$. In the additively separable case, and under the assumptions of the Theorem, all goods are normal and we can show that the order of magnitude of Δ^n is $1/n$. Proposition 1 states the results, a complete proof is given in the Appendix.

Proposition 1. *Under the assumptions of the Theorem suppose that only the price of good i changes and that we have n goods. Then the order of magnitude of the percentage error in approximating the HDL by the MDL ($\Delta^n/|HDL^n|$) is at most $1/\sqrt{n}$. In the additively separable case the order of magnitude of the percentage error is $1/n$.*

Example. Suppose that $U^n(x) = \sum_{i=1}^n \log x_i$ and that all the prices except the first are unity. Then p_1^{-1} and $p_1^{(1/n-1)}$ are respectively the Marshallian and Hicksian (evaluated at $U(1, \ldots, 1) = 0$) demands for good 1. Consider now an initial situation with $p_1 = 2$, and all other prices unity, and a final situation where the only change is that $p_1 = 1$. We may think of a situation where firms produce all goods at constant marginal cost equal to 1. All goods are priced competitively except the first one. We are evaluating the welfare gain of having p_1 decreased to the competitive level. We have,

$$|\text{MCS}^n| = \int_1^2 \frac{1}{p_1}\, dp_1 = \ln 2$$

and

$$|\text{HCS}^n| = \int_1^2 p_1^{(1/n-1)} dp = n(2^{1/n} - 1).$$

It is easily seen that $\lim_{n\to\infty}(n(2^{1/n} - 1) - \ln 2)n = (\ln 2)^2/2$ and therefore the order of magnitude of $\Delta^n = |\text{HCS}^n - \text{MCS}^n|$ is $1/n$. The loss in monopoly profits is $1/2$ since when $p_1 = 2$ $x_1 = 1/2$ and p_1 decreases one unity. Therefore $\text{MDL}^n = (\ln 2) - 1/2$ and $\text{HDL}^n = n(2^{1/n} - 1) - 1/2$. When $n = 3$ $\Delta^n/|\text{HCS}^n|$ and $\Delta^n/|\text{HDL}^n|$ are, approximately, 11% and 31%; when $n = 10$ they are 3% and 11%.

Multiple price changes

Suppose that we have n goods and k prices change (for simplicity suppose these are the k first goods). We are contemplating a price change from p^0 to $p^1 = (p_1^1, \ldots, p_k^1, p_{k+1}^0, \ldots, p_n^0)$. The HCS does not depend on the path one uses in going from p^0 to p^1, the MCS does. Suppose that prices are changed sequentially, each price being changed only once. We restrict attention to this type of *simple* sequences and denote by π_1 the ordering where p_1 is changed first, then p_2 and so on. The MCS is given by

$$\text{MCS}(p^0, p^1, m; \pi_1) = \sum_{i=1}^{k} \int_{p_i^0}^{p_i^1} f_i(p^{(i)}, m)\,dp_i$$

and the HCS is given by

$$\text{HCS}(p^0, p^1, m) = \sum_{i=1}^{k} \int_{p_i^0}^{p_i^1} h_i(p^{(i)}, u^1)\,dp_i$$

where m is the income of the consumer, $p^{(i)} = (p_1^1, \ldots, p_{i-1}^1, p_i, p_{i+1}^0, \ldots, p_n^0)$ and $u^1 = V(p^1, m)$.

The MCS depends on what particular simple sequence of price changes π we are considering. Nonetheless for n large enough the slopes of the marshallian and hicksian demand functions will be close and the MCS will be a good approximation of the HCS for *any* π.

Proposition 2. *Under the assumptions of the Theorem suppose that only k prices change and that we have n goods. It follows then that the approximation error done in measuring the HCS (or the HDL) using any simple sequence π of price changes to compute the MCS is bounded above by $\Delta p^\tau E \Delta p d/\sqrt{n}$ where d is an appropriate positive constant, $\Delta p = |p^0 - p^1|$ and E is a matrix of ones. If preferences are additively separable one may use $1/n$ instead of $1/\sqrt{n}$.*

Proof. See Appendix. ||

5. DOWNWARD SLOPING DEMAND

There are some results available in the literature about downward sloping *market* demand functions. Hildenbrand (1983) shows that if all individuals have a common demand function (which satisfies the Weak Axiom of Revealed Preference) and if the distribution of income is given by a decreasing density then all partial market demand curves are decreasing. Novshek and Sonnenschein (1979) consider a differentiated commodities model and decompose price-induced demand changes into three aggregate effects: substitution, income and change-of-commodity. They show that even if individual demand functions are upward sloping the change-of-commodity effect guarantees that market demand for a commodity must slope downward whenever there are differentiated commodities close to the given commodity. We give conditions under which *individual* demand is downward sloping and therefore it follows that *market* demand must be downward sloping too.

