4. Rational expectations and market microstructure in financial markets

In this chapter we review the basic static (or quasi-static) models of financial markets with asymmetric information in a competitive environment. Strategic traders are introduced in Chapter 5. The dynamic trading counterpart of the models in Chapters 4 and 5 is to be found in Chapters 8 and 9. We study both rational expectations models, where traders have the opportunity to condition on prices as in Chapter 3, and models where traders use less complex strategies and cannot condition on prices as in the simple market mechanisms of Chapter 1. We consider also mixed markets where some traders use complex strategies such as demand schedules or limit orders and some others simple strategies such as market orders.

We are mainly concerned with studying the determinants of market quality parameters, such as trading intensity, volatility, liquidity, informativeness of prices and volume, when the information about the fundamentals is dispersed among the traders in the market. The general theme of the chapter, as well as Chapter 5, is that private information and the market microstructure, i.e. the details of how transactions are organized or the specifics of trading mechanisms, matter a lot for market quality parameters. The chapter will present the basic noisy rational expectations model of a financial market with asymmetric information and study the main variants. We will try to answer questions such as:

- How are prices determined? Does it make a difference if informed or uninformed traders move first?
- Do prices reflect noise or information about the fundamentals?
- What determines the liquidity of a market?
- What drives the traded volume?
- What are the incentives to acquire information in an informationally efficient market?
- Does it make a difference if market makers are risk averse?
- What determines the volatility of prices?
- What are the incentives of traders to use different types of orders?

Section 4.1 reviews the diversity of market microstructures in financial markets and outlines the material covered in the chapter.
4.1 Market microstructure

The market microstructure of financial markets is very rich. Many different types of agents intervene in a trading system; for example, market makers, specialists, dealers, scalpers, and floor traders. Market making refers in general to the activity of setting prices to equilibrate demand and supply for securities at the potential risk of holding or releasing inventory to buttress market imbalances. Specialists in the New York Stock Exchange (NYSE) have the obligation to keep markets “deep, continuous in price and liquid”. They do this by handling orders and trading on their own account to smooth imbalances. The defining characteristic of a dealer is his obligation to accommodate trades at the set prices. A scalper is a type of broker who, dealing on his account, tries to obtain a quick profit from small fluctuations of the market. A floor trader is generally a stock exchange member trading for his own account or for an account he controls. Traders can place many different types of orders; price formation and market rules differ in different markets and the sequence of moves by the agents involved is market specific. Trading rules and the institutional structure define an extensive form according to which the game between market participants is played.¹

4.1.1 Types of orders

The main types of orders that traders can place are market orders, limit orders and stop (loss) orders. A market order specifies a quantity to be bought or sold at whatever price the market determines. There is no execution risk therefore but there is price risk. A market order is akin to the quantity strategy of a firm in a Cournot market (Chapter 1).

A limit order specifies a quantity to be bought (sold) and a limit price below (above) which to carry the transaction. A buy limit order can only be executed at the limit price or lower, and a sell order can only be executed at the limit price or higher. This reduces the price risk but exposes the trader to execution risk because the order will not be filled if the price goes above (below) the limit price. A stop order is like a limit order

but the limits are inverted, specifying a quantity to be sold (bought) and a limit price below (above) which to carry the transaction. The idea is that if the price goes below (above) a certain point the asset is sold (bought) to “stop” losses (to profit from raising prices). Markets typically impose price and time priority rules on the execution of the limit order: older orders and/or offering better terms of trade are executed first.

A demand schedule can be formed by combining appropriately a series of limit and stop orders. A demand schedule specifies a quantity to be bought (sold if negative) for any possible price level. The demand schedule is akin to the supply function of a firm (Chapter 3). The advantage of the demand schedule is that (in the limit) it eliminates both the price and the execution risk. The drawback is that it is more complex and therefore more costly to implement.

4.1.2 Trading systems
Trading systems can be classified as order-driven or quote-driven. In an order-driven system traders place orders before prices are set either by market makers or by a centralized mechanism or auction. Typically the orders of investors are matched with no intermediaries (except the broker who does not take a position himself) and provide liquidity to the market. Trading in an order-driven system is usually organized as an auction which can be continuous or in batches at discrete intervals. A batch auction can be an open-outcry (like in the futures market organized by the Chicago Board of Trade) or electronic. Recently crossing networks using prices derived from other markets have emerged. In many continuous systems there is order submission against an electronic limit order book where orders have accumulated. If a limit order does not find a matching order and execution is not possible, it is placed in the limit order book.

In most of these systems a batch auction is used to open continuous trading (like in Euronext – the successor in Paris of the Paris Bourse which merged with the NYSE in 2007, Deutsche Börse, or Tokyo Stock Exchange). For example, in the Deutsche Börse with the Xetra system there is an opening auction that begins with a call phase in which traders can enter and/or modify or delete existing orders before the price determination
phase. The indicative auction price is displayed when orders are executable. Furthermore, volatility interruptions may occur during auctions or continuous trading when prices lay outside certain predetermined price ranges. A volatility interruption is followed by an extended call phase. Intraday auctions interrupt continuous trading. There is also a closing batch auction.

In a quote-driven system (like NASDAQ or SEAQ at the London Stock Exchange) market makers set bid and ask prices or, more in general, a supply schedule, and then traders submit their orders. The latter mechanism is also called a continuous dealer market because a trader need not wait to get his order executed, taking a market maker as the counterpart. However, the term dealer market is probably best kept for use when dealers quote uniform bid and ask prices rather than market makers post schedules which build a limit order book. In a quote-driven system dealers provide liquidity.

Many trading mechanisms are hybrid and mix features of both systems. The NYSE starts with a batch auction and then continues as a dealer market where there is a specialist for each stock who manages the order book and provides liquidity. There is now competition at the NYSE between marker makers and the electronic limit order book to provide liquidity. The 2006 Hybrid NYSE Market Initiative aims at enhancing off-floor competition to the specialist. The London Stock Exchange (LSE) now uses an electronic limit order book for small orders while keeping the dealer mechanism for large orders. SETS at the LSE is an order-driven system for the most liquid stocks. In general markets have evolved towards a pure electronic limit order market or at least allowing for customer limit orders competing with the exchange market makers (e.g. Euronext Paris or the evolution in the NYSE –including the acquisition of the limit order market Archipelago).

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2 Otherwise the best bid/ask limit is displayed. See Xetra Market Model Release 3 at www.exchange.de. We will deal with the preopening auction in Chapters 8 and 9.

3 All these auctions have three phases: call in which orders can be entered or preexisting orders modified or cancelled; price determination, and order book balancing (which takes place only if there is a surplus).

4 Other markets organized as dealerships are the foreign exchange market, US Treasury bills secondary market, and the bond market.
In both order-driven and quote-driven systems market makers face potentially an adverse selection problem because traders may possess private information about the returns of the asset and may exploit the market makers.\(^5\) The order-driven system has a signaling flavor, because the potentially informed party moves first, while the quote-driven system has a screening flavor, because the uninformed party moves first by proposing a schedule of transaction to which potentially informed traders respond. Signaling models tend to have multiple equilibria while nonexistence of equilibrium is a possible feature of screening models.\(^6\) In this book we concentrate on asymmetric information as friction in the price formation of assets in the spirit of Bagehot (1971). Other frictions like order-handling costs (see e.g. Roll (1984)) and inventory effects (see e.g. Ho and Stoll (1983)) are not the focus of attention, although we do study models with risk averse market makers where inventory effects are prominent. Asymmetric information models have a parallel in auctions models with a common value component while the inventory models have a parallel in private-value auctions.

A further dimension along which trading mechanisms may differ is the pricing rules: every unit sold at the same price (\textit{uniform pricing}) or different units sold at different prices (\textit{discriminatory pricing}). Batch auctions typically involve uniform prices, while a trader submitting to the limit order book a large enough order will get different prices corresponding to different limit prices. This is because several limit orders are needed to fill his order. Transactions occur at multiple prices as the trader "walks up" the book getting worse terms. In many dealer markets the order of the customer is filled by a single dealer (who may retrade with other dealers) at a uniform price.

Trading mechanisms differ also in other dimensions like anonymity, ex ante and ex post transparency, or retrading opportunities. Transparency refers to information on quotes, quantities, and identity of traders. Ex post transparency refers to the disclosure rules

\(^5\) Adverse selection in an insurance context arises when the person or firm insured know more than the insurance company about the probability of the loss happening (i.e. about the risk characteristics). In general an adverse selection problem relates to the unfavorable consequences for uninformed parties of the actions of privately informed ones. This is akin to the lemons’ problem studied by Akerlof (1970) and the adverse selection problem in insurance markets of Rothschild and Stiglitz (1976).

after trading and ex ante transparency refers to information available in the trading process. In an open book all limit orders are observable for all investors while in a closed book traders do not see the book. The intermediate situation is when only some limit orders are observable. For example, often only members of the stock exchange observe the whole order book while investors observe only the best or some of the best quotes (e.g. in Xetra). The information disclosed about the identities of the traders varies across exchanges although there is a tendency to anonymity. In a fragmented dealer market a trader does not observe the quotations of dealers other than the one he is dealing with, while in a centralized limit order book market price quotations are observable.\footnote{Under some circumstances the two systems are equivalent (see Exercise 5.9 in Chapter 5).} Fragmented markets may impair liquidity but enhance competition.\footnote{See Pagano (1989), Battalio, Green and Jennings (1997), and Biais, Glosten and Spatt (2005).}

A canonical model of trade is the Walrasian auction where all traders are in a symmetric position and submit simultaneously demand schedules to a central market mechanism that finds a (uniform) market clearing price. In fact, in the XIX century Walras (1889) was inspired to build his market model by the batch auctions of the Paris Bourse.

It is worth recalling some terminology about the informational efficiency of prices from Chapter 3. Prices are said to be \textit{strongly informationally efficient} if the price is a sufficient statistic for the private information dispersed in the market. Prices are said to be \textit{semi-strong informationally efficient} if they incorporate all public information available.

4.1.3 Outline

We deal in this chapter with a competitive environment and in Chapter 5 with strategic traders. We look at competitive and strategic behavior in models of simultaneous submission of orders, be it demand schedules in the rational expectations tradition or market orders, and of sequential order submission considering the cases where informed traders or uninformed traders move first. As we have seen the former is typical of order-driven systems while the latter of quote-driven systems. In the models considered uninformed traders who submit limit orders (or generalized demand schedules), and
provide liquidity to the market, are identified with market makers. The basic benchmark
dynamic models of price formation with adverse selection of Kyle (1985) and Glosten
and Milgrom (1985) are dealt with in Chapter 9. In Chapters 4 and 5 we only deal with
static models. The models considered are highly stylized. The complexities of trading
with limit orders are finessed.9 In this chapter we consider only models of uniform
pricing and defer models of discriminatory pricing to Chapter 5 (Section 5.3).

The standard competitive noisy rational expectations financial market, with a model
that has as special cases virtually all the competitive models in the literature, is
presented in Section 4.2.1. The potential paradox of the existence of informationally
efficient markets when information is costly to acquire is addressed in Section 4.2.2.
Section 4.3 considers the case when informed traders move first and prices are set by
competitive risk neutral market makers. Informed traders may submit demand
schedules or market orders. Section 4.4 introduces producers that want to hedge in a
futures market and examines the impact of private information on the possibilities of
insurance and real decisions. This section does away with the presence of noise traders
in the market. In all models considered in the chapter we will assume constant absolute
risk aversion (CARA) utility functions and normally distributed random variables. The
CARA-Normal model is the workhorse model in the study of financial markets with
asymmetric information.

4.2 Competitive rational expectations equilibria
In this section we study the standard competitive rational expectations model with
asymmetric information in a setup that encompasses several variations developed by
Admati (1985) –who considers a multiasset market, and Vives (1995a). We start with
the canonical framework in which traders compete in demand schedules (Section 4.2.1)
and examine Grossman and Stiglitz’s paradox about the impossibility of an
informationally efficient market when information is costly to acquire (Section 4.2.2),

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9 See the survey on limit order markets by Parlour and Seppi (2007) for a state of the art account of
research in the area taking into account dynamic strategies.
4.2.1 The CARA-Gaussian model

Consider a market with a single risky asset, with random fundamental value $\theta$, and a riskless asset (with unitary return) which are traded by risk averse agents, indexed in the interval $[0,1]$ endowed with the Lebesgue measure, and noise traders. The utility derived by trader $i$ from the return

$$\pi_i = (\theta - p)x_i$$

of buying $x_i$ units of the risky asset at price $p$ is of the Constant Absolute Risk Aversion (CARA) type and is given by

$$U_i(\pi_i) = -\exp\{-\rho_i \pi_i\},$$

where $\rho_i$ is the coefficient of constant absolute risk aversion. The non-random initial wealth of traders is normalized to zero (this is without loss of generality with CARA preferences). Trader $i$ may be informed, i.e. endowed with a piece of information about the ex post liquidation value $\theta$, or uninformed, i.e. inferring information only from the price. Noise traders are assumed to trade in the aggregate according to a random variable $u$. This is justified typically by exogenous liquidity motives. We could also think that from the perspective of investors in a stock the number of shares that float is a random variable. In Section 4.4 we do away with the assumption of exogenous noise and introduce traders who have a hedging motive.

We will specify the model assuming that there is a proportion $\mu$ of traders who are informed (and receive a private signal $s_i$ about $\theta$) and a proportion $(1-\mu)$ who are uninformed (and can be considered market makers). Both types of traders can condition their trade on the price but only informed trader $i$ observes the signal $s_i$. We have thus that the information set of informed trader $i$ is $\{s_i,p\}$ while for an uninformed trader is $\{p\}$. For informed traders $\rho_i = \rho > 0$ and for uninformed traders $\rho_i = \rho_u \geq 0$.

It is assumed that all random variables are normally distributed: $\theta$ with mean $\bar{\theta}$ and variance $\sigma_\theta^2$; $s_i = \theta + \varepsilon_i$, where $\theta$ and $\varepsilon_i$ are uncorrelated, errors have mean zero, variance $\sigma_\varepsilon^2$ and are also uncorrelated across agents; $u$ has zero mean and variance $\sigma_u^2$ and is uncorrelated with the rest of random variables. The expected volume of noise
trading \( E[|u|] \) is proportional to its standard deviation \( \sigma_u \).\(^{10}\) Note also that we assume therefore that all informed traders receive signals of the same precision. As in the previous chapters the convention is made that given \( \theta \), the average signal of a positive mass \( \mu \) of informed agents with \( \sigma^2_e < \infty \), \( \left( \int_0^\mu s_i \, di \right) / \mu \) equals almost surely (a.s.) \( \theta \) (i.e. \( \int_0^\mu e_i \, di = 0 \)).\(^{11}\) The distributional assumptions made are common knowledge among the agents in the economy. Recall that we denote the precision of random variable \( x \) (that is, \( (\sigma^2_x)^{-1} \)) by \( \tau_x \).

The present formulation is a particular case of the more general model in which the degree of risk aversion and the precision of the private signal of traders are allowed to be different and are given by (measurable) functions: \( \rho_i : [0,1] \to \mathbb{R}_+ \) and \( \tau_i : [0,1] \to \mathbb{R}_+ \cup \{\infty\} \) with values, respectively, \( \rho_i \) and \( \tau_i \) for \( i \in [0,1] \). The parameter \( \rho_i^{-1} \) is the risk tolerance of trader \( i \). An important parameter is the risk-adjusted information advantage of trader \( i \rho_i^{-1} \tau_i \) and its population average. The results presented in this section are easily extended to the general case provided that \( \rho_i^{-1} \) and \( \tau_i \) are uniformly bounded across traders (Admati (1985) considers the general multiasset model). This boundedness assumption ensures that traders with very low risk aversion or very precise information do not loom large in the market outcome.\(^{12}\)

Under our symmetry assumptions we will be interested in symmetric equilibria with traders of the same type using the same trading strategy. Denote by \( X_I(s_i, p) \) the trade of informed trader \( i \in [0,\mu] \), and by \( X_U(p) \) the trade of uninformed trader \( i \in (\mu,1] \).

\(^{10}\) The natural measure of volume is \( E[|u|] \) because otherwise buys and sells would cancel \((E[u] = 0)\). Recall that for \( z \) normal with \( E[u] = 0 \) and variance \( \sigma^2_u \) we have that \( E[|u|] = (2/\pi)^{1/2} \sigma_u \).

\(^{11}\) See Section 3.1 of the Technical Appendix for more details.
A symmetric rational expectations equilibrium (REE) is a set of trades, contingent on the information that traders have, \( \{ X_i(s_i, p) \text{ for } i \in [0, \mu]; \ X_U(p) \text{ for } i \in (\mu, 1] \} \), and a (measurable) price functional \( P(\theta, u) \) (i.e., prices measurable in \( (\theta, u) \) ) such that:

(i) Markets clear:

\[
\int_0^\mu X_i(s_i, p) \, di + \int_{\mu}^1 X_U(p) \, di + u = 0 \text{ (a.s.)}
\]

(ii) Traders in \([0, 1]\) optimize:\(^{13}\)

\[
X_i(s_i, p) \in \arg \max_z E[ U_i((\theta - p)z)|s_i, p] \text{ for } i \in [0, \mu] \text{ and }
X_U(p) \in \arg \max_z E[ U_i((\theta - p)z)|p] \text{ for } i \in (\mu, 1].
\]

Traders understand the relationship of prices with the underlying uncertainty \((\theta, u)\). That is, they conjecture correctly the function \( P(\cdot, \cdot) \), and update their beliefs accordingly. Typically, the equilibrium will not be fully revealing due to the presence of noise \((u)\). We will have then an example of noisy REE. Without noise the REE may have paradoxical features.

