Expectations, Liquidity, and Short-term Trading*

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Abstract

Multiple equilibria arise in a finite horizon market whenever risk averse traders have incentives to speculate on short run price movements based on their private information. The characterization is tight and three ingredients, risk averse privately informed investors, short run price speculation, and persistent liquidity trading, are necessary for multiplicity with a finite horizon. In a two-period market with risk averse short term investors and heterogeneous information, when liquidity trading displays persistence, prices reflect average expectations about fundamentals and liquidity trading. Informed investors exploit a private learning channel to infer the demand of liquidity traders from first period prices to speculate on price changes. This yields multiple equilibria which can be ranked in terms of liquidity and informational efficiency with the high information one being unstable. Our results have implications to explain market anomalies and for the role of average expectations in asset pricing.

Keywords: Price speculation, multiple equilibria, average expectations, public information, momentum and reversal, price crash, Beauty Contest.

JEL Classification Numbers: G10, G12, G14

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Introduction

We study the drivers of asset prices in a two-period market where short-term, informed, competitive, risk-averse agents trade on account of private information and to accommodate liquidity supply, facing a persistent demand from liquidity traders.

Short term speculation ranks high on the regulatory agenda, testifying policy makers' concern with the possibly destabilizing impact it has on the market. For instance, the report on the causes of the "Flash-Crash" issued by the staffs of the CFTC-SEC highlights the role of High Frequency Trading (HFT) – a class of market players who engage in extremely short-term strategies – in exacerbating the sharp price drop that characterized the crash. Relatedly, policy makers' concern over the role played by financial markets in the current crisis has reanimated the debate over the means to curb short termism via the introduction of a transaction tax (the so-called "Tobin-tax"). The jury is still out on the impact of short term speculation on market liquidity. The issue has a long tradition in economic analysis. Indeed, short term trading is at the base of Keynes' beauty contest view of financial markets, according to which what matters are the average expectations of the average expectations of investors in an infinite regress of higher order beliefs. In this context it has been claimed that traders tend to put a disproportionately high weight on public information in their forecast of asset prices (see Allen, Morris, and Shin (2006)).

In this paper we present a two-period model of short term trading with asymmetric information in the tradition of dynamic noisy rational expectations models (see, e.g., Singleton (1987), Brown and Jennings (1989)). We advance the understanding of the effects of short horizons on market quality, establish the limits of the beauty contest analogy for financial markets, and deliver sharp predictions on asset pricing which are consistent with the received empirical evidence (including noted anomalies).

Assuming that informed investors have short horizons and that, due to persistence, the demand of liquidity traders is predictable, allows us to capture important features of actual markets. Indeed, a short term investor's concern over the price at which he unwinds, will make him more sensitive to the possibility to extrapolate patterns on the evolution of the *future* aggregate demand from the observation of the *current* aggregate demand for the stock. With correlated liquidity trading, this implies that not only "fundamentals" information, but also any information on the orders placed by uninformed investors becomes relevant to predict the future price.[1] The fact that both information on fundamentals and on liquidity trading matters is a crucial determinant of equilibrium and market quality properties.

We find that multiple equilibria in a two period market arise because of the combination of short-term privately informed investors, risk averse market making, and persistent liquidity trading. Under these conditions investors use their private information to speculate on short-run prices changes. Informed investors do so exploiting a private learning channel (as in Amador

[1]This appears to be especially true for HFTs that use "order anticipation" algorithms to uncover patterns within market data which provide information into large institutional order flows (see the January 2010 *SEC Concept Release on Equity Market Structure*, and Scott Patterson, June 30, 2010, "Fast Traders Face Off with Big Investors Over 'Gaming,'" *Wall Street Journal*).

and Weill (2010) and Manzano and Vives (2011)) using their private signals on the fundamentals *also* to infer the demand of liquidity traders from the first period price, to anticipate the impact that liquidity traders have on the price at period 2 at which they unwind their positions.