Under our assumptions individual demand must be downward sloping for n large enough from the Slutsky equation since the slope of the hicksian demand is bounded away from zero and the income effect goes to zero as n increases.

Proposition 3. *Under the assumptions of the Theorem for n large enough demand for any good is downward sloping.*

Proof. With n goods, consider the Slutsky equation of good i,

$$\frac{\partial f_i^n}{\partial p_i} = \frac{\partial h_i^n}{\partial p_i} - x_i^n \frac{\partial f_i^n}{\partial m}.$$

We know that $|\partial h_i^n/\partial p_i| \geqq b$, $x_i^n \leqq \bar{x}$ and $|\partial f_i^n/\partial m| \leqq c/\sqrt{n}$, where b, $\bar{x}$ and c are appropriate positive constants. Therefore

$$\frac{\partial f_i^n}{\partial p_i} \leqq -b + \frac{\bar{x}c}{\sqrt{n}}$$

which is negative for n large enough. ‖

It is clear that if we have L consumers all with individual downward sloping demand functions, the aggregate demand function will satisfy this property too.

Example. Consider the following case of a Giffen good due to Wold and Juréen ((1953), p. 102)

$$U(x, y) = \ln(x-1) - 2\ln(2-y).$$

where the domain of U, D, is $\{(x, y) \in R_{++}^2 : x > 1, y < 2\}$. It is easily checked that U is strictly quasiconcave and strictly monotone on D. Let p and q be the prices of the goods x and y respectively and let m be the income of the consumer. The demand for good x is

$$f(p, q, m) = \frac{2q-m}{p} + 2$$

and the demand for good y,

$$g(p, q, m) = 2\left(\frac{m-p}{q} - 1\right)$$

provided $2q + p > m > p + q$.

Good x is always inferior and it is Giffen if $m > 2q$ since $\partial f(p, q)/\partial p = (m-2q)/p^2$. The utility function $(\cdot)$ is not concave but the preferences it represents admit a concave representation on D. Let $V(x, y) = -e^{-tU(x,y)}$, then the Hessian of $V(\cdot)$ is negative definite on D if and only if $t > 1$, in which case $V(\cdot)$ is (differentiably) strictly concave on D. (By the way, this shows that Giffen goods are possible with additive separable preferences representable by a concave utility function.) Suppose now that we have n groups of two goods (x_i, y_i) as in the Wold and Juréen example and that total utility is given by

$$\textstyle\sum_{i=1}^n -e^{-tU(x_i,y_i)}.$$

Suppose as usual that the income available to the consumer in the n-group program is n, and that the prices of the x goods equal p and the prices of the y goods equal q where $(p, q) \in \{(p, q) \in R_{++}^2 : 2q + p > 1 > p + q\}$. By symmetry and since the utility function of each group, $V(x_i, y_i)$, is strictly concave it is clear that the demands arising from the n-group program

$$\text{Max} \textstyle\sum_{i=1}^n -e^{-tU(x_i,y_i)} \quad \text{subject to} \quad \sum_{i=1}^n (px_i + qy_i) = n$$

will be $x_i^n = f(p, q, 1)$ and $y_i^n = g(p, q, 1)$ for all i. The question is if good x_i, for instance, will be a Giffen good for all n and certain price configurations. Notice that strictly speaking Proposition 3 is not applicable to our example since the U.I.P. will not hold for the sequence of utility functions $\sum_{i=1}^n -e^{-tU(x_i,y_i)}$ (for instance, when y_i goes to zero its marginal utility does not go to infinity). Nonetheless we should get the result since we only use the U.I.P. to get bounded demands and in our example they are constant (independent of n). This is indeed the case, after some computations, the derivative of the demand function for good x_1 with respect to its own price p_1 equals (evaluated at $p_i = p$ and $q_i = q$ for all i)

$$\frac{1}{p}\left(\frac{2-x}{n}-\frac{2t-1}{t-1}(x-1)\left(1-\frac{1}{n}\right)\right)$$

where $x = 2+(2q-1)/p$. Where $q<1/2$, $1<x<2$ and $2-x>0$. When $n=1$ the expression reduces to $(2-x)/p$ and this is the Giffen case of Wold and Juréen. When $n \geqq 2$ we have a positive first term minus a positive second term since $t>1$. As n goes to infinity the slope of the demand function for good 1 converges to

$$-\frac{2t-1}{t-1}\frac{x-1}{p}.$$

We conclude thus that the demand for good x_1 is downward sloping for n large enough.