Grossman (1976) analyzed the market with a finite number of informed traders, no uniformed traders, and no noise (say \( u < 0 \) represents the non random supply of shares). The REE is defined similarly as above and a fully revealing equilibrium (FRREE) is shown to exist.\(^{14}\) The market is strongly informationally efficient. To show that this FRREE exists the competitive equilibrium of an artificial full information economy (in which all private signals are public) is computed and then it is checked that this equilibrium is also a (linear) REE of the private information economy (as in Chapter 3). The equilibrium has a puzzling property: demands are independent of private signals and prices! Demands are independent of private signals because the price is fully revealing, that is, the price is a sufficient statistic for \( \theta \). Demands are also independent

\[^{12}\] The assumption would be needed also to apply our conventions about the law of large numbers to the continuum economy.

\[^{13}\] Traders optimize almost everywhere in \([0, 1]\). As usual we will not insist on this qualification.

\[^{14}\] DeMarzo and Skiadas (1999) show that this FRREE is unique (in the CARA-normal context).
of prices because a higher price apart from changing the terms of trade (classical substitution effect) also raises the perceived value of the risky asset (information effect). In the model the two effects exactly offset each other (see Admati (1989)). However, this equilibrium is not implementable: the equilibrium cannot be derived from the equilibrium of a well-defined trading game. For example, given that demands are independent of private signals, how is it that prices are sufficient statistics for the private information in the economy? This cannot arise from a market clearing process of price formation. Indeed, in the Grossman economy each trader is not informationally small: his signal is not irrelevant when compared with the pooled information of other traders. (See Exercise 4.1 for the details of the Grossman example.)

There is a natural game in demand schedules which implements partially revealing REE in the presence of noise as a Bayesian equilibrium in the continuum economy. Note that with a continuum of traders each agent is informationally “small” (see Section 3.1). In the continuum economy there is always a trivial FRREE in which \( p = 0 \), traders are indifferent about the amounts traded and end up taking the counterpart in the aggregate of noise traders. This FRREE is not implementable and would not be an equilibrium if we were to insist that prices be measurable in excess demand functions as in Anderson and Sonnenschein (1982), see Section 3.1.

Let thus traders use demand schedules as strategies. This is the parallel to firms using supply functions as strategies in the partial equilibrium market of Chapter 3. At the interim stage, once each trader has received his private signal, traders submit demand schedules contingent on their private information (if any), noise traders place their orders, and then an auctioneer finds a market clearing price (as in (i) of the above definition of a REE). We will study the linear Bayesian equilibria of the demand schedule game.

In general we can allow traders to use general demand schedules which allow for market, limit and stop orders. Market clearing rules should be defined given the

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15 The schedules should have some continuity and convexity properties. A series of limit (stop) orders would be represented by a downward (upward) sloping schedule of step functions. “All or nothing”
strategies of traders. We can assume that if there is more than one market clearing price then the one with the minimum absolute value (and the positive one if there is also a negative price with the same absolute value) is chosen. If there is no market clearing price then the market shuts down (and traders get infinitely negative utility—for example, with the auctioneer setting a price equal to + or −∞).

When traders optimize they take into account the (equilibrium) functional relationship of prices with the random variables in the environment (θ and u). Trader i’s strategy is a mapping from his private information to the space of demand functions (correspondences more generally). Let \( X_i(s_i, \cdot) \) be the demand schedule chosen by an informed trader when he has received signal \( s_i \). When the signal of the trader is \( s_i \) and the price realization is \( p \) the desired position of the agent in the risky asset is then \( X_i(s_i, p) \). Similarly, for an uninformed trader the chosen demand schedule is represented by \( X_u(p) \). Noise traders' demands aggregate to the random variable \( u \).

Using standard methods we can characterize linear Bayesian equilibria of the demand function game. The CARA-normal model admits linear equilibria because optimization of the CARA expected utility under normality reduces—as we shall see—to the optimization of a mean-variance objective, and normality, as we know, yields linear conditional expectations.

We restrict attention to linear equilibria with price functional of the form \( P(\theta, u) \).
Linear Bayesian equilibria in demand functions will be necessarily noisy (i.e. \( \partial P/\partial u \neq 0 \)) since, as we have argued, a fully revealing equilibrium is not implementable. If traders receive no private signals then the price will not depend on the fundamental value \( \theta \) (see Example 1 below).

The profits of trader \( i \) with a position \( x_i \) are \( \pi_i = (\theta - p) x_i \), yielding a utility \( U_i(\pi_i) = -\exp\{\rho \pi_i\} \). Recall that if \( z \) is normally distributed \( N(\mu, \sigma^2) \) and \( r \) is

orders, in which a partial execution is not accepted, violate the assumption that the demand schedule is convex-valued (see Kyle (1989)).
a constant then \( E \left[ \exp \{ rz \} \right] = \exp \left\{ r \mu + \frac{r^2 \sigma^2}{2} \right\} \). Given \( x_i \), and when the price \( p \) is in the information set \( G \) of the trader, \( \pi_i | G \) is normally distributed (given that \( p \) is linear in \( \theta \) and \( u \) and all random variables have a joint normal distribution). Maximization of the CARA utility function conditional on the information set \( G \),

\[
E \left[ U (\pi_i | G) \right] = -E \left[ \exp \{ -\rho \pi_i \} | G \right] = -\exp \left\{ -\rho_i \left( E [\pi_i | G] - \rho \text{ var } [\pi_i | G] / 2 \right) \right\}
\]

is equivalent to the maximization of

\[
E [\pi_i | G] - \rho \text{ var } [\pi_i | G] / 2 = E [\theta - p | G] x_i - \rho \text{ var } [\theta - p | G] x_i^2 / 2. \quad 16
\]

This is a strictly (for \( \rho_i > 0 \)) concave problem which yields,

\[
x_i = \frac{E [\theta - p | G]}{\rho_i \text{ var } [\theta - p | G]} = \frac{E [\theta | G] - p}{\rho_i \text{ var } [\theta | G]},
\]

where \( G = \{ s_i, p \} \) for an informed trader, \( G = \{ p \} \) for an uninformed trader.

Because of the assumed symmetric ex ante signal structure and risk aversion \( (\rho_i = \rho_i) \) for informed traders, demand functions for the informed will be identical in equilibrium, and the same is true for the uninformed. It follows that

\[
E [\pi_i | G] - \rho \text{ var } [\pi_i | G] / 2 = \frac{\left( E [\theta - p | G] \right)^2}{2\rho_i \text{ var } [\theta - p | G]}
\]

and therefore the expected utility for a trader with information set \( G \) \( (-E \left[ \exp \{ -\rho \pi_i \} | G \right]) \) is given by

\[
-\exp \left\{ -\rho_i \left( E [\pi_i | G] - \rho \text{ var } [\pi_i | G] / 2 \right) \right\} = -\exp \left\{ -\frac{\left( E [\theta - p | G] \right)^2}{2 \text{ var } [\theta - p | G]} \right\}.
\]

This expression will be handy when we are interested in calculating expected utilities.

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16 The parallel to the marginal production cost in the Cournot model (with exogenous slope \( \lambda \), see chapter 1) is the term \( \rho_i \text{ var } [\theta - p | G] \) which is determined endogenously in equilibrium.
To solve for a linear REE a standard approach is the following. First, a linear price conjecture \( p = P(\theta, u) \) common for all agents is proposed. Second, using this conjecture, beliefs about \( \theta \) are updated; that is, expressions for \( E[\theta - p|\mathcal{G}] \) and \( \text{var}[\theta - p|\mathcal{G}] \) for informed and uninformed traders are derived. Third, asset demands are computed as above. Then market clearing yields the actual relation between \( p \) and \((\theta, u)\). Finally, the price conjecture must be self-fulfilling. This pins down the undetermined coefficients in the linear price functional. To this we should add the requirement that prices be measurable in excess demand functions.

To solve for a linear Bayesian equilibrium in demand schedules we follow similar steps. First, positing linear strategies for the traders we find, using the market clearing condition, an expression for \( p \) in terms of \( \theta \) and \( u \). Second, using this expression we update beliefs about \( \theta \). Third, we compute the asset demands for informed and uninformed types. Finally, we identify the coefficients of the linear demands imposing consistency between the conjectured and the actual strategies. If in the first step we want to consider potentially asymmetric strategies we must (as in Section 3.2.1) restrict attention to uniformly bounded signal sensitivities of the strategies across traders (as well as bounded average parameters in the linear strategies) in order to use our convention on the law of large numbers in the continuum economy and obtain a linear price functional of the form \( P(\theta, u) \). This restriction is equivalent to consider equilibria with linear price functional of the form \( P(\theta, u) \).

The following proposition characterizes the linear Bayesian equilibrium.

**Proposition 4.1**: Let \( \rho_1 > 0 \) and \( \rho_u > 0 \). There is a unique Bayesian linear equilibrium in demand functions with linear price functional of the form \( P(\theta, u) \). It is given by:

\[
X_i(s_i, p) = a(s_i - p) - b_i(p - \bar{\theta}),
\]

where \( a = \rho_1^{-1} \tau_u \), and \( b_i = \frac{\tau_u}{\rho_i + \mu \tau_u (\mu \rho_1^{-1} + (1 - \mu) \rho_u^{-1})} \).
\[ X_U(p) = -b_U(p - \bar{\theta}), \]

where \( b_U = \rho_t(\rho_U)^{-1} b_t \). In addition,

\[ P(0, u) = \lambda z + \bar{\theta}, \]

where \( z = \mu a(\theta - \bar{\theta}) + u \) and \( \lambda = (\mu (a + b_t) + (1 - \mu) b_U)^{-1}. \)

**Proof:**

Given the ex ante symmetric information structure and risk aversion for the informed and uninformed, who face a strictly concave optimization problem at a linear price equilibrium of the form \( P(0, u) \), we know that the demand functions of the informed and of the uninformed will be identical in each class:

\[ X_i(s, p) = a_i - c_i p + \hat{b}_i \]
\[ X_u(p) = -c_u p + \hat{b}_u. \]

Form the market clearing condition

\[ \int_{0}^{\theta} X_i(s, p) \, di + (1 - \mu) X_u(p) + u = 0 \]

and letting \( \hat{b} = \mu \hat{b}_i + (1 - \mu) \hat{b}_U \), and \( \lambda = (\mu c_i + (1 - \mu) c_U)^{-1} \) provided \( \mu c_i + (1 - \mu) c_U > 0 \), we obtain

\[ p = \lambda (\mu a_\theta + u + \hat{b}). \]

Let \( \mu a > 0 \), then the random variable \( \hat{z} \equiv \theta + \frac{1}{\mu a} u \) is informationally equivalent to the price and \( \hat{z} = \frac{p - \lambda \hat{b}}{\lambda \mu a}. \) Prices will be normally distributed because they are a linear transformation of normal random variables. We have that \( \text{var}[\theta | p] = \text{var}[\theta | \hat{z}] \) and it is immediate from the properties of normal distributions (see Sections 1 and 2 of the Technical Appendix) that the precision incorporated in prices in the estimation of \( \theta \), \( \tau \equiv (\text{var}[\theta | p])^{-1} \), is given by \( \tau = \tau_0 + \tau_u (\mu a)^2 \) (indeed –see Section 2 in the Technical Appendix).
Appendix, $\tau$ is the sum of the precision of the prior $\tau_0$ and the precision of the public signal, conditional on $\theta$ $\hat{z} \equiv \theta + \frac{1}{\mu a}$. It follows that

$$E[0|p] = E[0|\hat{z}] = \frac{\tau_0 \bar{\theta} + (\mu a)^2 \tau_u \hat{z}}{\tau} = \frac{\tau_0 \bar{\theta} + \mu a \tau_o \lambda^{-1} (p - \lambda \bar{b})}{\tau}. $$

From the optimization of the CARA utility of an uninformed trader we have that

$$X_u(p) = \frac{E[0|p] - p}{\rho_U var[0|p]} = -c_u p + \hat{b}_u,$$

and identifying coefficients it is immediate that

$$c_u = \frac{1}{\rho_U} \left( \tau - \frac{\mu_a \tau_o}{\lambda} \right) \text{ and } \hat{b}_u = \frac{\tau_0 \bar{\theta} - \mu a \tau_o \bar{b}}{\rho_U}. $$

From the optimization of the CARA utility of an informed trader we obtain

$$X_i(s_i, p) = \frac{E[0|s_i, p] - p}{\rho_i var[0|s_i, p]} = a_i - c_i p + \hat{b}_i,$$

where from the properties of normal distributions $\left( var[0|s_i, p] \right)^{-1} = \tau_e + \tau$.

Furthermore, proceeding in a similar way as before,

$$E[0|s_i, p] = E[0|s_i, \hat{z}] = \frac{\tau s_i + \tau_0 \bar{\theta} + (\mu a)^2 \tau_e \hat{z}}{\tau_e + \tau} = \frac{\tau s_i + \tau_0 \bar{\theta} + \mu a \tau_o \lambda^{-1} (p - \lambda \bar{b})}{\tau_e + \tau}. $$

Identifying coefficients it is immediate that $a = (\rho_i)^{-1} \tau_e$, $c_i = \rho_i^{-1} \left( \tau_e + \tau - \mu a \tau_o / \lambda \right)$ and $\hat{b}_i = \rho_i^{-1} \left( \tau_0 \bar{\theta} - \mu a \tau_o \bar{b} \right)$. It follows that
\[ \lambda = \frac{1 + \mu a (\mu \rho_i^{-1} + (1 - \mu) \rho_u^{-1}) \tau_u}{\mu a + (\mu \rho_i^{-1} + (1 - \mu) \rho_u^{-1}) \tau} > 0 \]

and

\[ \hat{b} = \frac{(\mu \rho_i^{-1} + (1 - \mu) \rho_u^{-1}) \tau_u \bar{\theta}}{1 + (\mu \rho_i^{-1} + (1 - \mu) \rho_u^{-1}) \mu a \tau_u} = (\lambda^{-1} - \mu a) \bar{\theta}. \]

From these expressions we obtain immediately \( \hat{b}_i = b_i \theta \) where

\[ b_i = \frac{\tau_u}{\rho_i + \mu \tau_u \rho_u (\mu \rho_i^{-1} + (1 - \mu) \rho_u^{-1})}, \quad c_i = a + b_i, \quad \text{and} \quad \hat{b}_u = b_u \theta \]

\( b_u = c_u = \rho_i (\rho_u)^{-1} b_i, \) and the expressions for \( X_i(s_i, p) = a(s_i - p) - b_i(p - \bar{\theta}) \) and \( X_u(p) = -b_u(p - \bar{\theta}) \) follow. The expression for the price \( p = \lambda z + \bar{\theta}, \) where \( z = \mu a(\bar{\theta} - \bar{\theta}) + u, \) follows from \( p = \lambda (\mu a \theta + u + \hat{b}) \) and the expressions for \( \lambda \) and \( \hat{b}. \)

This proves the proposition. It is worth noting that the computations can be shortened, with hindsight, if we postulate from the beginning the linear forms \( X_i(s_i, p) = a(s_i - p) - b_i(p - \bar{\theta}) \) and \( X_u(p) = -b_u(p - \bar{\theta}). \)

Since \( b_u > 0, \) uninformed traders sell (buy) when the price is above (below) the prior expectation of the asset value; i.e. they lean against the wind. This is the typical behavior of market makers. Uninformed traders face an adverse selection problem because they do not know if trade is mainly motivated by informed or by noise traders. For example, when \( \rho_i = \rho_u, \) it is immediate that the trading intensity of the uninformed \( (b_u) \) is decreasing in the proportion of informed traders \( \mu. \)

Informed agents trade for two reasons. First, they speculate on their private information, buying or selling depending on whether the price is larger or smaller than their signal according to a trading intensity directly related to their risk tolerance \( (\rho_i)^{-1} \) and precision of information \( \tau_e. \) The responsiveness to private information \( a \) is independent of the amount of noise trading and the (prior) variance of \( \theta. \) This is so
since an increase in noise trading $\tau_u^{-1}$ or prior variance $\tau_\theta^{-1}$ increases risk (by increasing $\text{var} [\theta | s, p]$) but also increases the conditional expected return $(E[\theta | s, p] - p)$, because it makes private information more valuable. The two effects exactly offset each other in the present framework. (The independence of $a$ with respect to $\tau_\theta^{-1}$ will not hold if traders have market power – see Exercise 5.1.) The second component of the trade of informed agents is related to their market making capacity, similar to the uninformed traders, with associated trading intensity $b_1$. The parameter $b_1$ depends positively on the amount of noise trading $\tau_u^{-1}$, and negatively on the average risk tolerance in the market $\mu \rho^{-1}_U + (1 - \mu) \rho^{-1}_U$. If uniformed traders are risk neutral (i.e., $\rho_U \to 0$) then $b_1 = 0$ and informed agents do not trade for market making purposes. If the prior on the fundamental value is uniform (improper with $\tau_\theta = 0$) then $b_1 = b_U = 0$ and there is no market making trade.

