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NASH EQUILIBRIUM WITH STRATEGIC COMPLEMENTARITIES

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Using lattice-theoretical methods, we analyze the existence and order structure of Nash equilibria of non-cooperative games where payoffs satisfy certain monotonicity properties (which are directly related to strategic complementarities) but need not be quasiconcave. In games with strategic complementarities the equilibrium set is always non-empty and has an order structure which ranges from the existence of a minimum and a maximum element to being a complete lattice. Some stability properties of equilibria are also pointed out.

1. Introduction

In this paper, we propose a powerful yet simple approach to study Nash equilibria in non-cooperative games. The central idea of this approach is to exploit order and monotonicity properties of the game using lattice-theoretical methods. With this new box of tools we are able, in the first place, to obtain results regarding the existence of Nash equilibria in games where payoff functions need not be quasiconcave. Those are out of reach when using the prevalent topologically-oriented techniques. In the second place, the lattice approach provides an order structure on the equilibrium set and some (tatonnement) stability properties independently of whether payoff functions are quasiconcave or not. The analysis is based on a fixpoint theorem due to Tarski (1955) and builds on the work of Topkis on the subject [Topkis (1979)].

The class of games where the lattice approach is most powerful is described by the presence of *strategic complementarities*, which yield monotone increasing best replies. In a differentiable setting the actions of two

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players are said to be strategic complements if the marginal profitability of a player increases with the action of the rival [see Bulow et al. (1983)]. Economic models where complementarities are important provide an environment conducive to *strategic* complementarities. Typical examples in differentiated oligopoly models include price competition with substitute products and quantity competition with complementary products. In macroeconomic models with imperfect competition strategic complementarities arise also naturally. In this context the ranking of the multiple equilibria will be very important. The economy can get stuck at a low activity equilibrium and there may exist a role for policy to move to a better equilibrium. [See Cooper and John (1985) and Heller (1985).]

The plan of this paper is as follows. Section 2 deals with lattices and Tarski's theorem. Section 3 with the monotonicity of optimal solutions in lattice programming. Section 4 considers abstract games in normal form and presents the basic existence results and order properties of the equilibrium set. Section 5 presents a note on (tatonnement) stability and section 6 considers Bayesian games. Section 7 gives examples and applications, including oligopoly games.

2. Lattices and Tarski's theorem[1]

Let $\geqq$ be a binary relation on a non-empty set S. The pair $(S, \geqq)$ is a *partially ordered set* (poset) if $\geqq$ is reflexive, transitive and antisymmetric.[2] A poset $(S, \geqq)$ is *(completely) ordered* if for x and y in S either $x \geqq y$ or $y \geqq x$. A *lattice* is a partially ordered set $(S, \geqq)$ in which any two elements have a least upper bound (supremum) and a greatest lower bound (infimum) in the set. For example, let $S \subset R^2$, $S = \{(1,0),(0,1)\}$, then S is not a lattice with the vector ordering since $(1,0)$ and $(0,1)$ have no joint upper bound in S. A lattice $(S, \geqq)$ is *complete* if every non-empty subset of S has a supremum and an infimum in S. Let $T \subset S$, where S is a complete lattice, and denote the least upper bound of T in S by $\sup_s T$ and the greatest lower bound of T in S by $\inf_s T$. A subset L of the lattice S is a *sublattice* of S if the supremum and infimum of any two elements of L belong also to L.

Let $(S, \geqq)$ be a poset. A function f from S to S is *increasing* (decreasing) if for x, y in S, $x \geqq y$ implies that $f(x) \geqq f(y)$ $(f(x) \leqq f(y))$. The following lattice-theoretical fixpoint theorem is due to Tarski (1955).

Theorem 2.1 (Tarski). *Let $(S, \geqq)$ be a complete lattice, f an increasing function from S to S and E the set of fixpoints of f, then E is non-empty and $(E, \geqq)$ is a complete lattice. In particular, this means that* $\sup_s E$ *and* $\inf_s E$ *belong to E.*

[1] For the theory of lattices see Birkhoff (1967).

[2] The binary relation $\geqq$ is antisymmetric if for x, y in S, $x \geqq y$ and $y \geqq x$ implies that $x = y$.

One may wonder whether a similar theorem holds for decreasing functions. It is trivial to see that, unfortunately, this is not the case.

Notice that Tarski's theorem is not asserting that the set E of fixpoints of $f\colon S \to S$ is a sublattice of S. That is, if x and y belong to E, it is not necessarily true that $\sup_s\{x,y\}$ and $\inf_s\{x,y\}$ also belong to E. What is true is that x and y have a supremum and an infimum in E. The following example[3] will clarify the issue. Let S be a finite lattice in R^2 consisting of the nine points (i,j) where i and j belong to $\{0,1,2\}$. Let $f\colon S \to S$ be such that all points are fixpoints except $(1,1)$, $(1,2)$ and $(2,1)$ which are mapped into $(2,2)$. S is a complete lattice and f is increasing. Consider $H=\{(0,1),(1,0)\}$, $H \subset E$. $\mathrm{Sup}_S H=(1,1)$ is not a fixpoint of f and therefore E is not a sublattice of S but certainly $\sup_E H=(2,2)$ does belong to E.

The conclusion in Tarski's theorem that the set of fixpoints E of f is a complete lattice is stronger than the assertion that $\inf_S E$ and $\sup_S E$ belong to E. Suppose that in our previous example all points in S are fixpoints with respect to a certain function g except $(1,1)$ which gets mapped into $(2,2)$. Then E would *not* be a complete lattice although $\inf_S E=(0,0)$ and $\sup_S E=(2,2)$ belong to E since $(0,1)$ and $(1,0)$ have no supremum in E. [$(2,2)$, $(1,2)$ and $(2,1)$ are all upper bounds of $(0,1)$ and $(1,0)$, but there is no least upper bound of $(0,1)$ and $(1,0)$ in E since $(1,1)$ is not a fixpoint of g.] Clearly g is not increasing since $g((1,1))=(2,2)$ but $g((1,2))=(1,2)$.

Theorem 2.1 can be improved upon when $(S,\geqq)$ is a completely and densely ordered lattice. That is, a completely ordered lattice for which for all x,y in S with[4] $x<y$, there is a z in S such that $x<z<y$. A function f from S to S is *quasi-increasing* if for every non-empty subset X of S, $f(\sup X) \geqq \inf f(X)$ and $f(\inf X) \leqq \sup f(X)$. $(f(X)=\{y \in S\colon y=f(x), x \in X\})$. If a function is quasi-increasing it cannot jump down, all the jumps must be up. A somewhat simplified version of Theorem 3 in Tarksi (1955, p. 250) follows.