6. CONCLUDING REMARKS

We have found sufficient conditions on the preferences of a consumer for the income effect to be small and formalized Marshall's idea that if all commodities are like tea, which represents only a small part of the total expenditure of the consumer, then income effects are small. These sufficient conditions require the preferences of the consumer to be smooth and symmetric enough, and prevent any two goods from becoming too close substitutes. The smallness of the income effect was used to justify the Marshallian Consumer Surplus approximation to the Hicksian measure and to get downward sloping demand (for large n). All these applications of our Theorem have a Partial Equilibrium flavor. Are there any implications for General Equilibrium Theory? If we have a sequence of endowments $\{e_i\}_{i=1}^{\infty}$ which are in a compact and positive interval and we let the income of the consumer in the n-good program be $p^n e^n$ where $e^n = (e_1, \ldots, e_n)$ it is clear that our Theorem holds. An open question is what properties will have the aggregate excess demand of an economy with a finite number of consumers satisfying the assumptions of our Theorem if n is large enough.

APPENDIX

Proof of Lemma 1. It follows from Gershgorin's Theorem (See Franklin (1968), p. 161, for example). The Theorem says that every eigenvalue μ of an $n \times n$ matrix A (suppose it is real and symmetric) satisfies at least one of the inequalities

$$|\mu - a_{ii}| \leqq \textstyle\sum_{j\neq 1} |a_{ij}| \quad \text{for } i = 1, \ldots, n.$$

Now if A is Dominant Diagonal, that is, if $|a_{ii}| > \sum_{j\neq 1} |a_{ij}|$ for all i, then for any eigenvalue μ of A

$$|a_{ii}| - \textstyle\sum_{j\neq 1} |a_{ij}| \leqq |\mu| \leqq |a_{ii}| = \sum_{j\neq 1} |a_{ij}| \quad \text{for some } i$$

and if $a_{ii} < 0$ for all i then all μ must be negative. Let $\bar{x} \geqq \underline{x} > 0$, then by hypothesis there exist constants b and c, $b \geqq c > 0$ such that $|\partial_{ii} U^n(x)| \leqq b$ and $\partial_{ii} U^n(x) + \sum_{j\neq 1} |\partial_{ij} U^n(x)| \leqq -c$ for all $x_i \in [\underline{x}, \bar{x}]$, for all $1 \leqq i \leqq n$, and for all n. Given U^n the eigenvalues of $D^2U^n(x)$ must be negative for all $x \in R^n_{++}$ since $D^2U^n(x)$ has a Dominant (and negative) Diagonal for all $x \in R^n_{++}$ and therefore U^n is differentiably strictly concave. Furthermore given that consumption of any good is in the interval $[\underline{x}, \bar{x}]$ the absolute value of the eigenvalues of $D^2U^n(x)$ will be in the interval $[c, 2b]$. ‖

Proof of Proposition 1. Suppose that the price of good i goes from p_i^0 to p_i^1. Without loss of generality assume $p_i^1 \geqq p_i^0$. Fix n (and forget about the superscript n), suppose the consumer has income n and let

$u' = V(p^1, m)$. We have,

$$|\mathrm{HCS}-\mathrm{MCS}| = \left| \int_{p_i^0}^{p_i^1} (h_i(p, u') - f_i(p, m))dp_i \right| \leqq \int_{p_i^0}^{p_i^1} |h_i(p, u') - f_i(p, m)| dp_i$$

and integrating the Slutsky equation, noting that $h_i(p^1, u') = f_i(p^1, m)$.