Prices equal the weighted average of the expectations of investors about the fundamental value plus a noise component reflecting the risk premium required for risk averse traders to absorb noise traders’ demands. The weights are according to the risk-adjusted information of the different traders. Indeed, from the fact that the demand of informed and uninformed traders can also be expressed, respectively, as

$$X_i(s, p) = \rho_i^{-1}(\tau_\varepsilon + \tau)[E[\theta | s, p] - p]$$

and

$$X_U(p) = \rho_U^{-1}\tau[E[\theta | p] - p],$$

and the market clearing condition, it is immediate that

$$p = \frac{\rho_i^{-1}(\tau_\varepsilon + \tau)\int_0^\mu E[\theta | s, p] \, \text{d}i + (1 - \mu)\rho_U^{-1}\tau E[\theta | p]}{\mu \rho_i^{-1}(\tau_\varepsilon + \tau) + (1 - \mu)\rho_U^{-1}\tau} + u.$$
If the uniformed traders become risk neutral ($\rho_u$ tends to 0) then $p$ tends to $E[0|p]$ and prices are semi-strong efficient (that is, they reflect all publicly available information).

The average expectation of informed traders is $\mu^{-1}\int_0^\mu E[0|s_i, p] \, d\mu$. With no informed traders ($\mu = 0$) prices are the average of the expectations of the uninformed traders plus a risk bearing term,

$$p = E[0|p] + \rho_u \tau^{-1} u = \bar{\theta} + \rho_u \tau_{\omega}^{-1} u$$

and with no uninformed traders ($\mu = 1$) prices are the average of the expectations of informed investors $\int_1^\mu E[0|s_i, p] \, d\mu$ plus a risk bearing term,

$$p = \int_1^\mu E[0|s_i, p] \, d\mu + \rho_i (\tau_{\varepsilon} + \tau)^{-1} u.$$

We examine now the following market quality parameters: depth, price informativeness and volatility, and expected volume traded by the informed.

**Market depth.** The depth of the market is given by the parameter $\lambda^{-1}$. A change of noise trading by one unit moves prices by $\lambda$. A market is deep if a noise trader shock is absorbed without moving prices much, and this happens when $\lambda$ is low. Market depth is equal to the average responsiveness of traders to the market price $\lambda^{-1} = \mu(a + b_{\omega}) + (1 - \mu)b_u$. Traders provide liquidity to the market by submitting demand schedules and stand willing to trade conditionally on the market price. The depth of the market $\lambda^{-1}$ increases with $\tau_{\omega}$, and decreases with $\tau_u$ and $\rho_u$. All this accords to intuition: more volatility of fundamentals (or equivalently a lower precision of prior information) and a higher degree of risk aversion of uniformed traders (market makers) decrease the depth of the market. Market makers protect themselves from the adverse selection problem by reducing market liquidity when they are more risk averse and/or there is less precise public information. At the same time more noise trading increases market depth as the market makers feel more confident that they are not trading against informed investors. In fact, as we will see, without noise traders the market collapses. The effect of the other parameters on $\lambda^{-1}$ is ambiguous. In particular,
an increase in the proportion of informed traders $\mu$ has an ambiguous impact on $\lambda^{-1}$. For example, when $\rho_u = \rho_i$ it is immediate that $b_u = b_i = b$ and $\lambda^{-1} = \mu a + b$. Increasing $\mu$ raises $\mu a$ (the informativeness of price) and decreases $b$ (the price sensitivity of traders). The increase in the informativeness of price tends increase market depth (since it diminishes the inventory risk for risk averse traders) but this is counterbalanced by a decreased price sensitivity of traders because of increased adverse selection.

There is evidence that market makers do face an adverse selection problem and that spreads reflect asymmetric information (see Glosten and Harris (1988) for evidence in the NYSE, Lee et al. (1993) find that depth decreases around earning announcements – when asymmetric information may be more important). Furthermore, there is evidence that trades have a permanent impact on prices, pointing towards the effects of private (or public) information (see Hasbrouck (2007)).

*Price informativeness and volatility.* The random variable $z = \mu a (\theta - \bar{\theta}) + u$ can be understood as the informational content of the price since $p = \lambda z + \bar{\theta}$. It follows immediately that $E[p] = \bar{\theta}$. This is so because we assume on average a zero supply of shares, $E[u] = 0$. With a positive expected supply $E[u] < 0$ there is a positive risk premium, $E[p] - \bar{\theta} = \lambda E[u] < 0$, because agents are risk averse and need a premium to absorb a positive supply of shares. The comparative statics of the risk premium therefore follow those of the liquidity parameter $\lambda$. (See Section 4.4 for results on risk premia in a more complex model.)

Prices are biased in the sense that $E[\theta | p]$ is smaller or larger than $p$ if and only if $p$ is above or below $\bar{\theta}$. Indeed, (see the proof of Proposition 4.1) from

$$X_U(p) = \frac{E[\theta | p] - p}{\rho_U \text{var} [\theta | p]} = -\rho_i (\rho_u)^{-1} b_i (p - \bar{\theta})$$

it follows that
\[ E[\theta|p] - p = \rho_i \tau^{-1} (\bar{\theta} - p) \]

since \( \tau^{-1} = \text{var}[\theta|p] \). We can write the demand of an informed trader also as

\[ X_i(s_i, p) = \rho_i^{-1} \tau \epsilon (s_i - p) + \rho_i^{-1} \tau (E[\theta|p] - p) \].

This highlights the fact that an informed agent trades exploiting his private information (with intensity according to the risk tolerance-weighted precision of private information \( (\rho_i)^{-1} \tau \epsilon \)) and exploiting price discrepancies with public information on the fundamental value, i.e. market making (with intensity according to the risk tolerance-weighted precision of public information \( (\rho_i)^{-1} \tau \)).

In their market making activity risk averse traders buy when \( E[\theta|p] > p \), and this happens when \( p < \bar{\theta} \). Prices with risk averse market makers exhibit reversal (negative drift) since \( \text{sign}(E[\theta - p|p - \bar{\theta}]) = \text{sign}((-\bar{\theta} - (p - \bar{\theta}))) \) (the price variation \( (\theta - p) \) goes in the opposite direction as the initial variation \( (p - \bar{\theta}) \)).

Price precision in the estimation of \( \theta \) is given by:

\[ \tau = \tau_o + \tau_u (\mu a)^2 \]

When prices are fully revealing, \( p = 0 \), and \( \tau \) is infinite; when they are pure noise \( \tau = \tau_o \). This can be easily seen if we recall that \( \tau = (\text{var}[\theta|p])^{-1} \). If \( p = 0 \) then \( \text{var}[\theta|p] = 0 \) and if \( p \) is pure noise \( \text{var}[\theta|p] = \text{var}[\theta] \). Price precision increases with \( (\rho_i)^{-1} \tau \epsilon \) (the risk tolerance adjusted informational advantage of informed traders), \( \tau_o \), \( \mu \) and \( \tau_u \), and is independent of \( \rho_u \). All these effects accord with intuition. Less volatility of fundamentals, a higher proportion of informed traders, more precise signals, less noise trading, and less risk aversion of the informed, contribute to a higher informativeness of prices. The fact that with more informed traders prices become more informative means that there is strategic substitutability in information acquisition since when public information is more precise there is less incentive to acquire a private signal.
The volatility of prices \( \text{var}[p] = \lambda^2 \left( (\mu a)^2 \sigma_a^2 + \sigma_\theta^2 \right) = \lambda^2 \tau \left( \tau_a \sigma_\theta \right) \) depends, ceteris paribus, negatively on market depth \( \lambda^{-1} \), and positively on price precision, prior volatility, and noise trading. Price precision depends on the aggregate trading intensity of informed and market depth depends on the price responsiveness of both informed and uninformed. Volatility increases with \( \rho \) and \( \sigma_a^2 \). In both instances market depth \( \lambda^{-1} \) decreases, and with increases in \( \sigma_a^2 \), there is a further direct impact. It can be checked that the effect of changes in the other parameters on \( \text{var}[p] \) is ambiguous (because of their indirect impact via market depth).

**Expected traded volume.** The expected (aggregate) volume traded by informed agents is given by \( \text{E} \left[ \int_0^\infty X_i(s_i,p) \text{d}i \right] \). Given that if \( x \sim N(0,\sigma^2) \) then \( \text{E}[x] = \sigma \sqrt{2/\pi} \), we have

\[
\text{E} \left[ \int_0^\infty X_i(s_i,p) \text{d}i \right] = \mu \left( a(\theta-p) - b_i(p-\bar{p}) \right) = \mu \left( \text{var}[a(\theta-p) - b_i(p-\bar{p})] \right)^{1/2} \sqrt{2/\pi}
\]

since \( \text{E}[a(\theta-p) - b_i(p-\bar{p})] = 0 \). Substituting the expression for \( \text{var}[a(\theta-p) - b_i(p-\bar{p})] \) we obtain that

\[
\text{E} \left[ \int_0^\infty X_i(s_i,p) \text{d}i \right] = \mu \left( \sigma_a^2 \left( 1 - (a + b_i) \lambda \mu \right)^2 + \sigma_a^2 \left( a + b_i \right)^2 \lambda^2 \right)^{1/2} \sqrt{2/\pi}.
\]

When noise trading vanishes \( (\sigma_u \to 0) \) then \( b_i \to 0 \) and \( \lambda \to 1/\mu a \) and the expected trade of informed speculators tends to zero. This is so because the precision incorporated into prices tends to infinity and therefore the information advantage of informed traders disappears. In fact, trade collapses as \( \sigma_u \) tends to 0 (\( b_u \) and \( b_i \) also tend to 0 and there is no market making based trade). There is trade because of the presence of noise traders. Without noise traders the no-trade theorem applies since the initial endowments are a Pareto optimal allocation for the risk averse traders (see Section 3.1 and Section 4 in the Technical Appendix).
When the precision of information \( \tau_e \) of informed agents tends to infinity so does their trade intensity \( a \), market depth \( \lambda^{-1} \), and price precision \( \tau \), with the result that the expected trade of informed agents tends to \( \sqrt{2/\pi} \sigma_u \) and market making vanishes (\( b_U \) and \( b_I \) tend to 0).

**Examples**

The following examples show that the above model encompasses several of the models presented in the literature as

**Example 1. No informed traders** \((\mu = 0)\). This corresponds to a REE with no asymmetric information. In this case traders condition only on the price, from which they learn the amount of noise trading: \( p = \lambda u + \bar{\theta} \) where \( \lambda = b_U^{-1} \) and \( b_U = \tau_\theta/\rho_U \). The price does not depend on the fundamental value \( \theta \) since demands do not depend on informative signals.\(^{17}\) It is worth noting that if \( \mu > 0 \) then \( b_U < \tau_\theta/\rho_U \). That is, in the presence of informed agents, uninformed traders react less to the price, decreasing market depth because of the adverse selection problem they face. Indeed, when an uninformed trader knows that informed traders are present in the market he will be more cautious responding to price movements because he does not know whether the price moves because of noise traders or because of the trades of informed.

**Example 2. No uninformed traders** \((\mu = 1)\). This case corresponds to the limit equilibrium of Hellwig (1980).\(^{18}\) Then there are only informed and noise traders. Informed traders have to “make the market”. We have then \( a = \left(\rho_1\right)^{-1} \tau_e, b_i = \frac{\tau_\theta}{\rho_1 + a \tau_u} \),

\[
\lambda = (a + b_1)^{-1} = \left(\frac{\tau_e + \tau}{\rho_1 + a \tau_u}\right)^{-1}, \text{ and } \tau = \tau_\theta + \tau_u a^2.
\]

\(^{17}\) Similarly, when the precision of information of informed traders \( (\tau_e) \) tends to zero so does their informational trade intensity \( a \). In the limit we have then \( b_U = \tau_\theta/\rho_U \) and \( b_I = \tau_\theta/\rho_I \) (the only potential difference between an informed (I) and uninformed (U) trader is in the degree of risk aversion).

\(^{18}\) The model in Diamond and Verrecchia (1981) is closely related.
Example 3. Competitive risk neutral market makers, $\rho_u \to 0$. This corresponds to the static model in Vives (1995a). Then $\mathbb{E}[\theta | p] = p$ in the limit as $\rho_u \to 0$. This follows immediately from $X_u(p) = \frac{\mathbb{E}[\theta | p] - p}{\rho_u \text{var}[\theta | p]}$. When $\rho_u \to 0$ it must be that $\mathbb{E}[\theta | p] = p$, otherwise uninformed traders would take unbounded positions. When $\mathbb{E}[\theta | p] = p$ then informed traders withhold from market making and only speculate.\(^{19}\) Taking the limit of equilibrium parameters as $\rho_u \to 0$ we obtain: $a = (\rho_t)^{-1} \tau e, b_1 = 0$, 
\[
b_u = \frac{\tau u}{(\mu a (1 - \mu) \tau_u)}, \quad \lambda = \frac{\mu a}{\tau} \tau_u, \quad \text{and} \quad \tau = \tau_0 + \tau_u (\mu a)^2.
\] Prices exhibit no drift since $\mathbb{E}\left[\theta - p | p - \theta\right] = 0$. This case is examined in more detail in Section 4.3.

4.2.2 Information acquisition and the Grossman-Stiglitz paradox

Let us consider a slightly different version of the model where all informed traders observe the same signal $s$ and where $\theta = s + \varepsilon$ with $s$ and $\varepsilon$ independent and $\mathbb{E}[\varepsilon] = 0$ (Grossman and Stiglitz (1980)). The liquidation value is now the sum of two components, one of which ($s$) is observable at a cost $k$. The random variables $(s, \varepsilon, u)$ are jointly normally distributed. Suppose also that $\rho_t = \rho_u = \rho$ and that noise trading has mean. $\mathbb{E}[u] = 0$ That is, there is a normalized mean exogenous supply of shares of 1. Apart from the above modifications the market microstructure is as in the previous section. $\mathbb{E}[u]$

With these assumptions the derivation of the demand of the informed is simplified because $s$ is a sufficient statistic for $(s, p)$. Indeed, the price cannot contain more

\(^{19}\) Similarly, in Kyle (1989) Theorem 6.1 it is shown that as the aggregate risk bearing capacity of uniformed speculators or “market makers” grows without bound, in the limit prices are unbiased in the sense that $\mathbb{E}[\theta | p] = p$. 

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information than the joint information of traders \( s \). We have that 
\[
E[\theta|s,p] = E[\theta|s] = s, \\
\text{var}[\theta|s,p] = \text{var}[\theta|s] = \sigma^2, \\
\text{var}[\theta|s,p] = \text{var}[\theta|s] = \sigma^2, \text{ and}
\]
\[
X_i'(s,p) = \frac{E[\theta|s,p] - p}{\rho \text{var}[\theta|s,p]} = a(s - p), \text{ with } a = \left(\rho \sigma^2 \right)^{-1}.
\]

For a given proportion of informed traders \( \mu \), market clearing implies that
\[
\mu X_i'(s,p) + (1 - \mu) X_u'(p) + u = 0,
\]
and therefore at the (unique) linear equilibrium the price will be informationally equivalent to \((\mu a) s + u\) or \( w = s + (\mu a)^{-1} u \). The price will be of the form
\[
P(s,u) = \alpha_1 + \alpha_2 w, \text{ where } w = s + (\mu a)^{-1} u
\]
for some appropriate constants \( \alpha_1 \) and \( \alpha_2 > 0 \). Once again, the price functional depends
on \( s \) and not on \( \theta \) because \( s \) is the joint information of traders. The more informative is the price about \( s \), the less is the incentive to acquire information. The price will be more informative the higher the precision of the noise term \((\mu a)^{-1} u\), which equals \((\mu a)^2 \tau_u \).

The precision of \( p \) or \( w \) in the estimation of \( s \) is \((\text{var}[s|w])^{-1} = \tau_s + (\mu a)^2 \tau_u \). If \( \mu = 0 \) then there is no information available on \( \theta \) and in equilibrium \( p = \overline{\theta} + \rho \sigma_u^2 u \) (this follows immediately from the demand of an uninformed trader \( X_u'(p) = (\overline{\theta} - p)/\rho \sigma_u^2 \) and the market clearing condition).

When there is no noise (\( \sigma_u^2 = 0 \)) there is a fully revealing equilibrium (in which \( p = s - a^{-1} \) and \( p \) reveals \( s \)) and both informed and uninformed traders have the same demands. It is worth noting that this is a FRREE which is implementable as a Bayesian equilibrium in demand functions. This is due to the residual uncertainty \( \varepsilon \) left in the liquidation value given \( s \). At the price \( p = s - a^{-1} \) all traders demand one unit and absorb

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the exogenous non-random unit supply of shares. With $\bar{u} = 0$ we would have $p = s$ and no trade.