Theorem 2.2 (Tarski). *Let $(S,\geqq)$ be completely and densely ordered lattice and f a quasi-increasing function on S to S. Denote the set of fixpoints of f by E. Then E is non-empty and $(E,\geqq)$ is a completely ordered lattice.*

3. Monotonicity of optimal solutions and lattice programming

Consider the following family of optimization problems indexed by a parameter t, $t \in T, \max\{g(x,t), x \in S\}$, where $\{S, \geqq_s)$ and $(T,\geqq_t)$ are non-empty lattices and $g\colon S \times T \to R$. Let $\phi(t)$ be the set of optimal solutions to the problem. We say that the correspondence ϕ from T to S is *increasing* if $t \geqq_t t', t \neq t'$, implies that for each s in $\phi(t)$ and each s' in $\phi(t'), s \geqq_s s'$.[5] If ϕ is

[3]The example was suggested by Andreu Mas-Colell.

[4]Let $(S,\geqq)$ be a poset and a and b be elements of S, then $a \geqq b$ and $a \neq b$. This is equivalent to require $a \geqq b$ to hold while $b \geqq a$ does not hold.

[5]When there is no risk of confusion we will drop the subindices of $\geqq$.

a function our definitions are the usual ones. Note that ϕ is increasing if and only if all selections of ϕ are increasing. Topkis (1978) examines the monotonicity properties of ϕ with respect to t.

Let $(S, \geqq)$ be a lattice and g a real valued function on S. We say that g is *supermodular* on S if for all x, y in S,

$$g(\min(x,y)) + g(\max(x,y)) \geqq g(x) + g(y).$$

g is *strictly supermodular* on S if the inequality is strict for all pairs x, y in S which cannot be compared with respect to $\geqq$.[6]

Let S and T be lattices and $g: S \times T \to R$. We say that g has (strictly) *increasing differences* in (s,t) if $g(s,t) - g(s,t')$ is (strictly) increasing in s for all $t \geqq t'$ $(t \geqq t', t \neq t')$. Decreasing differences are defined replacing 'increasing' by 'decreasing'. The concepts of supermodularity and increasing differences are closely related. As emphasized by Topkis the former is more convenient to work with mathematically while the latter is often more easily recognizable. They both formalize the idea of complementarity in a strategic setting. Supermodularity is a stronger property in general but for a function defined on a product of ordered sets the two concepts coincide [Topkis (1978, Theorems 3.1 and 3.2)].

For example, if $g: R^n \to R$ is twice-continuously differentiable then g is supermodular if and only if $\partial_{ij} g(x) \geqq 0$ for all x and $i \neq j$. If $\partial_{ij} g(x) > 0$ for all x and $i \neq j$, then g is strictly supermodular. The equivalence between the condition $\partial_{ij} g(x) \geqq 0$ and supermodularity for smooth functions can be motivated by thinking of the square with vertices $\{\min(x,y), y, \max(x,y), x\}$ and rewriting the definition of supermodularity as: $g(\max(x,y)) - g(x) \geqq g(y) - g(\min(x,y))$.

Lemma 3.1 below puts together some of Topkis' results.

Theorem 3.1. Let $g: S \times T \to R$ be supermodular on S for each t in T.

(*i*) *Then $\phi(t)$ is a lattice for all t.*
(*ii*) *If g has increasing (decreasing) differences in (s,t) and* $\sup\phi$ *and* $\inf\phi$ *exist and are selections of ϕ they are increasing (decreasing).*
(*iii*) *If g is strictly supermodular on S for each t in T, then $\phi(t)$ is ordered for all t.*
(*iv*) *If g has strictly increasing differences in (s,t), then ϕ is increasing.*

Proof. (i). Consider x and y in $\phi(t)$, then

$$0 \geqq g(\min(x,y),t) - g(x,t) \geqq g(y,t) - g(\max(x,y),t) \geqq 0.$$

The first and the last inequalities hold since $x \in \phi(t)$ and $y \in \phi(t)$ respectively,

[6]That is, neither $x \geqq y$ nor $y \geqq x$ holds.

the middle one since g is supermodular on S. We see that $\min(x, y)$ and $\max(x, y)$ belong to $\phi(t)$. Thus $\phi(t)$ is a lattice (in fact a sublattice of S).
(ii). Consider the case of increasing differences first. Let $x \in \phi(b)$ and $y \in \phi(t)$, $t \geqq b$, we claim that $\min(x, y) \in \phi(b)$ and $\max(x, y) \in \phi(t)$. Consider the following string of inequalities:

$$0 \geqq g(\max(x, y), t) - g(y, t) \geqq g(y, t) \geqq g(\max(x, y), b) - g(y, b)$$

$$\geqq g(x, b) - g(\min(x, b), b) \geqq 0.$$

The first and the last inequalities hold since $x \in \phi(t)$ and $y \in \phi(t)$ respectively, the second since g has increasing differences on $S \times T$, the third since g is supermodular on S. The claim follows.

Suppose now that $\sup \phi$ and $\inf \phi$ exist and are selections of ϕ. We show that $\sup \phi$ is increasing, that is, $t \geqq b$ implies that $\sup \phi(t) \geqq \sup \phi(b)$. We claim that $\sup \phi(t) \geqq x$, for all $x \in \phi(b)$. If $x \in \phi(b)$ then $\max(x, \sup \phi(t)) \in \phi(t)$ since $\sup \phi(t) \in \phi(t)$. Suppose it is not true that $\sup \phi(t) \geqq x$. Then $\max(x, \sup \phi(t)) \geqq \sup \phi(t)$ and $\max(x, \sup \phi(t)) \neq \sup \phi(t)$, which is a contradiction, since $\max(x, \sup \phi(t)) \in \phi(t)$. Similarly one shows $\inf \phi$ is increasing.

With decreasing differences the proof is analogous noticing that the claim above follows if $b \geqq t$.
(iii). Suppose now g is strictly supermodular on S. Let x and y belong to $\phi(t)$ and suppose they are not comparable with respect to $\geqq$. Since g is strictly supermodular on S we have

$$0 \geqq g(\max(x, y), t) - g(x, t) > g(y, t) - g(\min(x, y), t) \geqq 0,$$

which is a contradiction. Therefore, $\phi(t)$ is ordered for all t.
(iv). We show that $t \geqq b$, $t \neq b$ implies $y \geqq x$ for $x \in \phi(b)$ and $y \in \phi(t)$. Suppose it is not true that $y \geqq x$. Then $\max(x, y) \geqq y$ and $x \neq y$. Therefore the second inequality in the string considered in the proof of the claim in (ii) is strict because of strictly increasing differences, which provides the desired contradiction. Q.E.D.