$$\left| \int_{p_i^1}^{p_i} f_i(p, m) \frac{\partial f_i(p, m)}{\partial m} dp_i \right|$$

and therefore

$$|h_i(p, u') - f_i(p, m)| \leqq \int_{p_i^1}^{p_i} f_i(p, m) \left| \frac{\partial f_i(p, m)}{\partial p_i} \right| dp_i, \qquad p_i \in [p_i^0, p_i^1].$$

We know from our Theorem that $\|D_m f^n(p^n, n)\| \leqq c/\sqrt{n}$ where $c = \bar{\mu}/(\underline{\mu} \underline{p})$ and that $f_i^n(p^n, n) \leqq \bar{x}$. Therefore $\max_{1 \leqq i \leqq n} |\partial f_i^n(p^n, n)/\partial m| \leqq c/\sqrt{n}$ and $f_i^n(p^n, n)|\partial f_i^n(p^n, n)/\partial m| \leqq \bar{x}c/\sqrt{n}$. We have thus that

$$|h_i^n(p^n, u^{1,n}) - f_i^n(p^n, n)| \leqq |p_i - p_i^1| \frac{\bar{x}c}{\sqrt{n}} \quad \text{for } p_i \in [p_i^0, p_i^1]$$

where $u^{1,n}$ is the indirect utility corresponding to the n-good program when $p_i = p_i^1$. Therefore,

$$|\mathrm{HCS}^n - \mathrm{MCS}^n| \leqq \int_{p_i^0}^{p_i^1} |p_i - p_i^1| \frac{\bar{x}c}{\sqrt{n}} dp_i = \frac{1}{2} \frac{\bar{x}c}{\sqrt{n}} (p_i^1 - p_i^0)^2.$$

On the other hand from Remark 2 we know that the slopes of the hicksian demands are bounded away from zero and therefore the HDL^n is bounded away from zero. (It is easily checked that if $|\partial h_i^n/\partial p_i| \geqq b$ then $|\mathrm{HDL}|^n \geqq b(p_i^1 - p_i^0)^2/2$). Letting $\Delta^n = |\mathrm{HCS}^n - \mathrm{MCS}^n| = |\mathrm{HDL}^n - \mathrm{MDL}^n|$ we conclude that

$$\frac{\Delta^n}{|\mathrm{HDL}^n|} \leqq \frac{\bar{x}c}{b\sqrt{n}}.$$

Furthermore, if preferences are additively separable then all goods are normal (since the utility function for any good is concave) and from our Theorem we know that

$$\frac{\underline{x}c_2}{n} \leqq f_i^n(p^n, n) \frac{\partial f_i^n(p^n, n)}{\partial m} \leqq \frac{\bar{x}c_1}{n}$$

for appropriate constants c_1 and c_2, $c_1 \geqq c_2 > 0$. Since $\partial h_i^n/\partial p_i - \partial f_i^n/\partial p_i$ is always positive we can find a lower bound as well as an upper bound for $|h_i - f_i|$. We get

$$|p_i - p_i^1| \frac{\underline{x}c_2}{n} \leqq |h_i^n(p^n, u^{1,n}) - f_i^n(p^n, n)| \leqq |p_i - p_i^1| \frac{\bar{x}c_1}{n}, \qquad p_i \in [p_i^0, p_i^1]$$

and since $h_i^n - f_i^n \leqq 0$ if $p_i^1 \geqq p_i^0$ we get similarly as before

$$\frac{\underline{x}c_2}{2n} (p_i^1 - p_i^0)^2 \leqq |\mathrm{HCS}^n - \mathrm{MCS}^n| \leqq \frac{\bar{x}c_1}{2n} (p_i^1 - p_i^0)^2. \quad \|$$

Proof of Proposition 2. We suppose that the first three prices change according to the ordering π_1. Fix $n (n \geqq 3)$ and forget about the superscript n. We have

$$|\mathrm{MCS}-\mathrm{HCS}| = \left| \sum_{i=1}^3 \int_{p_i^0}^{p_i^1} (f_i(p^{(i)}, m) - h_i(p^{(i)}, u'))dp_i \right|$$

and therefore

$$|\mathrm{MCS}-\mathrm{HCS}| \leqq \sum_{i=1}^3 \left| \int_{p_i^0}^{p_i^1} |f_i(p^{(i)}, m) - h_i(p^{(i)}, u')| dp_i \right|.$$