Consider now a two-stage game in which first traders decide whether to purchase the signal $s$ at cost $k$ or remain uninformed. At the second stage we have a market equilibrium contingent on the proportion $\mu$ of traders who have decided to become informed. From Section 4.2 we know that the conditional expected utility of an informed trader is given by

$$-\exp\left\{ -\frac{(E[(\theta - p)s,p])^2}{2\text{var}[(\theta - p)s,p]} \right\} = -\exp\left\{ -\frac{(E[\theta s]-p)^2}{2\text{var}[\theta s]} \right\} = -\exp\left\{ -\frac{(s-p)^2}{2\sigma_s^2} \right\},$$

and for an uninformed trader

$$E[U(\pi_u)|p] = -\exp\left\{ -\frac{(E[(\theta - p)p])^2}{2\text{var}[(\theta - p)p]} \right\} = -\exp\left\{ -\frac{(E[\theta p]-p)^2}{2\text{var}[\theta p]} \right\}.$$

We claim (see the appendix to the chapter for a proof) that the expected utility of the informed conditional on public information (the price) and taking into account the cost $k$ of getting the signal is given by

$$E[U(\pi_i)|p] = \exp\{pk\}E\left[ -\exp\left\{ -\frac{(s-p)^2}{2\sigma_s^2} \right\} \right] = \exp\{pk\} \sqrt{\frac{\text{var}[\theta s]}{\text{var}[\theta w]}} E[U(\pi_u)|p].$$

Taking expectations on both sides and denoting by $EU_i(\mu)$ and $EU_u(\mu)$, respectively, the expected utility of an informed and uninformed trader, it follows that:

$$\frac{EU_i(\mu)}{EU_u(\mu)} = \phi(\mu) = e^{pk} \sqrt{\frac{\text{var}[\theta s]}{\text{var}[\theta w]}} = e^{pk} \sqrt{\frac{\sigma_s^2}{\mu a^2 + \tau_u + \sigma_s^2}}$$

given that $\text{var}[\theta p] = \text{var}[\theta w] = \text{var}[s p] + \sigma_s^2$. 

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Figure 1. Interior or corner equilibria depending on the position of $\phi(\cdot)$, the relative expected utilities of the informed and uninformed.

The right hand side $\phi(\mu)$ is increasing in $\mu$. Given that utilities are negative this means that as $\mu$ increases $EU_i$ goes down relative to $EU_u$. Indeed, we have that as the proportion of informed traders increases the informativeness of prices increases and the informational advantage of the informed decreases. This implies that there is strategic substitutability in information acquisition. Given that $\phi(\mu)$ is increasing we have a unique equilibrium in the two-stage game (see Figure 1). Any $\mu$ in $[0, 1]$ for which $EU_i(\mu) = EU_u(\mu)$ (or $\phi(\mu) = 1$) will be an equilibrium. Indeed, at an interior equilibrium the expected utility of both types of traders must be equalized. If $\phi(1) < 1(\phi(0) > 1)$ then $\mu = 1(\mu = 0)$ is an equilibrium. For $k$ large (small) no one (everyone) is informed, and for intermediate $k$ there is an interior solution. Note that in this case as noise trading vanishes ($\tau_u \to \infty$) the mass of informed traders must tend to zero ($\mu \to 0$) since the informativeness of the price system $\tau_u + (\mu a)^2 \tau_u$ is to be kept constant so that $EU_i(\mu) = EU_u(\mu)$ (or $\phi(\mu) = 1$).
An interesting case (the Grossman-Stiglitz paradox) arises when there is no noise trading. Then, unless \( k \) is large - in which case the equilibrium is \( \mu = 0 \), there is no equilibrium (in pure strategies). Indeed, consider an equilibrium candidate with \( \mu > 0 \). Then we know that the price must be fully revealing and the expected utility of both informed and uninform ed should be the same. However, the informed must pay \( k \) and the expected utilities derived from trade must be equal. Therefore, \( \mu > 0 \) is not possible in equilibrium. Suppose that \( k \) is not so large so that it pays for a trader to become informed if everyone else is uninformed; that is, \( \phi(0) < 1 \) or \( e^{\alpha k} < \left( (\sigma_u^2 + \sigma_e^2)/\sigma_e^2 \right)^{1/2} \). Then \( \mu = 0 \) can not be an equilibrium either. This is the Grossman-Stiglitz paradox about the impossibility of informationally efficient markets. If the price is fully revealing then it does not pay to acquire information and therefore the price can not contain any information. If no trader acquires information, and the cost of information is moderate, then there are incentives for a single trader to purchase information.

The paradox is resolved, obviously, if there is noise trading because then the price is not fully revealing and it pays, in equilibrium, to obtain information provided it is not excessively costly. For an interior equilibrium the informativeness of prices \( \tau \) decreases with \( k \) (as \( k \) increases the equilibrium \( \mu \) falls) and \( \rho \), and is independent of \( \sigma_u \). An increase in noise trading induces two effects that exactly balance each other: It increases the equilibrium proportion of informed traders and for a given \( \mu \) reduces price informativeness. It is possible to check that as noise trading \( \sigma_u \) tends to zero the proportion of informed traders \( \mu \) as well as expected trade tend to zero.

Muendler (2007) revises the paradox to find existence of a fully revealing equilibrium with information acquisition when there are a finite number of traders. Recall that in the context of a competitive market with a continuum of agents and constant returns to scale (Section 1.6.2) we already saw that no equilibrium with costly information acquisition would exist. However, the existence of equilibrium is restored with a finite number of agents (Section 2.3.2).

Other work has characterized information acquisition in the presence of noise in trading. Diamond and Verrecchia (1981) consider a CARA-normal model where each
trader receives an idiosyncratic endowment shock of the risky asset and information about the fundamental value is dispersed among the traders (i.e. each trader receives a private signal about \( \theta \) of the type \( s_i = \theta + \epsilon_i \)). The simplifying assumption is made that the correlation between the aggregate endowment shock \( u \) and the idiosyncratic one \( u_i \) is zero.\(^{20}\) This makes the model basically equivalent to the noise trader model. The authors show that there is a unique linear partially revealing equilibrium. Verrecchia (1982) extends the model to a population of traders that may differ in their levels of risk aversion and adds an information acquisition stage to the model (similarly as we did in Section 1.6 for the Cournot model) where to acquire precision \( \tau_e \) a trader incurs a cost \( C(\tau_e) \) where \( C(\cdot) \) is a strictly increasing convex function. The author finds that less risk averse traders purchase more precision of information. This is so since a trader with low risk aversion will trade more aggressively in the risky asset and therefore will be willing to spend more to protect his position. From this it follows that the informativeness of prices is decreasing with increases in risk aversion (measured by a first-order stochastic dominance shift in the distribution of risk aversion coefficients in the economy).\(^{21}\)

Strategic complementarity and multiplicity of equilibria

There are several attempts in the literature to introduce strategic complementarity in information on acquisition in variants of the Grossman and Stiglitz model. Barlevy and Veronesi (2000) in a model with risk neutral traders that face a borrowing constraint, with noise traders, and where the fundamentals follow a binomial distribution claimed that as more traders acquire information prices need not become more informative and, in consequence, traders may want to acquire more information. This has been proved incorrect by Chamley (2007) because of a mistake in the expression for the value of information. The strategic complementarity in information acquisition and the existence

\(^{20}\) Diamond and Verrechia (1981) assume that each trader receives an (independent) endowment shock with variance \( K\sigma_u^2 \) where \( K \) is the number of traders. As \( K \) tends to infinity the average per capita supply \( u \) has variance \( \sigma_u^2 \) (and is uncorrelated with individual supplies).

\(^{21}\) See Section 1.5 in the Technical Appendix for the definition of first-order stochastic dominance shift in a distribution.
of multiple equilibria may be restored if the fundamentals and noise trading are correlated (Barlevy and Veronesi (2007)).

Multiplicity of linear REE is also obtained by Lundholm (1988) in a variation of the model by Diamond and Verrecchia (1981) where traders, on top of their private conditionally independent signals, have available also a public signal with an error term which is correlated with the error terms in the private signals. In this case a sufficiently high correlation in the errors will imply that more favorable (public or private) news may lead to lower prices. This happens because correlated signals provide both direct and indirect information on $\theta$. Indeed, a higher value of one signal indicates a higher $\theta$ but, with positive correlation in the errors, it indicates also larger errors in the other signals. The latter indirect effect may come to dominate the first direct effect for large enough error correlation.\footnote{With imperfect competition the effect disappears and we obtain a unique equilibrium (Manzano (1999)).}

Ganguli and Yang (2007) consider another variation of the model of Diamond and Verrecchia (1981) with a positive correlation between the aggregate endowment shock $u$ and the idiosyncratic endowment shock of a trader $u_i = u + \eta_i$ (with $\sum \eta_i d_i = 0$) and with endogenous information acquisition. They obtain that either there is no linear partially revealing equilibrium or there are two linear partially revealing equilibria.\footnote{Linear equilibria exist if and only if $4\mu^2 \tau_\eta \leq 1$ (using the same notation as usual).} In one equilibrium prices become more informative about $\theta$ as the proportion of informed $\mu$ increases and in the other the opposite happens. The price in the second equilibrium has a higher informativeness about $\theta$ than in the first equilibrium. The first equilibrium shares the features of the equilibrium in Grossman and Stiglitz (1980) and information acquisition decisions about the fundamental value are strategic substitutes. In the second equilibrium they are strategic complements. As the correlation between the aggregate endowment shock and the idiosyncratic one tends to zero the first equilibrium converges to the (unique) partially revealing equilibrium in Diamond and Verrecchia (1981).
Multiplicity arises in the Ganguli and Yang model because the individual endowment shock of a trader helps reading information about \( \theta \) in the price (which depends on the aggregate endowment shock) even if the trader receives a private (noisy) signal about \( \theta \). A high level of price informativeness is self-fulfilling since it implies that an informed trader puts less weight on his endowment shock when trying to estimate \( \theta \) and this translates into a smaller weight to the aggregate endowment shock in the price, making it in turn less noisy. An analogous argument can be made for a low level of price informativeness. The implications for the cost of capital (i.e. the required return to hold the stock of the firm \( E[\theta - p] \) or negative of the risk premium) depend on which equilibrium obtains. For a positive expected supply of shares, the expected stock price is increasing (decreasing) in \( \mu \) in the first (second) equilibrium. This means that the cost of capital for a firm will decrease with \( \mu \) in the first equilibrium and increase with \( \mu \) in the second. Strategic complementarity in information acquisition may lead to multiple equilibria in the information market. The model can be extended to allow traders to receive (or purchase) information on both the payoff and supply/noise trading \( u \) (instead of each trader receiving a random endowment shock). With conditionally independent signals, information about \( u \) allows traders to extract more information about \( \theta \) from prices. In this model there are also two linear equilibria and in any one of them information acquisition decisions may be strategic complements or substitutes depending on information parameters. Strategic complementarity occurs because

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24 In Section 4.4 we present a model where there is also positive correlation between the individual and the aggregate endowment shocks of traders but where, as in the original Grossman and Stiglitz formulation, all the informed receive the same signal (and they do not learn anything new about \( \theta \) from the price). In this case there is always a unique linear partially revealing equilibrium.

25 Easley and O’Hara (2004) analyze the effects of public and private information on the cost of capital and conclude (analyzing an equilibrium similar to Grossman and Stiglitz) that shifting information from public to private increases the cost of capital to a firm. (Note that this is a different comparative statics exercise than increasing the proportion of informed traders where the total amount of information increases.)

26 Palomino (2001) allows also traders to have private information about aggregate supply/noise trading and obtains a unique equilibrium but in his model there is no common component in the endowments of traders and therefore with price-taking behavior it is akin to the Diamond and Verrechia (1981) model.
with more informed traders the identification problem for the uninformed may become worse.

As we will see in Section 8.4 models with non-monotone demand schedules and, more in general, with multiple equilibria (both in the financial and in the information markets) prove useful when trying to explain frenzies in asset prices and market crashes. In the Barlevy and Veronesi (2000) model, and contrary to the models in this chapter, the demand of the informed may be upward sloping. The reason is that a low price may be bad news and indicate to the uninformed that the asset value is very low (much as in Section 3.2.2 a high price may be bad news as indicator of high costs and may yield a downward sloping supply curve for firms). Admati (1985) also obtains upward-sloping demands in a standard CARA-normal noisy rational expectations model with multiple assets.

Veldkamp (2006) finds that a market for information in a multimarket extension of the Grossman-Stiglitz model introduces also a strategic complementarity that works through the price for information. It is claimed that when many investors buy a piece of information its price is lower in a competitive market (because information has a high fixed cost and low variable cost of production) and this entices other investors to purchase the same information—even though it may be less valuable. The result is that investors only buy information about a subset of the assets and asset prices com move because news about one asset will affect the prices of the other assets. High covariance of asset prices in relation to the covariance of fundamentals has remained a puzzle (e.g. Barberis, Shleifer, and Wurgler (2005)). Veldkamp claims that the model can explain observed patterns of price comovements in asset prices since asset price correlations are above what a model with a constant price of information or a model where traders would purchase information about all the assets would predict. An important tenet of the model, obtained assuming a perfectly contestable market for information, is that the price of information equals the average cost of producing the news.

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4.2.3 Summary

The main results to retain from the standard model with noise trading are the following:

- Informed agents trade both to profit from their private information and from deviations of prices from expected fundamental values given public information (i.e. for market making purposes).
- Uninformed traders act as market makers providing liquidity to the market and trade less aggressively in the presence of privately informed traders because of adverse selection.
- Prices equal a weighted average, according to the risk tolerance-adjusted information of traders, of the expectations of investors about the fundamental value plus a noise component.
- Market makers protect themselves from the adverse selection problem by reducing market liquidity (the depth of the market) when they are more risk averse and/or there is less precise public information. The opposite happens when there is more noise trading.
- The informativeness of prices increases with the risk tolerance-adjusted informational advantage of informed traders, with the proportion of informed traders, and decreases with the volatility of fundamentals and the amount of noise trading. There is strategic substitutability in information acquisition.
- The volatility of prices depends, ceteris paribus, negatively on market depth, and positively on price precision, prior volatility, and noise trading. In any case volatility increases with the degree of risk aversion of uninformed traders and with prior volatility.

Departures from the standard model introducing correlation between idiosyncratic and common endowment shocks, between fundamentals and noise trading, between the error terms of private and public signals, or with traders receiving private signals on the amount of noise trading, yield multiple (linear) equilibria in the financial market and, potentially, strategic complementarity in information acquisition. Another way to obtain strategic complementarity in information acquisition is with economies of scale in information production.
4.3 Informed traders move first and face risk neutral competitive market makers

In this section we consider a trading game in which informed traders move first, with a proportion of them using demand schedules and the complementary proportion market orders, and risk neutral market makers set prices. We will derive the pricing implications of risk neutral market makers and the differential impact of traders using market orders and demand schedules.

Let us go back to the model in Section 4.2.1 but now consider a situation where informed traders and noise traders move first.29 Informed traders can submit demand schedules or market orders (we will explain later why this may be so). Their orders are accumulated in a limit order book. The limit order book is observed by a competitive risk neutral market making sector which sets prices. This competitive sector can be formed of scalpers, floor traders, and different types of market makers. The competitive market making sector observes the book $L(\cdot)$ which is a function of $p$, and sets price (informationally) efficiently:

$$p = E\left[0|L(\cdot)\right].$$

The efficient pricing (zero expected profit) condition can be justified with Bertrand competition among risk neutral market makers who have the same information, and each one of them observes the limit order book. Introducing risk neutral market makers means that a risk averse agent will be willing to trade and hold the risky asset, only if he has an informational advantage. This market microstructure involves sequential moves; however, as we will see below, it can be seen equivalent to simultaneous submission of orders by all traders.

There is a continuum of informed traders, indexed in the interval $[0, 1]$, with common constant coefficient of absolute risk aversion $\rho$. The information structure is as in 4.2.1 but now a proportion $\nu$ of traders submit demand schedules while a proportion $1-\nu$ place market orders. We restrict attention to linear equilibria with price equilibrium of

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the form $P(0,u)$. Given the symmetric information structure and preferences of traders without loss of generality we concentrate attention on symmetric linear Bayesian equilibria where the same type of trader uses the same strategy. Demands will have the form derived from the CARA representation and will be identical within each class of informed traders: $X(s_i, p), i \in [0, v]$ and $Y(s_i), i \in (v, 1]$.

Suppose that the strategies of informed traders are given as follows:

$$X(s_i, p) = a(s_i - \bar{\theta}) + \zeta(p), \text{ and } Y(s_i) = c(s_i - \bar{\theta})$$

where $a$ and $c$ are trading intensities and $\zeta(\cdot)$ is a linear function. The noisy limit order book schedule is given by (using the convention that the average signal equals $\theta$ a.s.):

$$L(p) = \int_0^1 X(s_i, p) di + \int_v^s Y(s_i) di + u = z + v\zeta(p),$$

where $z = A(\theta - \bar{\theta}) + u$ and $A = va + (1 - v)c$. The competitive market making sector observes $L(\cdot)$, a linear function of $p$, and sets: $p = E[\theta|L(\cdot)] = E[\theta|z]$. Notice that the random intercept $z$ of the limit order schedule $L(\cdot)$ is what is informative about $\theta$. As before, the random variable $z$ is observationally equivalent to the market price and can be thought as representing the new information contained in the market price. From standard normal theory and $p = E[\theta|z]$ it follows that $p = \lambda z + \bar{\theta},$ where $\lambda = \tau_u A / \tau$ and $\tau = \tau_0 + \tau_u A^2$.