Remark 3.1. If g is (strictly) supermodular on $S + T$ then it has (strictly) increasing differences on $S \times T$ and, obviously, g is (strictly) supermodular on S for any t in T.

Under what conditions will $\inf \phi$ and $\sup \phi$ exist and be selections of ϕ? For this matter we need to introduce some topological concepts. If $(S, \geqq)$ is a lattice its *interval topology* is defined by taking the sets of the type $\{z \in S: z \leqq x\}$ and $\{z \in S: x \leqq z\}$ to form a sub-basis for closed sets. The interest of this topology lies in the following result: a lattice is compact in its interval topology if and only if it is complete [Birkhoff (1967, Theorem 20)].

Lemma 3.1. *Let $g: S \times T \to R$ be supermodular on S for each t in T. If S is a lattice which is compact in a topology finer than its interval topology and g is upper semicontinuous (u.s.c.) on S then $\phi(t)$ is a non-empty compact and complete lattice for all t and $\sup\phi$ and $\inf\phi$ are selections of ϕ.*

Proof. $\phi(t)$ is non-empty and compact since g is u.s.c. on S and S is compact. We know $\phi(t)$ is a lattice from Theorem 3.1(i). According to the result of Birkhoff it will be complete since it is compact. Therefore $\sup\phi$ and $\inf\phi$ exist and are selections of ϕ. Q.E.D.

4. Abstract games

Consider an n-player game in normal form where A_i is the strategy set of player $i, i \in N$, the set of players. We assume that $(A_i, \geqq_i)$ is a complete lattice for all i. Let $A = X_{i=1}^n A_i$ and for any a, b in A say that $a \geqq b$ if $a_i \geqq_i b_i$ for all i, then $(A, \geqq)$ is a complete lattice. Player i has a payoff or utility function which gives rise to a best reply correspondence Ψ_i. That is, Ψ_i assigns a (non-empty) set of best replies for player i to any combination of strategies of the other player. Let $a_{-i} = (a_j)_{j \neq i}$ and $A_{-i} = X_{j \neq i} A_j$, Ψ_i goes from A_{-i} to the non-empty subsets of A_i. Recall that we say that Ψ_i is *increasing* if for all $j \neq i$, $a_j \geqq_j b_j$, with strict inequality for at least one, implies that for each x_i in $\Psi_i(a_{-i})$ and y_i in $\Psi_i(b_{-i})$, $x_i \geqq_i y_i$. Let Ψ be the product of the best reply correspondences, $\Psi = X_{i=1}^n \Psi_i$, Ψ goes from A to the non-empty subsets of A. Let E be the set of fixpoints of Ψ, that is the set of Nash equilibria of our game, $E = \{a \in A : a \in \Psi(a)\}$.

If Ψ_i is an increasing function for all i, then Ψ will be an increasing function from A to A, and from Tarski's theorem we know that the equilibrium set E will be a non-empty complete lattice. Obviously, if Ψ_i is a correspondence and has an increasing selection for all i then Tarski's theorem can be used again to show that E is non-empty.[7] Similarly, in a two-person game, if there is a decreasing selection for the best reply correspondence of any player, say g_i of Ψ_i, $i = 1, 2$, then the composite best reply map, $f: A_2 \to A_2$, $f = g_1 \circ g_2$, will be an increasing function, being the composition of two decreasing functions. The function f will have a fixpoint, say $\bar{a}_2$, according to Tarski's theorem and $(g_1(\bar{a}_2), \bar{a}_2)$ will be the desired fixpoint of g.

The above arguments nevertheless are silent with respect to order structure of the equilibrium set E. Theorem 4.1 addresses this issue extending Tarski's theorem to correspondences for the case of Abstract Games.

[7]An analogous argument shows that in symmetric games there will exist symmetric equilibria. That is, if $(A_i \geqq_i) = (A_j \geqq_j)$, $\Psi_i = \Psi_j$ and Ψ_i has an increasing selection for all i and j then there is $a^* \in \Psi(a^*)$ and $a_i^* = a_j^*$ for all i and j. This follows by restricting Ψ to $\bar{A} \equiv \{a \in A : a_i = a_j$, all i and $j\}$ and noticing that $\bar{A}$ is a complete lattice.

Theorem 4.1. *Assume that* $(A_i, \geqq_i)$ *is a complete lattice for all i, then*

(*i*) *if* $\inf \Psi_i$ *and* $\sup \Psi_i$ *are increasing selections of* Ψ_i *for all i, then E has a largest and a smallest element;*

(*ii*) *if for all i* Ψ_i *is increasing and for all a in A,* $\Psi_i(a_{-i})$ *has a smallest element and* $\Psi_i(a_{-i}) \cap \{x_i \in A_i: a_i \geqq_i x_i\}$ *has a largest element if non-empty, then E is a (non-empty) complete lattice.*

Proof. (i). By assumption $\inf \Psi_i$ is an increasing selection of Ψ_i, and therefore $\inf \Psi$ is an increasing selection of Ψ. From Tarski's theorem, we know that $\underline{x} \equiv \inf\{x \in A: \inf \Psi(x) \leqq x\}$ belongs to E. We claim that $\underline{x} = \inf E$. Let $a \in E$, then $a \in \Psi(a)$ and $a \geqq \inf \Psi(a) \geqq \underline{x}$. Similarly with $\sup E$.

(ii). We construct an increasing selection g of Ψ with the property that $E = \{a \in A: a = g(a)\}$. The result then follows from Tarski's theorem since A is a complete lattice. Given any $a \in A$ let

$$g_i(a) = \begin{cases} \max\{\Psi_i(a_{-i}) \cap \{x_i \in A_i: a_i \geqq_i x_i\}\} & \text{if} \quad a_i \geqq_i \min \Psi_i(a_{-i}) \\ \min \Psi_i(a_{-i}) & \text{otherwise.} \end{cases}$$

Now, $E = \{a \in A: a = g(a)\}$ since by construction g is a selection of Ψ and if $a \in E$, $a \in \Psi(a)$ or $a_i \in \Psi_i(a_{-i})$ for all i and then $g_i(a) = a_i$ for all i. Furthermore g is increasing, that is, $a \geqq b$ implies that $g(a) \geqq g(b)$ for any a and b in A. If a and b are such that for some i $a_j \geqq_j b_j$, $j \neq i$, with strict inequality for at least one, then $g_i(a) \geqq_i g_i(b)$ since Ψ_i is increasing. If a and b are such that $a_{-i} = b_{-i}, a_i >_i b_i$ for some i, then $g_i(a) \geqq_i g_i(b)$ according to our construction. Q.E.D.