Furthermore, for p_1 between p_1^0 and p_1^1, and letting $\bar{p}^{(i)} = (p_1^1, \ldots, p_i^1, p_{i+1}^0, \ldots, p_n^0)$ we have

$$|f_1(p^{(1)}, n) - h_1(p^{(1)}, u')| = \left| f_1(\bar{p}^{(1)}, n) - h_1(\bar{p}^{(1)}, u') + \int_{p_1^1}^{p_1} \left(\frac{\partial f_1(p^{(1)}, n)}{\partial p_1} - \frac{\partial h_1(p^{(1)}, u')}{\partial p_1} \right) dp_1 \right|$$

and

$$f_1(\bar{p}^{(1)}, n) - h_1(\bar{p}^{(1)}, u') = f_1(\bar{p}^{(2)}, n) - h_1(\bar{p}^{(2)}, u') + \int_{p_2^1}^{p_2^0} \left(\frac{\partial f_1(p^{(2)}, n)}{\partial p_2} - \frac{\partial h_1(p^{(2)}, u')}{\partial p_2} \right) dp_2$$

and

$$f_1(\bar{p}^{(2)}, n) - h_1(\bar{p}^{(2)}, u^1) = \int_{p_3^1}^{p_3^0} \left(\frac{\partial f_1(p^{(3)}, n)}{\partial p_3} - \frac{\partial h_1(p^{(3)}, u^1)}{\partial p_3} \right) dp_3.$$

Using the Slutsky equation and the bounds on $|\partial f_i/\partial m|$ from our Theorem, we get

$$|f_i(p^{(1)}, n) - h_i(p^{(1)}, u')| \leqq \frac{\bar{x}c}{\sqrt{n}} (|p_1 - p_1^1| + |p_2^0 - p_2^1| + |p_3^0 - p_3^1|)$$

where p_1 is between p_1^0 and p_1^1. Similarly one gets

$$|f_2(p^{(2)}, n) - h_2(p^{(2)}, u')| \leqq \frac{\bar{x}c}{\sqrt{n}} (|p_2 - p_2^1| + |p_3^0 - p_3^1|)$$

and

$$|f_3(p^{(3)}, n) - h_3(p^{(3)}, u')| \leqq \frac{\bar{x}c}{\sqrt{n}} |p_3 - p_3^1|$$

where p_2 and p_3 are netween p_2^0 and p_2^1 and p_3^0 and p_3^1 respectively. Integrating these expressions and adding up for the three goods we obtain

$$|\text{MCS} - \text{HCS}| \leqq \frac{\bar{x}c}{\sqrt{n}} (\textstyle\sum_{i=1}^{3} (p_i^0 - p_i^1)^2 + \sum_{i \neq j} |p_i^0 - p_i^1||p_j^0 - p_j^1|).$$

The right hand side equals $\bar{x}c/\sqrt{n}\, \Delta p^\tau E \Delta p$, where $\Delta p = |p^0 - p^1|$ and E is a matrix of ones. Our bound does not depend on the ordering of the sequence of price changes and therefore it is good for any ordering. It is straightforward to generalize the result for k price changes. ||

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NOTES

1. Obviously, the income effect would not necessarily dissipate as n grows if prices tended to zero even if expenditure shares were $1/n$ and demand was bounded. For example if prices were restricted to be in the simplex we would have $p^n = 1/n$ (since by assumption all are equal) and income was fixed at M then demand for good i would equal M and the income derivative of demand would equal one.

2. We will use the following notation. The strictly positive orthant of R^n is denoted by R^n_{++}. For a function $U: R^n_{++} \to R$, $DU(x)$ will denote the vector of first derivatives, $(\partial_i U(x))_{i=1}^n$, and $D^2U(x)$ the Hessian matrix of U, with entries $\partial_{ij}U(x)$; all evaluated at the point x. The vector inequality $\gg$ means strict inequality for every component. Transposition is denoted by τ.

3. If preferences are additively separable by groups (although maybe they do not satisfy the U.I.P.) it is possible to show that the order of magnitude of the norm of the income derivative of demand for any group is $1/n$. One puts assumptions on each group utility function so that its indirect utility function satisfies the U.I.P. with respect to income and then our Corollary can be used (See Vives (1982)).

4. I am grateful to an anonymous referee for pointing out that the result of the corollary applied to homothetic preferences.

5. In this respect see Bewley's formulation of the Permanent Income Hypothesis (Bewley (1977)).

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