Since $p$ is a linear function of $z$ and is normally distributed we have that $p = E[\theta|z] = E[\theta|p]$. This implies that the set of Bayesian linear equilibria of the sequential game and the one with simultaneous placement of orders to an auctioneer (as in Section 4.2) will be equivalent if there is a positive mass of risk neutral market

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30 Without loss of generality (with hindsight) in the linear symmetric class for each class of traders.
makers. Indeed, in this case in equilibrium we have necessarily that $E[θ|p] = p$ because otherwise the competitive risk neutral market makers would like to take unbounded positions.

Market makers take the counterpart of the limit order book and clear the market. An important effect of the existence of a risk neutral competitive market making sector is that total volatility, the sum of the volatility of price increments $\text{var}[p - \bar{θ}] + \text{var}[θ - p]$ is constant and equal to prior volatility of $θ, \sigma_0^2$. In fact, since $\text{var}[θ|p] = \text{var}[θ - p]$ total volatility can be expressed also as the sum of conditional volatilities $\text{var}[p|θ] + \text{var}[θ|p]$. This is a direct consequence of semi-strong efficient pricing: $p = E[θ|p]$. Note that $\text{var}[θ - p] = \text{var}[θ] - 2\text{cov}[θ, p] + \text{var}[p]$ but $\text{cov}[θ, p] = \text{var}[p]$. This follows since $p = E[θ|p], E[θp] = E[E[θ|p]] = E[pE[θ|p]] = E[p^2],$ and $E[p] = E[θ]$. We conclude that $\text{var}[θ - p] = \text{var}[θ] - \text{var}[p]$. Now, it is well known (see De Groot (p. 69, 1970)) that if $\text{var}[θ]$ is finite, then $\text{var}[θ] = E[\text{var}[θ|p]] + E[\text{var}[θ|p]]$. Therefore, if $p = E[θ|p]$ then $\text{var}[θ] = \text{var}[θ|p] + \text{var}[p]$ (recall that $(θ, p)$ are jointly normally distributed and consequently $\text{var}[θ|p]$ is non-random). Therefore, ex ante price volatility is given by

$$\text{var}[p] = \text{var}[θ] - \text{var}[θ|p] = (τ_0)^{-1} - τ^{-1}$$

and is increasing in the precision incorporated in prices $τ$. Prices are more volatile if they are more informative. We have therefore that

$$\text{var}[p - \bar{θ}] + \text{var}[θ - p] = \text{var}[p|θ] + \text{var}[θ|p] = \text{var}[θ].$$

An increase in the informativeness of prices only brings forward the resolution of uncertainty, increasing $\text{var}[p]$ and decreasing $\text{var}[θ|p]$, leaving the sum constant. This is the result of risk neutral competitive market making.
When market makers are risk averse, as in Section 4.2, then the direct link between price informativeness and volatility is broken. As we have seen then \( \text{var}[p] = \lambda^2 \tau (\tau_0 \tau_u)^{-i} \) and prices may be more volatile because there is more noise trading or because the market is shallower, for a given level of price precision.

The demands of informed traders are derived easily (see Exercise 4.2). The following proposition states the results:

**Proposition 4.2**: There is a unique linear Bayesian equilibrium. It is given by:

\[
X(s_i, p) = a(s_i - p), \quad \text{where} \quad a = \rho^{-1} \tau_c,
\]

\[
Y(s_i) = c(s_i - \bar{\theta}), \quad \text{where} \quad c = \left(\rho \left(\sigma_c^2 + \text{var}[p]\right)\right)^{-1}, \quad \text{and} \quad p = \lambda z + \bar{\theta},
\]

where

\[
z = \left[ A \left(\theta - \bar{\theta}\right) + u, \quad \lambda = \tau_u A / \tau, \quad \tau = \tau_o + \tau_u A^2, \quad \text{var}[p] = (\tau_o)^{-1} - \tau^{-1}, \quad \text{and} \quad A = \nu a + (1 - \nu) c.
\]

The parameter \( c \) is the unique solution to the cubic equation \( \tau_u A / \tau 
\]

\[
c \left( \frac{\tau_0 + \tau_c \tau_u (\nu \rho^{-1} \tau_c + (1 - \nu) c)^2}{\tau_0 + \tau_u (\nu \rho^{-1} \tau_c + (1 - \nu) c)^2} \right) = \rho^{-1} \tau_c \tau_o.
\]

**Market orders versus limit orders**

The strategies of traders using demand schedules are as before in its private information speculative component. Those of traders using market orders depend on the discrepancy between the private signal realization and the prior mean, weighted by a trading intensity \( c \) which is inversely related to risk aversion \( \rho \), noisiness in the signal \( \sigma_c^2 \), and the volatility of prices, \( \text{var}[p] \). Indeed, risk averse traders using market orders dislike price volatility. It is precisely because market order traders have to bear price risk that their trading intensity is smaller than the one of a limit order trader: \( c < a \).
The precision incorporated into prices depends now on the average responsiveness of traders to private information $\Lambda$. As the proportion $\nu$ of traders using demand schedules increases $\Lambda$ increases. As a result price precision $\tau$ as well as volatility $\text{var}[p]$ increases also. The direct effect of traders increasing their trading intensity as they switch from using market orders to demand schedules is larger than the indirect effect reducing the trading intensity of those using market orders due to the increased price volatility. The effect of $\nu$ on market depth $\lambda^{-1}$ is ambiguous. An increase in $\delta$ tends to provide more liquidity to the market and make prices more informative but at the same time worsens the adverse selection problem faced by market makers because the order book is more likely to contain information-based orders.\footnote{Note that $\frac{\partial(\lambda^{-1})}{\partial \nu} = \left(1 - \tau_\nu \left(\tau, A^2\right)^{-\epsilon}\right)(a - c)$. Since $a - c > 0$, we have that $\lambda^{-1}$ is increasing in $\nu$ when the incremental precision incorporated in the price due to trade $\tau A^2$ is larger than the...}

It should be clear that if the cost of placing a market order or a demand schedule were to be the same all traders would prefer to place the schedule. However, we need a series of limit and stop orders to construct a demand schedule. To place a demand schedule is bound to be more costly. Suppose that to place a demand schedule involves a (differential) fixed cost and that traders can choose whether to place a schedule and incur the cost or to place a market order at no cost. It is possible to show (Medrano (1996)) that, provided the differential fixed cost is neither too high or too low, there will be at least one interior equilibrium, in which traders partition themselves according to the type of orders they place. Informed traders with a high risk tolerance-adjusted informational advantage (that is, with high $\rho_i^{-1}\tau_{i\nu}$) place demand schedules while the rest of traders place market orders. Traders with a high $\rho_i^{-1}\tau_{i\nu}$ are willing to pay more to obtain the information contained in the price (that is, they benefit more from observing the price). This is akin to the result in the endogenous information model of Verrecchia (1982) where traders with high $\rho_i^{-1}\tau_{i\nu}$ are willing to spend more to improve their information. The reason is that traders with high $\rho_i^{-1}\tau_{i\nu}$ trade more aggressively and therefore benefit more from extra information.
We analyze next the comparative statics of the equilibria for two extreme cases: \( \nu = 1 \) and \( \nu = 0 \).

When all informed traders use demand schedules \( (\nu = 1) \) an increase in noise trading reduces directly the precision of prices \( \tau \) (and price volatility) even though the trading intensity of informed agents is not affected. An increase in risk aversion or in the noisiness of private information induces a decrease in \( \tau \) via a decreased trading intensity of informed agents. The depth of the market \( \lambda^{-1} \) is increasing in noise trading \( (\tau_u)^{-1} \) and non-monotonic in \( \rho \) and \( \tau_c \). In equilibrium, and depending on parameter values, the depth of the market may be increasing in the risk tolerance and the precision of information of informed agents. The explanation is that these changes increase the trading intensity of informed agents, which tends to decrease market depth, but this may be more than compensated by the induced increase in the precision of prices.\(^{32}\) The result is that we have that \( \lambda^{-1} \) is increasing in \( a = \rho^{-1} \tau_c \) if and only if \( \tau_0 - \tau_u a^2 < 0 \). An increase in the precision of private information, leading to an increase in \( a \), will imply a larger \( \lambda^{-1} \) if noise trading is small \( (\tau_u \text{ large}) \) and the levels of risk tolerance and precision of private information large (which imply that \( a \) is large).

The expected (aggregate) volume traded by informed agents is
\[
\mathbb{E} \left[ \int_0^1 X(s, p) \mathrm{d}i \right] = \left( \frac{2}{\pi} \right)^{1/2} a \sqrt{\tau^{-1}} \quad \text{(see Exercise 4.3).}
\]
Expected trading volume of the informed is increasing with noise trading \( (\tau_u)^{-1} \). When \( \tau_c \) tends to 0 so does the expected trade of the informed. The market makers then offer the counterpart to noise traders, \( \lambda = 0 \) and \( p = \bar{0} \).

When all informed traders use market orders \( (\nu = 0) \) we have a model which is a financial market counterpart of the Cournot model of Chapter 1 (while the rational

\(^{32}\) This explanation is therefore different from the one in Subrahmanyan (1991), where the phenomenon is attributed to an increase in the degree of competition among a finite number of insiders.
expectations model in Section 3.1 or the case \( \nu = 1 \) is a financial market counterpart of the supply function competition model of Chapter 3). Exercise 4.4 provides the comparative statics of the model.

There are several studies of the choice between market and limit orders in the literature. Brown and Zhang (1997) show, consistently with the model in this section, that a market with traders using limit orders induces more informational price efficiency than one with traders using market orders since in the former execution price risk is moderated. Chakravaty and Holden (1995) analyze this choice by an informed trader in a quote-driven system. In this case the informed trader may exploit limit orders by submitting a market order. Foucault (1999) analyzes the choice in a dynamic model and concludes that it is better to place a limit (market) order when the spread is large (tight). This analysis is extended by Goettler, Parlour and Rajan (2005). Harris and Hasbrouck (1996) and Biais, Hillion and Spatt (1995) provide evidence consistent with the last two theoretical pieces. Wald and Horrigan (2005) analyze the choice of a risk averse investor between a limit and a market order and estimate the parameters of the model with NYSE data.

Summary
The main learning points of the section are:

- The presence of a competitive risk neutral market making sector induces prices to reflect all publicly available information.\(^{33}\) That is, the market is semi-strong informationally efficient and prices are volatile because they are informative.
- As a consequence, total volatility is constant and equal to the volatility of fundamentals. An increase in informativeness of prices only brings forward the resolution of uncertainty.
- Sequential and simultaneous order placement need not yield different outcomes. This is so in the presence of competitive risk neutral market makers.

\(^{33}\) However, the precision of prices (i.e. their informational content) is the same than in a model where market makers are privately informed and risk averse. In other words, if we let \( \mu = 1 \) in the model of Section 4.2.1 and \( \nu = 1 \) in the model of Section 4.2.3, the informational content of the equilibrium price is the same in the two models: \( \tau = \tau_\mu + \tau_\nu \) with \( \tau \).
• Risk averse traders using market orders are more cautious when responding to their information than limit order (demand schedule) traders because they are subject to price volatility.
• As a result, when the proportion of traders using demand schedules increases, so does price precision and volatility (and the impact on market depth is ambiguous).
• Whenever there is a differential fixed cost to submit a demand schedule instead of a market order, traders with a large risk tolerance-adjusted informational advantage place demand schedules while the others place market orders.

4.4 Hedgers and producers in a futures’ market
Up to know we have considered markets where some exogenous noise traders are present and drive the trade. Their presence is motivated by unspecified liquidity reasons and allows for REE not to be fully revealing as well as trade in the presence of asymmetric information. This is unsatisfactory because the decisions of noise traders are not modeled, it is not explained why these traders are willing to lose money in the market, and consequently a proper welfare analysis cannot be performed. In this section we endogenize the presence of noise traders with risk averse hedgers. We present a variation of the model of Section 4.2 replacing noise traders by risk-averse competitive hedgers and assuming that all informed traders receive the same signal (as in Grossman and Stiglitz (1980), see Section 4.2.2; we follow here Medrano and Vives (2007)).

We want to examine the relationships between information and insurance in a financial market populated by risk averse traders and the derived incentives for real investment. The model will allow us to study the welfare consequences of improvements in private information and of public information release. Several papers in the literature emphasize the allocational role of financial prices guiding production and investment decisions (e.g. Leland (1992), Dow and Gorton (1997), and Subrahmanyam and Titman (1999)). Chen, Goldstein and Jiang (2007) display empirical evidence on how managers learn from the (private) information in the stock price and incorporate it in their corporate investment decisions. In the model of the present section the informativeness of prices in the financial market will affect production incentives only indirectly since real decisions are taken before the financial market opens.
The risky asset is a futures contract for a good (say agricultural product or raw material) with future random spot price $\theta$. The futures contract trades at price $p$. Producers want to hedge their production in the futures market at $t = 2$ and obtain private information at $t = 1$ about the future value of the product once the production process has been set (say, the seeds have been planted) at $t = 0$ (as in Bray (1981, 1985), see the timeline below). We first set up the model and the demands for the different traders, then analyze the market equilibrium and study the effect of information on production. A preview of results follows. The private information of producers can not help production decisions, because it comes too late, but allows them to speculate in the futures market. This creates adverse selection in the future's market where uninformed speculators (market makers) and other hedgers operate. This will tend to diminish the hedging effectiveness of the futures market and diminish consequently the output of risk averse producers (since they will be able to hedge less of their production). The adverse selection is aggravated with more precise private information. Adverse selection is eliminated if the signal received by producers is made public. However, more public information may decrease production (and the expected utility of all traders) because it destroys insurance opportunities (this is the “Hirshleifer effect”, see Section 3.1 and Section 5.4 also). The model also shows under what circumstances hedgers have demands of the “noise trader” form.

4.4.1 A futures market with hedgers
We have a single risky asset (the futures contract), with random liquidation value $\theta$ (the future spot price), and a riskless asset, with unitary return, traded among a continuum of risk-averse competitive uninformed speculators (market makers), a continuum of risk-averse competitive hedgers, and a continuum of risk-averse competitive informed speculators. The risky asset is traded at a price $p$ and thus generates a return $\theta - p$.

Informed traders. There is a continuum of informed traders with mass one. They are producers of a good with random future spot price $\theta$. The representative informed trader:

- Receives a private signal $s = \theta + \varepsilon$, where $\theta$ and $\varepsilon$ are independent, and $E[\varepsilon] = 0$.
- Has a level of production $q$ with cost function $C(q) = c_1q + c_2q^2/2$ where
\( c_1 \geq 0 \) and \( c_2 \geq 0 \).

- Is risk averse with CARA utility: 
  \[ U_1(W_1) = -\exp\{-\rho W_1\} \], \( \rho > 0 \), where 
  \[ W_1 = 0q - C(q) + (\theta - p)x_1 \] is his final wealth when buying \( x_1 \) futures contracts. His position in the futures market is then \( q + x_1 \).

- Submits a demand schedule contingent on the private information \( s \) he observes. If \( x_1 > 0 \) he is a net buyer of futures while he is a net supplier if \( x_1 < 0 \). In equilibrium we will see that \( \operatorname{E}[x_1] < 0 \) and informed traders will sell on average.

An informed trader has three motives to trade in the market for futures. First, he is interested in trading in order to hedge part of the risk coming from his production \( q \) (\( 0q - C(q) \) is the random value of the producer's endowment before trading in the securities' market that needs to be hedged). Secondly, he may trade for speculative reasons in order to exploit his private information about \( \theta \). Finally, he will speculate also on the differences between prices and the expected value of \( \theta \) (i.e. for market making purposes).

*Market makers.* There is a continuum of competitive uninformed speculators (or market makers) with unitary mass also. The final wealth of a representative market maker buying \( x_u \) shares at price \( p \) is given by \( W_u = (\theta - p)x_u \), where his initial non-random wealth is normalized to zero.\(^{34}\) Market makers trade in order to obtain profits by absorbing some of the risks that the informed traders and hedgers try to hedge (their trades are not motivated by any informational advantage or any need of hedging). A representative market maker is risk averse with CARA utility \( U_u(W_u) = -\exp\{-\rho_u W_u\} \), \( \rho_u > 0 \) and submits a demand schedule. Since they have rational expectations, they use their observation of the price to update their beliefs about \( \theta \).

\(^{34}\) Recall that with constant absolute risk aversion, a trader's demand for a risky asset does not depend on his initial non-random wealth, and we can assume (without loss of generality) that speculators have zero initial wealth.
Hedgers. There is a continuum of competitive hedgers with unitary mass, indexed in the interval [0, 1]. Hedger j:

- Has an initial endowment $u_j$ of an asset with future (random) value $z$ correlated with $\theta$. This could be the random production of a related good which is not traded in a futures market.
- Has final wealth $W_j = u_j z + (\theta - p)x_j$ when buying $x_j$ shares at price $p$.
- Is risk averse with CARA utility $U(W_j) = -\rho H_j H_j U(W) \exp{-\rho H_j H_j W_j}$, $\rho > 0$.
- Privately observes $u_j$ and places a demand schedule contingent on his private information $u_j$.