Remark 4.1. A similar theorem could be stated for general correspondences, providing thus an extension of Tarski's theorem. Nevertheless, the analog of result (ii) for general correspondences would not be useful in the context of Abstract Games since even if all individual best reply correspondences Ψ_i are increasing the product of them $\Psi \equiv X_i \Psi_i$ will *not* be necessarily increasing. This is easily understood. If for some i $a_{-i} = b_{-i}$ and $a_i >_i b_i$ then $\Psi_i(a_{-i}) = \Psi_i(b_{-i})$ and, obviously, it is not true that $x_i \geqq_i y_i$ for each x_i in $\Psi_i(a_{-i})$ and y_i in $\Psi_i(b_{-i})$. Therefore Ψ cannot be increasing unless it is a function.

Remark 4.2. If we endow the complete lattice $(A_i, \geqq_i)$ with a topology finer than its interval topology (note that this makes A_i compact) and assume $\Psi_i(a_{-1})$ to be closed and ordered for all a_{-i} in $X_{j \neq i} A_j$ then $\Psi_i(a_{-j})$ has a smallest element and $\Psi_i(a_{-i}) \cap \{x_i \in A_i: a_i \geqq_i x_i\}$ has a largest element if non-empty. This is clear. (a) $\Psi_i(a_{-i})$ is compact since it is a closed subset of the compact set A_i. (b) $\Psi_i(a_{-i}) \cap \{x_i \in A_i: a_i \geqq_i x_i\}$ is also closed (being the intersection of closed sets) and therefore compact; furthermore, it is ordered since $\Psi_i(a_{-i})$ is ordered. Both sets are compact ordered sets and therefore have smallest and largest elements.

We can now put together the results on games with monotone best responses with the characterization of payoffs which yield the appropriate monotonicity conditions.

Theorem 4.2. Let A_i be a lattice compact in a topology finer than its interval topology and $\pi_i: A \to R$, $A = X_{i=1}^n A_i$, upper semicontinuous on A_i, for all i. Then

(*i*) *if π_i is a supermodular on A_i and has increasing differences in (a_i, a_{-i}) the equilibrium set is non-empty and a largest and smallest equilibrium point exist;*
(*ii*) *if π_i is strictly supermodular on A_i and has strictly increasing differences in (a_i, a_{-i}) the equilibrium set is a non-empty complete lattice;*
(*iii*) *if $n=2$ and for $i=1,2$ π_i is supermodular on A_i and has decreasing differences in (a_i, a_j), $j \neq i$, then an equilibrium exists.*

Proof. Under the assumptions the best response correspondence of player *i*, Ψ_i, is compact valued.

(i) According to Theorem 3.1 and Lemma 3.1 $\sup \Psi_i$ and $\inf \Psi_i$ are increasing selections of Ψ_i. Thoerem 4.1(i) implies then that a largest and smallest equilibrium point exist.
(ii) Theorem 3.1 implies that Ψ_i is increasing and that $\Psi_i(a_{-i})$ is ordered for all $a_{-i} \in X_{j \neq i} A_j$. Theorem 4.1(ii) and Remark 4.2 imply then that E is a (non-empty) complete lattice.
(iii) From Theorem 3.1 and Lemma 3.1 we know that $\sup \Psi_i$ will be a decreasing selection of Ψ_i, For $n=2$ then Tarski's theorem can be used on the composite best reply map to yield the existence of an equilibrium point. Q.E.D.

Remark 4.3. Part (i) of the theorem is due to Topkis (1979).

Remark 4.4. If each A_i is a product of compact intervals of the reals and π_i is smooth (twice continuously differentiable) then π_i will be supermodular on A if and only if for all a in A.

$$\partial^2 \pi_i / \partial a_{ih} \partial a_{ik} \geqq 0 \quad \text{for all } k \neq h \quad \text{and}$$

$$\partial^2 \pi_i / \partial a_{ih} \partial a_{jk} \geqq 0 \quad \text{for all } j \neq i \text{ and for all } h \text{ and } k.^8$$

[8] a_{ih} denotes the h action of player i.

If the condition is satisfied (i) in the theorem will hold. If the inequalities are strict then π_i will be strictly supermodular on A and (ii) in the theorem will hold. For (iii) to hold reverse the second set of the above inequalities.

Remark 4.5. Under the assumptions of (i) in the theorem if the payoff to a player is increasing in the strategies of the other players then the payoffs associated to the largest ($\sup E$) and smallest ($\inf E$) equilibrium points provide bounds for equilibrium payoffs for each player. If (ii) holds then tighter bounds on payoffs associated to any subset of equilibria, $A \subset E$, may be provided by $\sup_E A$ and $\inf_E A$, which are themselves equilibria since E is a complete lattice.

5. A note on stability

Equilibria of games with supermodular payoffs, yielding monotone increasing best responses, have nice stability properties. This contrasts with the possible 'chaotic' dynamics associated with games with non-monotone best responses. [See, for example, Rand (1978) for an analysis of duopoly models.]

A *Cournot tatonnement* is defined by the process: $a^0 \in A$, $a^t \in \Psi(a^{t-1})$, $t = 1, 2, \ldots$, where, as before, Ψ is the product of the best reply correspondences of the players. We make the convention that if for some t and i, $a_{-i}^{t+1} = a_{-i}^t$ then player i chooses $a_i^{t+2} = a_i^{t+1}$. That is, if the rivals of player i choose the same strategies in t and $t+1$ then player i also chooses the same strategy in $t+2$ as in $t+1$.

Let

$$A_i^+ = \{a \in A : a_i \geqq \sup \Psi_i(a_{-i})\}, \qquad A_i^- = \{a \in A : a_i \leqq \inf \Psi_i(a_{-i})\},$$

$$A^+ = \bigcap_{i=1}^n A_i^+ \quad \text{and} \quad A^- = \bigcap_{i=1}^n A_i^-.$$

The following theorem establishes monotone convergence to an equilibrium point of the game whenever the starting point is 'below' or 'above' all the best reply correspondences of the players, that is whenever $a^0 \in A^-$ or $a^0 \in A^+$.