We assume that $u_j$ may be written as $u_j = u + \eta_j$, where $u$ and $\eta_j$ are independent (and $\eta_j$ is independent of $\eta$ for all $j \neq i$). The usual convention that errors cancel out in the aggregate, $\int_0^1 \eta_j dj = 0$ a.s., will be used. As a result, $\int_0^1 u_j dj = \int_0^1 (u + \eta_j) dj = u + \int_0^1 \eta_j dj = u$ a.s., so that $u$ is the aggregate risky endowment of the hedgers. A hedger uses the observation of the price to update his beliefs about $\theta$. Hedgers' main motive to trade is to reduce risks. However, the endowment shock to hedger $j$ is his private information and therefore their demand has also a speculative component.

Timing. At $t = 0$, producers choose the level of production $q$. The level of production $q$ is public information. At $t = 1$, each producer receives a private signal $s$ about $\theta$ and hedger $j$ an endowment shock $u_j$, and the demand schedules of all traders placed. At $t = 2$ the market clearing price is set and trade takes place. Finally, at $t = 3$, the terminal values $z$ and $\theta$ and payoffs are realized.
Distributional assumptions. All random variables are assumed to be normally distributed: \( \theta \sim \mathcal{N}(\bar{\theta}, \sigma^2_\theta), z \sim \mathcal{N}(z, \sigma^2_z), u \sim \mathcal{N}(0, \sigma^2_u), \varepsilon \sim \mathcal{N}(0, \sigma^2_\varepsilon), \) and \( \eta_j \sim \mathcal{N}(0, \sigma^2_\eta) \) for all \( j \). Without loss of generality, we assume that \( z \) may be written as \( z = \theta \left( \frac{\theta}{\sigma_\theta} \right) + \sqrt{1 - r^2_{z \theta}} \gamma \), where \( r_{z \theta} \) is the correlation coefficient between \( z \) and \( \theta \), and \( \gamma \sim \mathcal{N}(0,1) \) is independent of any other variable in the model. Moreover, we assume that \( \text{cov} \left[ \theta, u \right] = \text{cov} \left[ s, u \right] = \text{cov} \left[ \theta, u_j \right] = \text{cov} \left[ s, u_j \right] = \text{cov} \left[ \theta, \varepsilon \right] = \text{cov} \left[ \eta_j \right] = 0 \) for all \( j \) and \( \text{cov} \left[ \eta_j, \eta_i \right] = 0 \) for all \( j \neq i \).

Let \( \xi \) denote the square of the correlation coefficient between \( s \) and \( \theta \), \( \xi = \frac{\sigma_{\theta s}^2}{\sigma_\theta^2 + \sigma_s^2} \), and let \( \xi_u \) denote the square of the correlation coefficient between \( u \) and \( u_j \), \( \xi_u = \frac{\sigma_{u u}^2}{\sigma_u^2 + \sigma_u^2} \).

Throughout the section, the subscript \( I \) will refer to the informed traders; the subscript \( U \) will refer to the uninformed speculators, and the subscript \( H \) will refer to the hedgers.

4.4.2 Equilibrium in the futures market

We will restrict attention to Bayesian linear equilibria with price functional of the form \( P(s, u) \). In order to find the linear equilibrium we follow the standard procedure.

We posit candidate linear strategies, derive the linear relationship between prices and the underlying random variables, work through the optimization problems of traders to derive their demands and finally determine the coefficients of the linear strategies. Given the information structure and preferences of the different types of traders, and as in Section 4.2.1, the equilibrium will be symmetric (i.e. same strategy for each class of traders).

The strategies may be written (with hindsight and without loss of generality) as follows. For an informed trader,

\[
X_I(s, p) = a(s - \bar{\theta}) + b_I(\bar{\theta} - p) - \gamma_I q
\]

where \( a, b_I, \) and \( \gamma_I \) are endogenous non-random parameters.\(^{35\text{a}}\)

\(^{35\text{a}}\) That is, we should write \( X_I(s, p) = -b, p + \varphi_i \) but in equilibrium we will have \( \varphi_i = (-a + b_I)\bar{\theta} - \gamma_i q \) (and similarly for the other types of traders).
For a market maker,

$$X_U(p) = b_U(\bar{\theta} - p) - \gamma_U q$$

Where $b_U$, and $\gamma_U$ are endogenous non-random parameters.

For hedger $j$,

$$X_h(p, u_j) = b_h(\bar{\theta} - p) - \delta u_j - \gamma_h q$$

where $b_h$, $\gamma_h$, and $\delta$ are endogenous non-random parameters.

The market clearing condition is

$$X_U(p) + X_h(p, u) + X_I(p, s) = 0,$$

where $X_h(p, u) = \int_0^1 X_h(p, u) \, dj = b_h(\bar{\theta} - p) - \delta u - \gamma_h q$ (since $\int_0^1 u_j \, dj = u$) is the hedgers' aggregate demand. Given the linear strategies posited above the equilibrium price is a linear function of the private information $s$, the hedgers' random aggregate endowment $u$ (errors $\eta_j$ cancel in the aggregate), and production $q$:

$$p = \bar{\theta} - \Gamma q + \frac{\{a(s - \bar{\theta}) - \delta u\}}{\Lambda}$$

where $\Gamma = (\gamma_1 + \gamma_U + \gamma_H) / \Lambda$, and $\Lambda = b_i + b_U + b_h$.

Hedger $j$ will choose $x_j$ to maximize $E[U_{hi}(W_j) \mid p, u_j]$, where $W_j = u_j z + (\theta - p)x_j$, or (since all random variables, including the price, are normally distributed)

$$\exp\{-p_{hi}(E[W_j \mid p, u_j] - \rho_{hi} \text{var}[W_j \mid p, u_j]/2)\}.$$  
We have that $E[W_j \mid p, u_j] = u_j E[z \mid p, u_j] + (E[\theta \mid p, u_j] - p)x_j$ and

$$\text{var}[W_j \mid p, u_j] = u_j^2 \text{var}[z \mid p, u_j] + x_j^2 \text{var}[\theta - p \mid p, u_j] + 2u_j x_j \text{cov}[z, \theta - p \mid p, u_j]$. From the first order condition, hedger $j$'s optimal demand for shares is given by

$$X_{hi}(p, u_j) = \frac{E[\theta - p \mid p, u_j] - \rho_{hi} u_j \text{cov}[z, \theta - p \mid p, u_j]}{\rho_{hi} \text{var}[\theta - p \mid p, u_j]}.$$  

Hedger $j$'s demand may be decomposed in two terms:
• Speculative demand: 
  \[ \frac{E\left[ \theta - p \mid p, u_j \right]}{\rho_{u \theta} \text{ var}\left[ \theta - p \mid p, u_j \right]} \], 

  which will depend on \( q \) (because this helps reading the information about \( s \) in the price) and on \( u_j \) provided that \( \xi_{\text{u}} > 0 \) (because then \( u_j \) contains information on \( u \) which in turn helps to recover information about \( s \) in the price) and

• Hedge supply: 
  \[ \frac{-\text{cov}\left[ z, \theta - p \mid p, u_j \right]}{\text{var}\left[ \theta - p \mid p, u_j \right]} u_j, \] 

  which can be seen to be equal to \( -\frac{\sigma_{\text{u}}}{\sigma_z} u_j \). \(^{36}\) The amount of the hedger's initial endowment \( (u_j) \) that is hedged in the market is proportional to the correlation between the value of the hedger's asset \( z \) and the return of the risky security \( \theta - p \) conditional on the hedger's information \( \{p, u_j\} \).

Similarly (just looking at the speculative component), optimization of the CARA utility for an uninformed yields

\[ X_{U}(p) = \frac{E[0 - p]}{\rho_{u \theta} \text{ var}[0 - p]}, \]

which is linear in \( p \) since \( \text{var}[0 - p] \) is constant and \( E[0 - p \mid p] \) is linear in \( p \) due to the normality assumption. All the speculators will place the same demand schedule (since all of them have the same information), so that the speculators' aggregate demand \( X_{U}(p) \) will be given by the same expression. The demand will depend on \( q \) because the knowledge of \( q \) is needed to infer information about \( s \) from the price.

The representative informed trader’s maximization problem is the following:

\(^{36}\) Since \( z = \sigma_z \left( \frac{z}{\mu} \right) \theta + \sqrt{1 - \rho_{z\theta}^2} \) \( y \) and \( y \) is independent of any other random variable we have that

\[ \frac{\text{cov}[z, \theta - p \mid p, u_j]}{\text{var}[\theta - p \mid p, u_j]} = \frac{\rho_{u \theta}}{\rho_{u \theta}} \left( \frac{\sigma_z}{\sigma_{\theta}} \right) \frac{\text{cov}[\theta, \theta - p \mid p, u_j]}{\text{var}[\theta - p \mid p, u_j]} \]

which gives the result since \( \text{cov}[\theta, \theta - p \mid p, u_j] = \text{var}[\theta \mid p, u_j] \), \( \text{var}[\theta - p \mid p, u_j] = \text{var}[\theta \mid p, u_j] \) and

\[ \rho_{u \theta} = \sigma_{u \theta} / \sigma_{\theta} \sigma_z. \]
\[
\max_{s_i} E[-\exp\{-\rho_1 W_i\}|s,p]
\]
where \(W_i = 0q - C(q) + (\theta - p)x_i\). Given normality this is equivalent to maximizing
\[
E[W_i|s,p] - \frac{\rho_1}{2} \text{var}[W_i|s,p] = qE[\theta|s] + x_i \left\{ E[\theta|s] - p \right\} - \frac{\rho_1}{2} (x_i + q)^2 \text{var}[\theta|s].
\]

Note that the price does not provide an informed trader with any further information about \(\theta\) over and above the signal \(s\) and therefore \(E[\theta|s,p] = E[\theta|s]\) and \(\text{var}[\theta|s,p] = \text{var}[\theta|s]\). However, although the price has no information to aggregate, it is still useful from the informed trader's point of view since it allows him to infer the exact amount of noise trading (and thus eliminate the price risk it creates). If \(\rho_1 \text{var}[\theta|s] > 0\), then
\[
X_i(s,p) = \frac{E[\theta|s] - p}{\rho_1 \text{var}[\theta|s]} - q
\]
where \(E[\theta|s] = \bar{\theta} + \xi (s - \bar{s})\) and \(\text{var}[\theta|s] = (1 - \xi)^2 \sigma^2_\theta\). We may write the demand as
\[
X_i(s,p) = \frac{1}{\rho_1 \sigma^2_\xi} (s - p) + \frac{1}{\rho_1 \sigma^2_\theta} (\bar{\theta} - p) - q = a(s - \bar{\theta}) + b_1(\bar{\theta} - p) - q
\]
where \(a = 1/\left(\rho_1 \sigma^2_\xi\right)\), \(b_1 = 1/\left(\rho_1 (1 - \xi)^2 \sigma^2_\theta\right)\).

An informed trader's asset position can be decomposed into two terms:

- **Speculative demand**: \(\frac{E[\theta|s] - p}{\rho_1 \text{var}[\theta|s]}\), according to which the informed trader buys (sells) if his estimate of the asset liquidation value is greater (lower) than the price.
- **Hedge supply**: \(q\). Since the representative informed agent is strictly risk averse and price-taker he hedges all the endowment risk, \(\gamma_i = 1\) (provided that he is imperfectly informed, i.e. \(\sigma^2_\xi > 0\) or \(\xi < 1\)).

In order to characterize a linear equilibrium, using the expression for the price, the expressions for \(E[\theta|p,u_j]\), \(\text{var}[\theta|p,u_j]\), \(E[\theta|p]\), and \(\text{var}[\theta|p]\) are plugged back in \(X_{hi}(p,u_j)\) and \(X_{ui}(p)\) and a solution is found for the undetermined coefficients of the
linear strategies. The result is presented in the following proposition (see Medrano and Vives (2007) for a proof).

**Proposition 4.3.** If $\xi < 1$, there is a unique linear Bayesian equilibrium. It is characterized by

$$X_{u}(p) = b_{u}(\overline{\theta} - p) - \gamma_{u}q$$
$$X_{h}(u,p) = b_{h}(\overline{\theta} - p) - \delta u - \gamma_{h}q$$
$$X_{i}(s,p) = a(s - \overline{\theta}) + b_{i}(\overline{\theta} - p) - q$$
$$p = \overline{\theta} - \Gamma q + \frac{\{a(s - \overline{\theta}) - \delta u\}}{\Lambda}$$

where

$$\Lambda = b_{i} + b_{u} + b_{h}, \quad \Gamma = (1 + \gamma_{u} + \gamma_{h}) / \Lambda,$$

$$a = \frac{1}{\rho_{i}\sigma_{e}^{2}}, \quad b_{i} = \frac{1}{\rho_{i}(1 - \xi)\sigma_{\theta}^{2}}$$

$$b_{u} = \frac{\delta^{2}\sigma_{u}^{2}\sigma_{\theta}^{2}}{(\rho_{u}\left[\delta^{2}\sigma_{u}^{2} + a^{2}\sigma_{e}^{2}\right] + a)}\left[\rho_{u}\left[1 + \delta^{2}\sigma_{u}^{2} + a^{2}\sigma_{e}^{2}\right] + a\right] - a^{2}$$

$$b_{h} = \frac{\delta^{2}\sigma_{u}^{2}\sigma_{\theta}^{2}}{\left[\rho_{u}\left[1 + \delta^{2}\sigma_{u}^{2} + a^{2}\sigma_{e}^{2}\right] + a\right]}\left[\rho_{u}\left[1 + \delta^{2}\sigma_{u}^{2} + a^{2}\sigma_{e}^{2}\right] + a\right] - a^{2}$$

$$\gamma_{u} = \frac{-a}{\rho_{h}\left[1 + \delta^{2}\sigma_{u}^{2} + a^{2}\sigma_{e}^{2}\right]}(1 + aE), \quad \gamma_{h} = \frac{-a}{\rho_{u}\left[1 + \delta^{2}\sigma_{u}^{2} + a^{2}\sigma_{e}^{2}\right]}(1 + aE),$$

with

$$E = \frac{1}{\rho_{o}}\left[\delta^{2}\sigma_{u}^{2} + a^{2}\sigma_{e}^{2}\right]^{-1} + \frac{1}{\rho_{u}}\left[1 + \delta^{2}\sigma_{u}^{2} + a^{2}\sigma_{e}^{2}\right]^{-1}.$$

The parameter $\delta$ is the unique solution of the (implicit) cubic equation

$$\delta = \left(\frac{\sigma_{\theta}^{2}}{\sigma_{\phi}^{2}}\right)^{-1} \left[1 + \rho_{h}\overline{\xi}_{u}\left(1 - \overline{\theta}_{u}\delta^{2}\sigma_{u}^{2} + \rho_{i}\right)\right]^{-1}.$$

From the solution to the cubic equation for $\delta$ we obtain the rest of the equilibrium parameters.
The expected price is equal to the prior expected liquidation value minus a risk premium, $\bar{p} = \bar{\theta} - \Gamma q$. The risk premium is positive and is directly proportional to the level of the endowment of informed traders (production), where $\Gamma = (\gamma_U + \gamma_H + 1)/\Lambda$. The equilibrium parameter $\Lambda = b_i + b_u + b_H$ is related to market depth. In terms of our previous lambda we have that $\lambda = \left| \frac{\partial p}{\partial u} \right| = \delta / \Lambda$. The market is deeper the more traders respond to price movements and the less hedgers react to their endowment shock (where $\sigma_\theta^2 < \delta < \sigma_a^2$).

The price is informationally equivalent to $\{a(s - \bar{\theta}) - \delta u\}$ and therefore information (s) and the aggregate endowment shock (u) are the sources of price volatility. As before the price precision is $\tau = (\text{var}[\theta|p])^{-1}$ where, since the price is informationally equivalent also to $\{\theta + \varepsilon - \frac{\delta}{a} u\}$ given that $a > 0$,

$$\tau = \tau_\theta + \frac{1}{\tau_\varepsilon^{-1} + \delta^2 \left(a^2 \tau_u\right)^{-1}}.$$

The price contains information about $\theta$ if and only if traders with information on fundamentals trade on the basis of that information (i.e. $a > 0$). Thus, it is natural to expect that the higher the traders' sensitivity to information on fundamentals, the more informative the price. This is true in equilibrium.

Producers, on average, are net suppliers of the risky asset. That is, $\mathbb{E}[x_i] = q((a + b_i)\Gamma - 1) < 0$. Since the risk premium is positive, the ex ante expected value of the speculative demand is positive but the hedge supply $-q$ is larger in equilibrium.