Theorem 5.1. Let A_i be a lattice compact in a topology finer than its interval topology and $\pi_i : A \to R$, $A = X_{i=1}^n A_i$, continuous on A (endowed with the product topology), supermodular on A_i and with strictly increasing differences in (a_i, a_{-i}) on $A_i \times A_{-i}$ for all i. Then a Cournot tatonnement starting at any a^0 in A^+ (A^-) converges monotonically downwards (upwards) to an equilibrium point of the game.

Proof. Let $a^0 \in A^+$, then for any i, $a_i^0 \geqq \sup \Psi_i(a_{-i}^0) \geqq a_i^1$ since $a_i^1 \in \Psi_i(a_{-i}^0)$.

Any best reply correspondence is increasing since payoffs show strictly increasing differences [Theorem 3.1(iv)]. Therefore $a_i^1 \geqq a_i^2$ since $a_i^1 \in \Psi_i(a_{-i}^0)$, $a_i^2 \in \Psi_i(a_{-i}^1)$ and either $a_{-i}^0 \geqq a_{-i}^1$, $a_{-i}^0 \neq a_{-i}^1$ or $a_{-i}^0 = a_{-i}^1$ and then $a_i^1 = a_i^2$ according to our convention. We have therefore $a^0 \geqq a^1 \geqq a^2$. The Cournot tatonnement defines thus (reasoning by induction) a monotone decreasing sequence $\{a^t\}$, $a^t \geqq a^{t+1}$ for all t. This decreasing sequence defines in turn a nested sequence of (non-empty) closed sets $C^t = \{a \in A: a \leqq a^t\}$ in the compact space A which satisfies the finite intersection property. Therefore the intersection of the collection of closed sets C^t is non-empty and equal to the infimum of the sequence. The point $\hat{a} = \inf\{a^t\}$ is a limit point of the sequence $\{a^t\}$. This point must also be an equilibrium point, $\hat{a} \in \Psi(\hat{a})$, by continuity of the payoffs. For any t, $\pi_i(a_i^t, a_{-i}^{t-1}) \geqq \pi_i(a_i, a_{-i}^{t-1})$ for all a_i in A_i since $a_i^t \in \Psi_i(a_{-i}^{t-1})$. Since Π_i is continuous on A and $a^t \xrightarrow[t]{} \hat{a}$ we have that $\pi_i(\hat{a}_i, \hat{a}_{-i}) \geqq \pi_i(a_i, \hat{a}_{-i})$ for all a_i in A_i, and therefore $\hat{a}_i \in \Psi_i(\hat{a}_{-i})$. If $a^0 \in A^-$ the proof follows along the same lines. Q.E.D.

Remark 5.1. A similar argument was used in Vives (1985a, b). Topkis (1979) obtains related results.

Remark 5.2. Suppose that strategy spaces are compact intervals and that best replies are strictly increasing continuously differentiable functions $g_i(\cdot), i = 1, \ldots, n$ (that is, we have $\partial g_i / \partial a_j > 0, j \neq i$). The results of Hirsch (1985, Theorem 5.1) imply then that the *continuous* Cournot tatonnement

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = g_i(a_{-i}(t)) - a_i(t), \qquad i = 1, \ldots, n,$$

converges to an equilibrium point of the game for almost all starting points a^0 in A. When $n = 2$ and best replies are either strictly increasing or strictly decreasing convergence everywhere, as opposed to almost everywhere, obtains [Hirsch (1985, Corollary 2.8)].

6. Bayesian games

Let the action spaces be compact lattice subsets of Euclidean spaces and T_i the set of types of player i, a non-empty complete separable metric space. Denote by T the cartesian product of the sets of types of the players, $T = X_{i=1}^n T_i$. The common beliefs of the players are represented by μ, a probability measure on the Borel subsets of T. The measure μ_i will represent the marginal on T_i. The payoff to player i is given by $\pi_i: A \times T \to R$, Borel measurable and bounded. A (pure) strategy for player i is a (Borel measurable) map $\sigma_i: T_i \to A_i$ which assigns an action to every possible type of the player. Let $\sum_i(\mu_i)$ denote the strategy space of player i when we identify strategies σ_i and τ_i if they are equal μ_i-almost surely (a.s.)

Let

$$P_i(\sigma)=\int_T \pi_i(\sigma_1(t_1),\dots,\sigma_n(t_n),t)\mu(\mathrm{d}t).$$

The function $P_i(\cdot)$ is the expected payoff to player i when agent j uses strategy σ_j, $j\in N$. A Bayesian Nash equilibrium is a Nash equilibrium of the game where player i's strategy space is $\sum_i(\mu_i)$ and its payoff function P_i.

The first step to use the lattice machinery on the Bayesian game is to show that $\sum_i(\mu_i)$ is a complete lattice for some appropriate ordering. We will say that $\sigma_i\leqq\tau_i$ if $\sigma_i(t_i)\leqq\tau_i(t_i)$ for μ_i-a.s. T_i, and we will refer to this ordering as the natural ordering. We have to show that every non-empty subset of $\sum_i(\mu_i)$ has a supremum and an infimum under the natural ordering. This is not immediate since the supremum of an uncountable set of functions need not be measurable. Lemma 6.1 states the result.

Lemma 6.1. *$\sum_i(\mu_i)$ is a complete lattice under the natural ordering.*

Proof. We have to show that every non-empty subset of $\sum_i(\mu_i)$ has a supremum and an infimum. Let $\Omega\subset\sum_i(\mu_i)$ clearly $\sup\Omega$ (let $\omega=\sup\Omega$) exists since A_i is compact. We have to check that every component of ω is measurable, then ω is measurable [see Hildenbrand (1974, p. 42)]. Let $\sum_{ih}(\mu_i)=\{\sigma_{ih}: T_i\to A_{ih}, \sigma_{ih}$ Borel measurable$\}$ (identify functions which are equal μ_i a.s.) where A_{ih} is the projection of A_i on the hth coordinate. Let Ω_h be the subset of $\sum_{ih}(\mu_i)$ consisting of the hth components of the functions of Ω, then $\omega_h=\sup\Omega_h$. Note that $\sum_i(\mu_i)\subset L^1(\mu_i)$ [$L^1(\mu_i)$ stands for the quotient space of the set of μ_i-integrable real valued function on T_i] since $\mu_i(T_i)=1$ and A_{ih} is compact. $L^1(\mu_i)$ is a conditionally complete lattice, that is, every bounded non-empty subset of $L^1(\mu_i)$ has a supremum and an infimum [see Birkhoff (1967, p. 51 and p. 241)]. Also $\Omega_h\subset L^1(\mu_i)$ and therefore $\sup\Omega\in\sum_i(\mu_i)$. Similarly one shows that $\inf\Omega\in\sum_i(\mu_i)$. Q.E.D.

The second step is to realize that supermodularity is preserved under integration. Theorem 6.1 states the result.

Theorem 6.1. *Let action sets be compact lattice subsets of Euclidean spaces, type sets be complete separable metric spaces and $\pi_i: A\times T\to R$ be bounded, upper semicontinuous on A_i and Borel measurable for all i. Then*

(*i*) *if for any i π_i is supermodular on A for all t in T the equilibrium set is non-empty and has a largest and a smallest point;*

(ii) *if* $n=2$ *and* $g_i(a_i, a_j, t) \equiv \pi_i(a_i, -a_j, t)$, $j \neq i$, $i=1,2$, *is supermodular on* A *for all* t *in* T *an equilibrium exists.*

Proof. T_i is a complete separable metric space and therefore it is a Borel space (that is, there is a one-to-one map between T_i and some Borel subset of $[0,1]$ which is Borel measurable in both directions). T_{-i} is also a Borel space and consequently there exists a regular conditional distribution on T_{-i} given t_i. [See Ash (1972, p. 265.] Denote by $\sigma_{-i}(t_{-i})$ the vector $(\sigma_1(t_1), \ldots, \sigma_n(t_n))$ except the ith component and let the expected payoff to player i conditional on t_i when the other players use σ_{-i} and player i uses a_i be $\mathrm{E}\{\pi_i(a_i, \sigma_{-i}(t_{-i}), t) \,|\, t_i\}$. Let $\Psi_i(\sigma_{-i})$ be the set of best responses of player i to the strategy profile of the other players, σ_{-i}. The action $\sigma_i(t_i)$ maximizes over A_i the conditional payoff $\mathrm{E}\{\pi_i(a_i, \sigma_{-i}(t_{-i}), t) \,|\, t_i\}$ μ_i a.s. T_i. $\mathrm{E}\{\pi_i(a_i, \sigma_{-i}(t_{-i}), t) \,|\, t_i\}$ is upper semicontinuous on A_i since π_i is bounded and upper semicontinuous on A_i (this follows easily from Fatou's lemma). Furthermore, it is supermodular on A_i since $\pi_i(a, t)$ is supermodular in a_i for all t and all a_{-i} and supermodularity is preserved by integration. It follows from Lemma 3.1 that the set of maximizers given t_i is a non-empty compact and complete lattice and its supremum and its infimum are themselves maximizers. We have then that $\sup \Psi_i(\sigma_{-i})$ and $\inf \Psi_i(\sigma_{-i})$ belong to $\Psi_i(\sigma_{-i})$. In case (i) $\pi_i(a,t)$ is supermodular in a for all t and $P_i(\sigma)$ is also supermodular in σ ($\sigma \in \sum, \sum = X_{i=1}^{n} \sum_i$, and recall that $\sum$ is a complete lattice). Theorem 3.1(ii) and Remark 3.1 establish then that $\sup \Psi_i$ and $\inf \Psi_i$ are increasing selections of Ψ_i. Theorem 4.1(i) implies that there exist a largest and a smallest equilibrium point. In case (ii) $\pi_i(a_i, -a_j, t)$ is supermodular on A for all t, and consequently $P_i(\sigma_i, -\sigma_j)$ is supermodular on $\sum$, $j \neq i$, and $P_i(\sigma_i, \sigma_j)$ has decreasing differences on $\sum_i \times \sum_j$. In that case, $\sup \Psi_i$ is a decreasing selection of Ψ_i [Theorem 3.1(ii)] and existence follows applying Theorem 2.1 (Tarski) to the composite best reply map. Q.E.D.

Remark 6.1. There are several results available in the literature on the existence of pure strategy equilibria in Bayesian games [e.g. Radner and Rosenthal (1982) and Milgrom and Weber (1985)]. In these papers restrictions are put on the action space (A_i finite, for example) and on the distributions allowed. Furthermore the complete information counterpart of the games considered may not have pure strategy equilibria. By contrast, our conditions imply existence of pure strategy equilibria in the certainty games, and this translates, with no distributional restrictions, into the existence of pure strategy Bayesian equilibria.

7. Applications and examples

Models where *complementarities*, in a strategic sense, are fundamental constitute the ground where the tools provided by the lattice approach prove useful. This should be clear since, precisely, we say that the actions of players in a game are complementary from a strategic point of view when best responses are monotone increasing.[9]

Oligopoly pricing and oligopolistic competition in general are examples where the lattice theory approach can be applied successfully.

Non-existence of Nash equilibrium is a pervasive problem in oligopoly models. Examples of duopoly models where firms can produce at no cost and where demands arise from well-behaved preferences in which no Nash equilibrium (in pure strategies) exists are easily produced. In these examples payoffs are not quasiconcave and the best response correspondence of one firm (which gives the profit-maximizing response to the action of the other firms) is not convex-valued, that is, it has at least one jump.[10]

There are several results available in the oligopoly literature about existence of Nash equilibrium without quasiconcave payoffs. In a homogeneous product setting McManus (1964) and Roberts and Sonnenschein (1976) showed the existence of a symmetric Cournot equilibrium allowing for a general downward sloping demand when there are n *identical* firms with convex costs. In this context, the best response correspondence of a firm may slope up or down but all jumps up and the existence of a *symmetric* equilibrium is established. The essence of the McManus, Roberts–Sonnenschein result is a fixpoint theorem which says that a function from $[0,1]$ to $[0,1]$ has a fixpoint if the only discontinuities it has are jumps up. This result follows quite directly from the work of Tarski: just let $S=[0,1]$ in Theorem 2.2. Bamon and Fraysse (1985) and Novshek (1985) have shown, using a different approach from the one presented in this paper, existence of a Cournot equilibrium with n firms in a market for a homogeneous good if each firm's marginal revenue is declining in the aggregate output of the other firms.[11]

7.1. *Oligopoly games*

Consider an n-player oligopoly game where the strategy space of player (firm) i, A_i, is a compact interval, and where its payoff function, Π_i, can be decomposed as the sum of a revenue function $R_i\colon A \to R_+$ and a cost function $C_i\colon A_i \to R_+ : \pi_i(a) = R_i(a) - C_i(a_i)$. Strategies can be prices, quantities or R&D or advertising expenditure levels, for example.