The following patterns can be shown (see Medrano and Vives (2007)): increasing $\xi$ increases the trading signal sensitivity of informed traders (a) and this drives price precision $\tau$ upward (the latter follows from the expression for $\tau$ and the easily checked fact that $\delta$ decreases with $\tau_\varepsilon$ or $\xi$). Simulations (see footnote 38 for the parameter grid) show that increases in $\xi$ decrease the price responsiveness of market makers ($b_{U}$).
and hedgers ($b_h$). Uninformed traders protect themselves by attempting to reduce market depth (increasing $\Lambda^{-1}$) when the informed have a signal of better quality. This effect together with the increase in $a$ dominates in the simulations other effects (such as the decrease in $\delta$) and drives price volatility \[ \text{var}[p] = \Lambda^{-1} \left( a^2 \left( \sigma_u^2 + \sigma_x^2 \right) + \delta^2 \sigma_x^2 \right) = \Lambda^{-1} \left( \rho_1^2 \left( \tau_x \left( 1 - \xi \right) \right) + \delta^2 \tau_u^{-1} \right) \] up. We have that $\delta/\Lambda$ is hump-shaped as a function of $\xi$. Note that $\Lambda = b_i + b_u + b_h$, where $b_u$ and $b_h$ are strictly decreasing in $\xi$ and $b_i = \frac{1}{\rho_1(1-\xi)\sigma_u^2}$ is strictly increasing in $\xi$. For $\xi$ low, the first effect dominates and $\delta/\Lambda$ is increasing in $\xi$, while the opposite occurs for $\xi$ high. (As $\xi \to 1$ we have that $b_i$ as well as $\Lambda$ tend to infinity, and $b_u$ and $b_h$ tend to zero.) In consequence, market depth $(\delta/\Lambda)^{-1}$ is U-shaped as a function of information precision $\xi$.

If $\xi=1$ (perfect information) or $\rho_1=0$ (risk neutrality for the informed), the only possible equilibrium would be characterized by $p = E[\theta|s]$. The informed are indifferent about what to trade since $p = E[\theta|s]$. The market makers are also indifferent if $\xi=1$ (since then $p = E[\theta|s] = E[\theta|p] = 0$ and they face no risk $\text{var}[\theta|p] = 0$), and they do not trade if $\rho_1 = 0$ (since then $p = E[\theta|s] = E[\theta|p]$ but they face risk). This would constitute a fully revealing REE but it is not implementable in demand functions.

4.4.3 Hedgers and noise traders

The market microstructure models that we have studied assume the existence of noise traders, agents that trade randomly for unspecified liquidity reasons. Are there circumstances in which rational expected utility maximizing agents give rise to demands for assets of the “noise trader” form? Are expected losses an appropriate measure of their welfare? The answer is that the order flow will contain an exogenous supply $u$ (independent of any deep parameter of the model) whenever $z$ is perfectly correlated with $\theta$ and the risk tolerance-adjusted informational advantage of a hedger is vanishingly small ($\xi_u/\rho_h$ tending to 0). This happens if hedgers are infinitely risk averse ($\rho_h \to \infty$) or if there is no correlation between each individual endowment
shock $u_j$ and the average $u$ ($\xi_u \to 0$).\(^{37}\) In the first case hedgers just get rid of all the risk associated to their endowment and supply $u$ in the aggregate. In the second, hedgers are exactly like market makers because they have no informational advantage. In the aggregate they supply again $u$ but now they take a speculative position also. In both cases we can evaluate their expected utility. Indeed, according to the proposition above as $\xi_u / \rho_H$ or $\xi / \rho_i$ tend to 0, $\delta$ tends to $\sigma_{oz} / \sigma_\varrho^2$ and if in addition $\sigma_{oz} = \sigma_u^2$, the equilibrium demand of the hedgers tends to $X_H(u) = b_H(\tilde{\theta} - p) - u - \gamma_H q$, with $b_H \geq 0$ and $\gamma_H < 0$. (Furthermore, as $\xi / \rho_i \to 0$ we have that $\gamma_H \to 0$.) If $\rho_H \to \infty$ then $b_H, \gamma_H \to 0$ and $X_H(u) \to -u$.

In summary, “noise trader” type demands arise when hedgers are very risk averse (or when their personal shock is almost uncorrelated with the aggregate one). This will have important implications for the welfare analysis of the impact of parameter changes on the utility of hedgers. Their utility is typically evaluated in noise trader models in terms of the losses they make, i.e. as if they were risk neutral. In the usual CARA-normal models the expected losses of noise traders (trading $u$) are $\lambda \sigma_u^2$ where $\lambda$.\(^{37}\) is market depth.

4.4.4 Production, insurance, and private information

For a given $q$, a producer’s ex ante expected utility, after long and tedious manipulations (see Exercise 4.5), can be seen to be given by the product of three terms: The utility derived from the speculative demand $|SG_i|$, the utility derived from the insurance achieved via the hedge supply $|IG_i|$, and the utility coming from production

\[
\exp\{-\rho_i[q\bar{\theta} - C(q) - (\rho_i/2)q^2\sigma_\varrho^2]\}\). That is,

\[
J_i(q) \equiv E[-\exp\{-\rho_i W_i\}] = \left|SG_i\right| \left|IG_i\right| \exp\{-\rho_i[q\bar{\theta} - C(q) - (\rho_i/2)q^2\sigma_\varrho^2]\}
\]

where

\(^{37}\) Sarkar (1994) presents results in related models. As stated in Section 4.2.2 the case of no correlation between the individual and aggregate endowment shocks is considered in Diamond and Verrecchia (1981) and Verrecchia (1982).
\[
|SG_i| = \left(1 + \frac{\rho_i^2(1-\xi)\sigma_u^2(\xi\sigma_u^2 + \delta^2 (b_u + b_{hi})^{-2} \sigma_u^2)}{[\rho_i(1-\xi)\sigma_u^2 + (b_u + b_{hi})^{-1}]^2}\right)^{-1/2}
\]

and

\[
|IG_i| = \exp\{-\left(\frac{\rho_i^2}{2}\right)\sigma_u^2 d q^2\}.
\]

The key endogenous parameter \(d\) represents the hedging effectiveness of the market. It is a complicated expression of the deep parameters of the model (see Exercise 4.5). The speculative term has two components. The term \(\xi\sigma_u^2\) is associated to gains from private information and the term \(\delta^2 (b_u + b_{hi})^{-2} \sigma_u^2\) with gains from market making. The private information gains disappear, obviously, when there is no private information (\(\xi = 0\)).

The optimal production level solves \(\max_q J_1(q)\) or, equivalently,

\[
\max_q [q\bar{\theta} - C(q) - \frac{1}{2} \rho_i \sigma_u^2 q^2 (1-d)].
\]

The optimal level of production is obtained by equating (expected) marginal value \(\bar{\theta}\) to marginal cost \(C'(q) + \rho_i \sigma_u^2 (1-d)q\), which is the sum of the marginal production costs \(C'(q) = c_1 + c_2 q\) and the (opportunity) cost related to the riskiness of real investment \(\rho_i \sigma_u^2 (1-d)q\). The optimal level of real investment is increasing in \(d\), which is a measure of hedging effectiveness of the asset market from a producer's point of view (see Figure 2):

\[
q^* = \frac{\bar{\theta} - c_1}{c_2 + \rho_i \sigma_u^2 (1-d)}.
\]

To perform comparative statics with the model is complicated because of the complexity of the equations determining the endogenous equilibrium parameters.
However, using simulations we can obtain results.\footnote{We take postulate as central case that \( \rho_{H} \geq \rho_{I} > \rho_{U} \) and that volatilities are not too far from market values (similar to those in Leland (1992), for example). The ranking of coefficients of risk aversion seems reasonable: hedgers are the most risk averse and market makers the least. The base case has \( \rho_{H} = 3, \ \rho_{I} = 2, \) and \( \rho_{U} = 1; \) volatilities are given by \( \sigma_{H} = .2, \ \sigma_{I} = .1, \ \sigma_{U} = .2 \) and covariances by \( \sqrt{\xi_{0}} = .1, \ r_{0} = \frac{\sigma_{0}}{\sigma_{H} \sigma_{I}} = .91; \) and \( \bar{\theta} = 1, \ \bar{\xi} \) ranges from 0 to 1. We consider a case with positive production costs, \( c_{1} = .9, c_{2} = .02, \) and another with no costs \( c_{1} = c_{2} = 0. \) We also consider variations in \( \rho_{H}, \ \rho_{I}, \ \sqrt{\xi_{0}}, \ \sigma_{I} \) and \( \sigma_{U} : \) a volatility of the fundamental value of \( \sigma_{0} = 0.6, \) which is of the NASDAQ type in contrast with the base case of \( \sigma_{0} = 0.2, \) which is of the NYSE type; \( \rho_{H} = 6, \ \rho_{I} \in \{1, 2, 5, 1.5\}, \) high noise scenarios with \( \sigma_{I} \in \{0.5, 0.6, 0.7\} \) and \( \sqrt{\xi_{0}} \in \{0.4, 0.5\}. \)}

Figure 2: Investment (q) and hedging effectiveness of the market (d)

The direct impact of an increase in risk aversion \( \rho_{I} \) or underlying risk \( \sigma_{0}^{2} \) is to decrease
q∗. There are other indirect effects operating through d but the simulations performed with the model indicate that the direct effects prevail. An increase in the cost parameters c1, c2 unambiguously decreases production. When the market is totally ineffective in hedging, or there is no future's market, d = 0 and q∗ = qo = \frac{\Theta - c_1}{c_2 + \rho_1 \sigma_0} \ (see \ Figure \ 2). \ This \ happens \ as \ \xi \to 1 \ (see \ Exercise \ 4.5). \ The \ parameter \ d \ is \ decreasing \ in \ \xi \ according \ to \ the \ simulation \ of \ the \ model. \ The \ better \ the \ private \ information \ of \ producers \ the \ more \ the \ futures \ market \ faces \ an \ adverse \ selection \ problem \ and \ its \ hedging \ effectiveness \ is \ reduced.

A producer's ex ante expected utility may be written as the product of the speculative component with production and insurance gains

\[ J_1(q^*) = -|SG_1| \exp\{-\rho_1(\Theta - c_1)q^*/2\}. \]

The speculative component of utility is hump-shaped in \xi. \ For \ low \ \xi \ an \ increase \ in \ signal \ precision \ improves \ speculative \ benefits \ but \ for \ high \ \xi \ the \ opposite \ happens \ because \ information \ revelation \ is \ “too \ strong”. \ Production \ and \ insurance \ gains \ are \ decreasing \ in \ \xi \ because \ q^* \ is \ decreasing \ in \ \xi. \ The \ result \ is \ that \ J_1(q^*) \ is \ decreasing \ with \ \xi \ for \ “normal” \ values \ of \ parameters \ or \ hump \ shaped \ with \ \xi \ for \ more \ extreme \ parameter \ configurations (high noise scenarios).

The uninformed speculators’ ex ante expected utility \( EU_U \) can be seen (see Exercise 4.5) to increase, for given \( \text{var}\left[ E[\Theta | p] - p \right] \) and \( \text{var}[\Theta | p] \), with the risk premium \( \Gamma q \), which is nothing else but the expected margin \( E[\Theta - p] = \Theta - \bar{p} = \Gamma q \). \ The \ risk \ premium \ decreases \ as \ \xi \ increases \ and \ this \ leads \ to \ a \ decrease \ in \ \( EU_U \). \ In \ all \ cases \ considered \ in \ the \ simulations \ we \ find \ that \ \( EU_U \) \ is \ decreasing \ in \ \xi.

The expressions for the expected utility of a hedger \( EU_H \) are complicated (and we need to assume that \( \xi_u / \rho_H \) is small since otherwise \( EU_H \) diverges to minus infinity) but an increase in \( \xi \) typically decreases \( EU_H \) because \( q \) is decreasing in \( \xi \). \ This \ happens \ for \ all \ simulations \ performed. \ Note \ that \( EU_H \) tends to increase also with the risk premium.
or, equivalently, decrease with $\eta = \theta - \Gamma q$. Indeed, when a hedger hedges his endowment the return is precisely $p$ and a higher expected level of $p$ increases the risk borne by the agent.$^{39}$

Interestingly, when the precision of information is high market depth increases with $\xi$ but $\text{EU}_{11}$ decreases. This means that looking at the usual cost of trading in noise trading models is misleading and this happens precisely when the demands of hedgers are close to the noise trader form, that is, when $\xi_u / \rho_{11}$ is small.

In short, for a very wide range of parameter values we have that more private information is Pareto inferior because it aggravates the adverse selection problem and reduces the hedging effectiveness of the futures market and production. This means that all market participants would prefer that there is no private information in the market. The question arises whether this is true also with public disclosure of information. With public disclosure adverse selection is eliminated and this should increase market depth but the impact on production is ambiguous ex ante. The reason is that a public signal also reduces insurance opportunities (this is nothing else but the “Hirshleifer effect”, see Section 3.1). Again there are scenarios where all market participants end up losing with more public information. (See Exercise 4.6.)$^{40}$ Section 5.4 provides further discussion of the welfare economics of informed trading in production economies.

### 4.4.5 Summary

The typical (and unmodeled) noise trader behavior corresponds to risk averse rational hedgers with a high degree of risk aversion and/or when correlation between individual and aggregate endowment shocks is very low. This implies that the usual welfare analysis of noise trader models based on the losses that those traders make, and which

39 If $\theta = z$, so that $\sigma_{v_{u}} = \sigma_{o}^2$, and the endowment is completely hedged $x_{j} = -u_{j}$, then $W_{j} = u_{j}z + (\theta - p)x_{j} = u_{j}p$.

40 Interestingly, the welfare effects of public information depend on whether information is dispersed or not. Diamond (1985), in a variation of the models of Diamond and Verrecchia (1981) and Verrecchia (1982) with endogeneous dispersed information, finds that releasing public information
depends on market depth, may be misleading. Indeed, market depth may increase but still the expected utility of hedgers may decrease. Private information creates adverse selection and may decrease the welfare of all market participants because it reduces the hedging effectiveness of the market. The same may happen with public information because of the destruction of insurance opportunities. The consequence is that having more information may yield Pareto inferior outcomes.

Summary
In this chapter we have examined static financial market models in the frame of rational expectations with asymmetric information. A general theme of the chapter is that market microstructure matters when it comes to the informational properties of prices in financial markets and how uniformed traders protect themselves from informed trading by making the market less liquid. A recurrent result is how risk aversion for competitive traders makes agents cautious when trading and responding to their private signals. In Chapter 5 we will see how market power for strategic traders plays a similar role to risk aversion for competitive traders.

The main insights from the standard model with a unique linear REE are as follows.

- Prices reflect private information about the returns of the asset through the trades of investors but typically not perfectly. Indeed, prices reflect the fundamentals and noise or shocks to preferences of investors.
- A perfect informationally efficient market is impossible whenever information is costly to acquire.
- In the presence of traders with private information market makers and other uninformed agents face an adverse selection problem and protect themselves by increasing the bid-ask spread and reducing market depth. If market makers are risk averse then price volatility increases with their degree of risk aversion.
- The informativeness of prices increases with the risk tolerance-adjusted informational advantage of informed traders, with the proportion of informed, and decreases in the volatility of fundamentals and the amount of noise trading.
- Information acquisition displays strategic substitutability.

improves welfare because it reduces costly private information acquisition by traders and makes
• The presence of a risk neutral competitive fringe of market makers with no privileged information makes prices reflect all public available information. That is, it makes the market semi-strong informationally efficient market and the price of the risky asset equals the expected fundamental value given publicly available information.

• In a semi-strong efficient market:
  • Prices are volatile because they are informative about fundamentals.
  • Total volatility is constant and a more informative price just advances the resolution of uncertainty.

• Risk averse traders using market orders are more cautious than limit order traders because the former bear price risk. As a consequence, if the proportion of traders using limit orders or demand schedules (instead of market orders) increases prices are more informative and more volatile (and the impact on market depth is ambiguous).

• Whenever there is a differential fixed cost to submit a demand schedule instead of a market order, traders with a large risk tolerance-adjusted informational advantage place demand schedules while the others place market orders.

• Noise trader demands are close to demands by rational utility maximizing hedgers with a large degree of risk aversion. It is possible then that market depth increases coexist with decreases in the expected utility of hedgers.

• An increase in either private or public information may be Pareto inferior because the hedging effectiveness of the market is impaired. With private information this happens because of adverse selection and with public information because of the Hirshleifer effect.

Departures from the standard model (such as having private signals on noise trading, or correlation between individual and aggregate endowment shocks, fundamentals and noise trading, or error terms of private and public signals) introduce multiple (linear) equilibria in the financial market and, potentially, strategic complementarity in information acquisition. Another way to obtain strategic complementarity in traders’ beliefs more homogeneous.
information acquisition is with economies of scale in information production. The end result may be multiple equilibria in both the financial market and the market for information acquisition. This allows for a rich pattern of explanations of different phenomena in financial markets.
Appendix

Claim (Section 4.2.2): The expected utility of an informed trader conditional on public information (the price) and taking into account the cost \( k \) of getting the signal is given by

\[
E[U(\pi_1) | p] = \exp\{\rho k\} \sqrt{\frac{\text{var}[s]}{\text{var}[p]}} \left( E[U(\pi_u) | p] \right).
\]

Proof: We know from Section 4.2.2 that

\[
E[U(\pi_1) | p] = \exp\{\rho k\} E\left[-\exp\left(-\frac{\frac{(s - p)^2}{2\sigma^2}}{}\right)\right] | p.
\]

Let \( y = \frac{s - p}{\sqrt{2\sigma^2}} \). It follows that

\[
E[y | p] = \frac{1}{\sqrt{2\sigma^2}} (E[s | p] - p) = \frac{1}{\sqrt{2\sigma^2}} (E[\theta | p] - p)
\]

\[
( E[s | p] = E[E[\theta | s] | p] = E[\theta | p] \text{ because } s = E[\theta | s] \text{ and } p \text{ is a noisy version of } s)
\]

and

\[
\text{var}[y | p] = \frac{1}{2\sigma^2} \text{var}[s | p] = \frac{1}{2\sigma^2} \left( \text{var}[\theta | p] - \sigma^2 \right)
\]

since from \( \theta = s + \varepsilon \) and the independence of \( s \) and \( \varepsilon \) we have that \( \text{var}[\theta | p] = \text{var}[s | p] + \sigma^2 = \text{var}[s | p] + \text{var}[\theta | s] \).