Suppose for a moment that we are in a very nice case: π_i is twice-

[9] Bulow et al. (1983) say then that actions are 'strategic complements'.

[10] See Roberts and Sonnenschein (1977) and Friedman (1983, p. 67–69) for non-existence examples. Dasgupta and Maskin (1986) give an argument to put the blame for non-existence on the lack of quasiconcavity of payoffs.

[11] Nishimura and Friedman (1981) also examine the existence problem without quasiconcave payoffs.

continuously differentiable and the ith player best reply to a_{-i} is unique, interior and equal to $r_i(a_{-i})$. We know then that the first-order condition for profit maximization will be satisfied: $\partial_i \pi_i(r_i(a_{-i}), a_{-i}) = 0$. Furthermore, if $\partial_{ii}\pi_i(r_i(a_{-i}), a_{-i}) < 0$ the best reply function r_i is differentiable and $\partial r_i/\partial a_j = -\partial_{ij}R_i/\partial_{ii}\pi_i$, $j \neq i$. We see that r_i is monotonically increasing if and only if $\partial_{ij}R_i \geqq 0$. The profit function π_i need not be single peaked in general but as long as $\partial_{ij}R_i \geqq 0$ for all a in A, $j \neq i$, it will be supermodular and consequently an equilibrium will exist. Obviously the cost function need only be lower semicontinuous for the result to obtain. According to Theorem 4.2(i), $\partial_{ij}R_i \geqq 0, j \neq i$, implies that the equilibrium set E is non-empty and a largest and a smallest equilibrium point exist. According to Theorem 4.2(ii), $\partial_{ij}R_i > 0, j \neq i$, implies that E is a complete lattice. If $n=2$ or 3 it can be shown that E is in fact ordered. For the case $n=2$ and $\partial_{ij}R_i \leqq 0, j \neq i$, an equilibrium can be shown to exist since then π_i has increasing differences on $A_i \times A_j$, $j \neq i$ [Theorem 4.2(iii)].

These results extend in straightforward way to the case of multi-dimensional strategy spaces provided strategy sets are products of compact intervals and the cost function of any firm is additively separable.

7.2. Examples

7.2.1. Bertrand competition in differentiated markets

Consider an n-firm oligopoly with product differentiation. Every firm produces a single product. Firm i's strategy set is a compact interval of prices, $[o, \bar{p}_i]$ and there are no fixed costs.[12] Given a demand system $x_i = h_i(p)$, $i = 1, \dots, n, p \in R^n_+$, profits of firm i are given by $\pi_i(p) = p_i h_i(p) - C_i(h_i(p))$. Assuming that they are a smooth function of prices, the condition $\partial_{ij}\pi_i \geqq 0, j \neq i$, means that the marginal profitability of firm i increases with the prices charged by rival firms. This is reasonable if the goods are gross substitutes ($\partial_j h_i > 0, j \neq i$), demand is downward sloping ($\partial_i h_i < 0$) and costs are convex ($C_i'' \geqq 0$); since $\partial_{ij}\pi_i = (p_i - C_i')\partial_{ij}h_i + (1 - \partial_i h_i C_i'')\partial_j h_i$, all that is needed is that the second summand (which is always positive) dominates the first. When a rival increases its price, firm i wants to increase its price too. Nevertheless, even with product differentiation, it is a strong assumption to suppose that revenues are smooth on the cartesian product of the price spaces. Demands may have kinks when one firm is priced out of the market [see Friedman (1983)]. Supermodularity (increasing differences) again will be a natural assumption to make with gross substitutes.[13] In a multiproduct

[12] We take Bertrand competition to mean that when firm i announces a price p_i, it is committed to sell whatever demand is forthcoming at that price, even if it has to produce where marginal cost exceeds p_i.

[13] Alternatively the condition $\partial_{ij}\pi_i \geqq 0, j \neq i$, can be imposed on the set of prices for which all firms have positive demand and best reply correspondences characterized directly outside this region. An example of this approach is given in Vives (1985b, Proposition 1).

context the required conditions are more stringent. [See Spady (1984) for an example of price-setting multiproduct firms where best responses are increasing.]

7.2.2. Cournot competition in differentiated markets

Consider an n-firm oligopoly. Each firm produces one differentiated product. Costs of firm i are given by a lower semicontinuous increasing function C_i. (Notice avoidable fixed costs are allowed). The inverse demand system f satisfies: (a) $f_i: R^n_+ \to R^+$ is a continuous function for all i. Let $X_i = \{X \in R^n_+ : f_i(x) > 0\}$ and $\bar{x}_i = \sup\{x_i \in R_+ : x \in x_i\}$. Assume that $0 < \bar{x}_i < \infty$ for all i. (b) f_i is twice-continuously differentiable on X_i, $\partial_i f_i < 0$ and for $j \neq i$, $\partial_j f_i < 0$ if the goods are substitutes or $\partial_j f_i < 0$ if they are complements, for all i. Under (a) and (b), $A_i = [0, \bar{x}_i]$, $R_i(x) = f_i(x) x_i$ and R_i is twice-continuously differentiable on x_i.[14]

If $n = 2$ and $\partial_{ij} R_i(x) \leqq 0$ for all $x \in X_i$, $j \neq i, i = 1,2$, then according to Theorem 4.2(iii) a Nash equilibrium exists. Notice that $\partial_{ij} R_i = \partial_j f_i + x_i \partial_{ij} f_i$. This is likely to be non-positive if the goods are *substitutes* ($\partial_j f_i < 0$).

If $\partial_{ij} R_i(x) \geqq 0$ for all $x \in X_i$, $j \neq i, i \in N$, then an equilibrium exists. If the inequality is strict then the equilibrium set E is a complete lattice. The cross partial $\partial_{ij} R_i$ is likely to be non-negative when the goods are complements ($\partial_j f_i > 0$). In fact, Spence calls two complementary goods i and j *strongly complementary* if $\partial_{ij} R_i > 0$ [Spence (1976, p. 220)]. There is always a Cournot equilibrium and the equilibrium set is a complete lattice if the goods are strongly complementary.

7.2.3. Product selection and complementary products

It is well known that complementary products tend to be undersupplied in a Cournot equilibrium, that is, there are too few products and quantities are too low.[15] Spence (1976) claims that if products are strongly complementary, then there should be an equilibrium in which all quantities are below the optimal quantities and some of the optimal products are not produced. Obviously, this proposition makes sense only if an equilibrium is guaranteed to exist. Spence did not address this issue. As we have seen existence follows from our approach though.

Suppose that the inverse demand system comes from the following

[14]Note that $\Psi_i(x_{-i}) \subset X_i$ for all $x_{-i} \in A_{-i}$ since out of X_i firm i gets no revenue and by setting a smaller x_i in a way that $x \in X_i$ the revenue of firm i is positive and its costs are less since C_i is increasing.

[15]'The intuitive reason is that when a monopolistically competitive firm holds back output and raises price above marginal cost, it reduces the demand for other complementary products. That induces further quantity cast-backs and possibly the exit of products from the markets as well. That cycle reinforces itself and leads to an equilibrium where all outputs are below the optimum and some of the products in the optimal set are not produced at all.' [Spence (1977, p. 220).]

maximization program, $\max\{U(x)-px, x\in R^n_+\}$, where $U(\cdot)$ is a three times continuously differentiable concave utility function. The potential product set is N and the costs to firm i of producing a positive amount x_i are $F_i+V_i(x_i)$ where $F_i \geqq 0$ and V_i is a twice-continuously differentiable, increasing and convex variable cost function. If the firm decides not to produce F_i is avoidable. Furthermore, assume that revenue net of variable cost for firm i is strictly concave in x_i, that the goods are complementary and $\partial_{ij}R_i \geqq 0, j \neq i$, so that the best reply map of firm i is an increasing function of the quantities produced by the rivals whenever the firm produces a positive amount. The following proposition strengthens Spence's result and takes care of the existence problem.

Proposition 7.1. Under the assumptions above there is a Cournot equilibrium with less products and less production than any welfare optimum.

Sketch of Proof. Given any welfare optimum, Theorem 4.2(i) and an argument similar to Spence (1976, p. 221) and Vives (1985a, p. 172) ensures the existence of a Cournot equilibrium with less products and production than the welfare optimum. Theorem 4.2(i) ensures also that the equilibrium set has a smallest element. This is the desired Cournot equilibrium. See Vives (1985b) for a detailed account.

7.2.4. Bertrand and Cournot equilibria

Bertrand and Cournot equilibrium prices have been compared for a market with n differentiated products which are gross substitutes. The following proposition is a strengthening of Proposition 2 in Vives (1985a).

Proposition 7.2. Suppose that $\pi_i(p)$ is strictly quasiconcave in p_i for all p_{-i} in $X_{j\neq i}[0,\bar{p}_j]$ whenever the demand for the ith goods is positive and that the Bertrand best response functions r_i are increasing for all i. Then there is a Bertrand equilibrium with lower prices than any interior Cournot equilibrium price vector.

Proof. In Vives (1985a) it was shown that any interior Cournot equilibrium price vector one could find a Bertrand equilibrium with lower prices. Since the Bertrand best response map is increasing, Tarski's theorem can be used and the smallest Bertrand equilibrium is the desired one. Q.E.D.

References

Ash, R., 1972, Real analysis and probability (Academic Press, New York).

Bamon, Ramon and Jean Fraysse, 1985, Existence of Cournot equilibrium in large markets, Econometrica 53, May, 587–597.

Birkhoff, Garrett, 1967, Lattice theory, 3rd ed., American Mathematical Society 15 (Colloquium Publications, Providence, RI).
Bulow, Jeremy, John Geanakoplos and Paul Klemperer, 1983, Multimarket oligopoly: Strategic substitutes and complements, Journal of Political Economy 93, 488–511.
Cooper, R. and A. John, 1985, Coordinating coordination failures in Keynesian models, Cowles discussion paper no. 745R (Cowles Foundation, Yale University, New Haven, CT).
Dasgupta, Partha and Eric Maskin, 1986, The existence of equilibria in discontinuous economic games, 1: Theory, Review of Economic Studies 53, Jan., 1–26.
Friedman, James W., 1983, Oligopoly theory (Cambridge Surveys of Economic Literature).
Heller, Walter P., 1985, Coordination failure under complete markets with applications to effective demand, Working paper (University of California, San Diego, CA).
Hildenbrand, Werner, 1976, Core and equilibria of a large economy (Princeton University Press, Princeton, NJ).
Hirsch, Morris, 1985, Systems of differential equations that are competitive or cooperative II: Convergence almost everywhere, SIAM Journal of Mathematical Analysis.
McManus, M., 1964, Equilibrium, numbers and size in Cournot oligopoly, Yorkshire Bulletin of Social and Economic Research 16.
Milgrom, Paul and Robert Weber, 1985, Distributional strategies for games with incomplete information, Mathematics of Operations Research 10, no. 4, Nov., 619–632.
Nishimura, K. and J. Friedman, 1981, Existence of Nash equilibrium in n-person games without quasiconcavity, International Economic Review, Oct., 637–648.
Novshek, William, 1985, On the existence of Cournot equilibrium, Review of Economic Studies 52, no. 1, 85–98.
Radner, Ray and Robert Rosenthal, 1982, Private information and pure-strategy equilibria, Mathematical Operations Research 7, no. 3.
Rand, David, 1978, Exotic phenomena in games and duopoly models, Journal of Mathematical Economics 5, no. 2, 173–184.
Roberts, John and Hugo Sonnenschein, 1976, On the existence of Cournot equilibrium without concave profit functions, Journal of Economic Theory 13.
Roberts, John and Hugo Sonnenschein, 1977, On the foundations of the theory of monopolistic competition, Econometrica 45, 101–113.
Spady, Robert, 1984, Noncooperative price setting by asymmetric multiproduct firms, Mimeo. (Bellcore).
Spence, Michael, 1976, Product selection, fixed costs and monopolistic competition, Review of Economic Studies 43, June.
Tarski, Alfred, 1955, A lattice-theoretical fixpoint theorem and its applications, Pacific Journal of Mathematics 5, 285–308.
Topkis, Donald, 1978, Minimizing a submodular function on a lattice, Operations Research 26, no. 2.
Topkis, Donald, 1979, Equilibrium points in nonzero-sum n-person submodular games, SIAM Journal of Control and Optimization 17, Nov., 773–787.
Vives, Xavier, 1985a, On the efficiency of Cournot and Bertrand equilibria with product differentiation, Journal of Economy Theory 36, June, 166–175.
Vives, Xavier, 1985b, Nash equilibrium in oligopoly games with monotone best responses, CARESS working paper no. 85-10 (University of Pennsylvania, Philadelphia).