Recall that (see Section 2.4 in the Technical Appendix) if \( y \) conditional on \( p \) is normally distributed with mean \( E[y | p] \) and \( \text{var}[y | p] \) then

\[
E[e^{-y^2} | p] = \frac{1}{\sqrt{1 + 2 \text{var}[y | p]}} \exp\left\{-\frac{(E[y | p])^2}{1 + 2 \text{var}[y | p]}\right\}
\]

and it follows that

\[
E\left[\exp\left(-\frac{(s - p)^2}{2\sigma^2}\right) | p\right] = \sqrt{\frac{\text{var}[\theta | s]}{\text{var}[\theta | p]}} \exp\left\{-\frac{(E[\theta | p] - p)^2}{2 \text{var}[\theta | p]}\right\}.
\]

The result follows since
Exercises

4.1. A paradoxical fully revealing equilibrium. Consider a variation of the CARA-normal model of Section 4.2.1 with n informed traders, no uninformed traders and no noise (with \( u = \bar{u} < 0 \) with probability one). Find a linear fully-revealing REE using Grossman’s method (using the competitive equilibrium of an artificial shared information economy). Check the result with the usual procedure of positing a linear REE price function and work through the optimization and updating rules of traders. Check that in equilibrium the demands of traders are independent of private signals and the price. How can prices be a sufficient statistic for the information of traders and the demand of traders be independent of the signals of traders?

Solution: In a shared-information economy the CARA demand of a trader is

\[
E[\theta|s_n] - p = \frac{\rho var[\theta|s_n]}{\rho var[\theta|s_n]}\]  

where \( s_n = \frac{1}{n} \sum_{i=1}^{n} s_i \) is a sufficient statistic for the signals of traders. Market clearing implies then that \( p = E[\theta|s_n] + \rho var[\theta|s_n]\bar{u}/n \) which again is a sufficient statistic for the signals of traders (because \( E[\theta|s_n] \) is). It is immediate that this price is a FRREE price for the asymmetric information economy. The properties of the demands follow by simple manipulation. (See Grossman (1976)).

4.2. Derivation of CARA demands. Consider the competitive market in Section 4.3 and derive the demands for both classes of informed traders.

Solution: At a linear equilibrium both \( \theta \) and \( p \) are normally distributed. The demand of a trader with information set \( G \), with \( G = \{s_i, p\} \) if the trader places a demand schedule and \( G = \{s_i\} \) if he places a market order, is then

\[
E[\theta|s_i] = \left(\tau_s + \tau_p\right) \text{ and } \left(\frac{\text{var}[\theta|s_i, p]}{\text{var}[\theta|s_i, p]}\right)^{-1} = \tau_s + \tau.
\]  

It
follows then that $E\left[ \theta | s, p \right] - p = \tau_x (\tau_x + \tau)^{-1} (s_i - p)$, and therefore $X(s_i, p) = p^{-1} \tau_x (s_i - p)$ for a trader placing a demand schedule. For traders placing market orders we find that $E\left[ \theta - p | s_i \right] = (1 - \lambda A) \tau_x (\tau_x + \tau)^{-1} (s_i - \bar{0})$ and $\text{var}\left[ \theta - p | s_i \right] = (1 - \lambda A)^2 \text{var}\left[ \theta | s_i \right] + \lambda^2 \sigma^2_\theta$ with $\text{var}\left[ \theta | s_i \right] = (\tau_x + \tau_\theta)^{-1}$. It follows then that $Y(s_i) = E\left[ \theta - p | s_i \right] / (\rho \text{var}\left[ \theta - p | s_i \right]) = \left( \rho \left( \sigma^2_\theta + \text{var}[p] \right) \right)^{-1} (s_i - \bar{0})$ and therefore $c$ is given implicitly by $c = \left( \rho \left( \sigma^2_\theta + \text{var}[p] \right) \right)^{-1}$. This is a cubic equation $c = \frac{\bar{a} \tau_\theta}{\tau_x + \tau_\theta A^2}$, with $A = v + (1 - v) c$ and $\bar{a} = \rho^{-1} \tau_x$, with a unique solution in $c$.

4.3. Trading volume in a competitive market. In the same market as in Exercise 4.2 show that when all informed traders use demand schedules ($\nu = 1$) the expected (aggregate) volume traded by informed agents is $E\left[ \int_0^1 X(s, p) d_i \right] = (2 / \pi)^{1/2} \bar{a} \sqrt{\tau^{-1}}$.

Solution: For $z$ normal with mean 0 and standard deviation variance $\sigma_z$ we know that $E[z] = (2 / \pi)^{1/2} \sigma_z$. We have then $\text{var}\left[ \int_0^1 X(s, p) d_i \right] = \bar{a}^2 \text{var}[\theta - p]$ and it is easily checked that when $p = E[\theta | p]$, $\text{var}[\theta - p] = \sigma^2_z - \text{var}[p] = \text{var}[\theta | p]$. Indeed, the first equality follows from $\text{var}[\theta - p] = \text{var}[\theta] + \text{var}[p] - 2\text{cov}[\theta, p]$ and the projection theorem for normal random variables: $\text{cov}[\theta - E[\theta | p], E[\theta | p]] = 0$ because with $E[\theta | p] = p$ we have that $\text{cov}[\theta, p] = \text{cov}[E[\theta | p], p] = \text{var}[p]$. For the second equality we know that $\text{var}[p] = \sigma^2_z - \text{var}[\theta | p]$.

4.4. Comparative statics of basic market parameters. In the same market as in Exercise 4.2 perform a comparative static analysis of the relevant market parameters when all informed traders use market orders ($\nu = 0$) and interpret the results.

Answer (see Vives (Proposition 2.2, 1995b)): 

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(i) The responsiveness of informed agents to private signals (c) decreases with $\rho$, $\sigma^2_x$ and $\sigma^2_\theta$, and increases with $\sigma^2_u$.

(ii) The informativeness of the price ($\tau$) decreases with $\rho$, $\sigma^2_x$, $\sigma^2_\theta$, and $\sigma^2_u$.

(iii) The ex-ante volatility of prices ($\text{var}[p]$) decreases with $\rho$, $\sigma^2_x$ and $\sigma^2_\theta$, and increases with $\sigma^2_u$.

(iv) The (expected) volume traded by informed agents

$$\left( \mathbb{E}\left[ \int_0^t X(s) \, ds \right] = \left( \frac{2}{\pi} \right)^{1/2} c \sigma_\theta \right)$$

decreases with $\rho$ and $\sigma^2_x$ and increases with $\sigma^2_u$.

**Interpretation.** The effects with respect to $\rho$ and $\sigma^2_x$ accord to intuition. Increases in $\sigma^2_\theta$ decrease the price precision for a fixed $c$ and induce market makers to raise $\lambda$. Informed agents respond by trading less intensely accentuating the decrease in $\tau$. The ex-ante volatility of prices is positively related to their informativeness: $\text{var}[p] = \tau^{-1} - \tau$. Therefore, all factors (except $\sigma^2_\theta$) which increase $\tau$ will also increase $\text{var}[p]$. A higher $\sigma^2_\theta$ has a double impact on $\text{var}[p]$: A negative indirect effect, since it decreases $\tau$, and a positive direct effect which dominates. Increases in noise ($\sigma^2_u$) increase $c$ since in equilibrium they induce a lower volatility of prices. Noise has a negative direct effect on the informativeness of prices (that is for a constant parameter $\alpha$) and a positive indirect effect through $c$. The direct effect dominates. This is due to the presence of risk aversion.

**4.5. Expected utilities in the futures market model.** Consider the model in Section 4.4 and derive first the expressions for the expected utility of the informed $J_I(q)$ as stated in Section 4.4.4 with

$$d = (1-\xi) \left( \frac{1 - \frac{\xi}{\rho_I} \text{cov}[E[q \mid s], X_i(s,p)]}{1 + \text{var}[X_i(s,p)]} \right)^2 = (1-\xi) \left( \frac{1 - \frac{\xi}{\rho_I(1-\xi)} \sigma_\theta + \frac{\xi}{(1-\xi)\sigma^2_u} \left( \frac{\xi - \alpha}{\xi} \right)^2}{1 + \frac{(\xi\Lambda - a)^2 \sigma^2_\theta + \delta^2 \sigma^2_u}{\Lambda^2(1-\xi)\sigma^2_u}} \right)$$
where the endogenous parameters $a$, $\Lambda$ and $\Gamma$ are as in Proposition 4.3. Show that $d \to 0$ as $\xi \to 1$. Why is this so?

Show next that the expected utility of an uniformed trader is given by

$$EU_{U} = E[-\exp\{-\rho_{U}W_{U}\}] = -|SG_{U}| \exp\left\{-\frac{1}{2} \left(\frac{(\Gamma q)^{2}}{\text{var}[E[0|p] - p] + \text{var}[0|p]}\right)\right\}$$

where

$$|SG_{U}| \equiv \left\{1 + \frac{\text{var}[E[0|p] - p]}{\text{var}[0|p]}\right\}^{-1/2},$$

$$\text{var}[E[0|p] - p] = \frac{\delta^{2}\sigma_{u}^{2} + a^{2}\left(\sigma_{\theta}^{2} + \sigma_{\varepsilon}^{2}\right) - a\Lambda\sigma_{\theta}^{2}}{\Lambda^{2}\delta^{2}\sigma_{u}^{2} + a^{2}\left(\sigma_{\theta}^{2} + \sigma_{\varepsilon}^{2}\right)}.$$

Finally, derive the following expression for the ex ante expected utility of a hedger with endowment shock $u_{j}$ when he does not trade in the future's market and his wealth is $u_{j}z$:

$$\left(-1 - \rho^{2}_{H}\sigma_{z}^{2}\sigma_{u}^{2}\xi_{u}\right)^{-1/2} \exp\left\{\frac{\rho^{2}_{H}z\sigma_{u}^{2}}{2\left(\xi_{u} - \rho^{2}_{H}\sigma_{z}^{2}\sigma_{u}^{2}\right)}\right\},$$

provided that $\rho^{2}_{H}\sigma_{z}^{2}\sigma_{u}^{2}\xi_{u}\rho^{2}_{H}\sigma_{z}^{2}\sigma_{u}^{2}<1$. Otherwise the expected utility diverges to $-\infty$.

**Hint:** Use the following result (see Section 2.4 in the Technical Appendix) and follow a similar procedure as in the proof of the claim in Section 4.2.2. If $x \sim N(\overline{x}, \sigma_{x}^{2})$ and $y \sim N(\overline{y}, \sigma_{y}^{2})$, then

$$E[\exp\{x - y^{2}\}] = \frac{1}{\sqrt{1 + 2\sigma_{y}^{2}}} \exp\left\{x + \frac{\sigma_{x}^{2}}{2} - \frac{(\overline{y} + \text{cov}[x, y])^{2}}{1 + 2\sigma_{y}^{2}}\right\}.$$

For example, from $X_{U}(p) = \frac{E[0|p] - p}{\rho_{U}\text{var}[0|p]}$ obtain

$$E[-\exp\{\rho_{U}W_{U}\} | p] = -\exp\left\{-\frac{(E[0|p] - p)^{2}}{2\text{var}[0|p]}\right\}.$$
To obtain the unconditional expected utility it suffices to apply the result taking $z = 0$ and $y = (E[θ | p] - p) / \sqrt{2 \text{var}[θ | p]}$, so that

$$E\left[ -\exp\left\{-\rho_{W_{U}} W_{U}\right\}\right] = E\left[ E\left[ -\exp\left\{-\rho_{W_{U}} W_{U}\right\} | p \right]\right] = E\left\{ -\exp\left\{-\frac{(E[θ | p] - p)^2}{2 \text{var}(θ | p)}\right\}\right\} =$$

$$\left\{1 + \frac{\text{var}(θ | p)}{\text{var}(θ | p)}\right\}^{-\frac{1}{2}} \exp\left\{-\frac{1}{2 \text{var}(θ | p) + \text{var}(E[θ | p] - p)}\right\}$$

and the result follows since $E\left[ E(θ | p) - p\right] = \tilde{θ} - \tilde{p} = \Gamma q$. (See Medrano and Vives (2007) for details on the computation of other expected utilities).

**4.6 Public disclosure and the Hirshleifer effect.** Consider the model in Section 4.4 and assume now that the signal received by producers becomes public information at $t = 1$.

(i) Show that if $ξ < 1$ then there is a unique linear REE in the futures market characterized by

$$X_{U}(s, p) = a_{U}(s - \bar{θ}) + b_{U}(\bar{θ} - p)$$

$$X_{H}(u) = a_{H}(s - \bar{θ}) + b_{H}(\bar{θ} - p) - δu$$

$$X_{I}(s, p) = a_{I}(s - \bar{θ}) + b_{I}(\bar{θ} - p) - q$$

$$p = E[θ | s] - \frac{q}{\Lambda} - \frac{δu}{\Lambda}$$

where $a_{i} = \frac{ξ}{ρ_{i}(1-ξ)σ_{θ}^2}$, $b_{i} = \frac{1}{ρ_{i}(1-ξ)σ_{θ}^2}$, $a_{U} = \frac{ξ}{ρ_{U}(1-ξ)σ_{θ}^2}$, $b_{U} = \frac{1}{ρ_{U}(1-ξ)σ_{θ}^2}$, $a_{H} = \frac{ξ}{ρ_{H}(1-ξ)σ_{θ}^2}$, $b_{H} = \frac{1}{ρ_{H}(1-ξ)σ_{θ}^2}$, $δ = σ_{θ} / σ_{θ}^2$, and $Λ = \frac{1}{ρ_{U}(1-ξ)σ_{θ}^2}\left(\frac{1}{ρ_{H}} + \frac{1}{ρ_{U}} + \frac{1}{ρ_{H}}\right)$.

The equilibrium level of production chosen is $q = \frac{π_{e} - ξ}{e_{e}π_{e}(1 - d)}$ where

$$d = (1 - ξ)\left(\frac{\frac{1}{ρ_{U}} + \frac{1}{ρ_{H}}}{\frac{1}{ρ_{H}} + \frac{1}{ρ_{U}} + \frac{1}{ρ_{H}}}\right)^2 \frac{1}{\left[1 + (1 - ξ)σ_{θ}^2\left(\frac{1}{ρ_{U}} + \frac{1}{ρ_{H}} + \frac{1}{ρ_{H}}\right)^2 δ^2σ_{θ}^2\right]}.$$

(ii) What happens when $ξ = 1$?

(iii) Show that if $ξ$ increases then:
1) Trading intensities increase.
2) The market becomes deeper (Λ increases).
3) The level of production decreases.
4) The future's price becomes more informative.
5) The risk premium (ξ) is lower.
6) The expected utility of both producers and market makers decrease.

(iv) Conjecture the comparative statics properties of price volatility and the expected utility of hedgers with respect to ξ.

(v) How do you interpret the results? What happens to the hedging effectiveness of the market as ξ tends to 1? Is it possible that providing more public information all traders lose? Is it possible that in a deeper market hedgers see their expected utility diminished?

(vi) Let σ_0^2 / σ_0^2 = 1 and ρ_{it} → ∞ so that we are in the noise trader case, and ρ_u → ∞. Then the depth of the market is \( \frac{\rho_t^{-1}}{(1-\xi)^2} = \rho_t^{-1}(\tau_0 + \tau_\xi) \). Show that this is larger than when there is dispersed information (i.e. than \( \lambda^{-1} \) in Proposition 4.1 when \( \mu = 1 \)).


4.7 The effects of overconfidence. Speculate about the consequences for price informativeness, market depth and traded volume of the presence of overconfident informed traders (this is the situation where an informed trader believes that the precision of his private signal is larger than what actually is). Do the exercise assuming first that the information endowments are given and second allowing for endogenous information acquisition.

Hints: Kyle and Wang (1997), Odean (1998) and Benos (1998), in (strategic) models where informed traders submit market orders, find a positive effect of overconfidence on the above mentioned market parameters because of the positive externality overconfident traders have on other investors. García, Urosevic and Sangiorgi (2007) find in a competitive model à la Grossman-Stiglitz an irrelevance result of
overconfidence on informational efficiency when endogenous information acquisition is allowed. The reason is that when information acquisition decisions are strategic substitutes rational traders purchase less information in the presence of overconfident traders (and exactly compensate the increase in price precision because of the presence of overconfident traders). Other applications with overconfident traders are Caballé and Sàkovics (2003), and Daniel, Hirshleifer and Subrahmanyam (1998, 2001).
References