The intuition for the multiple equilibria result is as follows. Due to short horizons, first period investors unwind their position in the second period, which implies that they need to forecast the second period price. To this aim they use their private signal on the fundamentals in two distinct ways. On the one hand, as the liquidation price is related to the fundamentals, their signal directly provides them with useful information to anticipate it. On the other hand, as long as liquidity trading displays persistence, their private signal allows them to improve their reading of first period liquidity trading from the first period price to anticipate the impact of liquidity trading on the second period price. The less the second period price reflects the fundamentals, the less their private signal is useful to predict it, and the lower is their response to private information. This, in turn, makes the first period price reflect more liquidity trading, and less private information. Thus, the effect of private information impounded in the price by second period investors prevails over that coming from first period investors and liquidity trading has a large and positive price impact in the second period. Indeed, an increase in liquidity trading demand in period 2 needs a higher price to be accommodated since a buy order from liquidity traders is confounded with "good news" about the asset. This augments first period investors' uncertainty over the liquidation price, further lowering their response to private information. In this equilibrium, the response to private information is low, and the second period market is thin. Conversely, the more the second period price reflects the fundamentals, the more investors' private signal is useful to predict the liquidation price, and the larger is the response to private information. This makes the first period price reflect less liquidity trading and more private information. Thus, the effect of private information impounded in the price by first period investors prevails over that coming from second period investors, and liquidity trading has a small but negative price impact. In this case, an increase in liquidity trading demand in period 2 needs a *lower* price to be accomodated since a buy order from liquidity traders is confounded with "bad news" about the asset. This lowers first period investors' uncertainty over the liquidation price, further boosting their response to private information. When liquidity trading is transient, the dual role of private information vanishes, and a unique equilibrium arises.

Thus, with persistent liquidity trading, two self-fulfilling equilibria arise: in one equilibrium the market is thin, and prices are poorly informationally efficient (Low Information Equilibrium, LIE). In the other equilibrium, the opposite occurs, with a thick market and highly informationally efficient prices (High Information Equilibrium, HIE). Studying the stability of the equilibrium solutions, we show that the HIE is unstable according to the best reply dynamics. We also show that in the HIE (LIE) equilibrium, first period investors engage in a "conditional" momentum (reversal) strategy. Indeed, in equilibrium investors anticipate the price at which they trade, so that factoring out the impact of first period public information, the covariation of future returns could be either due to liquidity trading or fundamentals information. In the HIE (LIE), as prices are closer to (farther away from) fundamentals, the second (first) effect prevails, and returns positively (negatively) covary around their means. As a consequence, when estimating a positive order imbalance investors chase the market (take the other side of the market), anticipating a trend (reversal) in the price at which they unwind their positions.

We perform two robustness exercises, extending the model to encompass the presence of residual uncertainty on the liquidation value, and allowing for the existence of a public signal in the second period. With residual uncertainty there are potentially three equilibria and it is possible to show that an unexpected increase in residual uncertainty may induce a price crash.[2] A public signal in the second period can instead "stabilize" the HIE.

Interestingly, we show also that with long-term risk averse investors and residual uncertainty (as in He and Wang (1995), and Cespa and Vives (2012)) there may also be multiple equilibria. This arises because in those cases informed traders have incentives to use their private information to speculate on short-term price movements. Thus, long-term traders may behave as short-term ones with the result that we may have situations with a negative price impact in the second period. A similar situation arises when there is a common shock in the private signals (Grundy and McNichols (1989)).

Our results are related to and have implications for three strands of the literature.

First, our paper is related to the literature that investigates the relationship between the impact of short-term investment horizons on prices and investors' reaction to their private signals (see, e.g. Singleton (1987), Brown and Jennings (1989), Froot, Scharfstein, and Stein (1992), Dow and Gorton (1994), Vives (1995), Cespa (2002), Albagli (2011) and Vives (2008) for a survey). If prices are semi-strong efficient (as in Vives (1995)) then there is no private learning channel from prices since the price is a sufficient statistic for public information. Brown and Jennings (1989), instead analyze a model in which prices are not semi-strong efficient, with short term investors and where liquidity trading can be correlated. Their work provides a rationale for "technical analysis," showing how in the absence of semi-strong efficiency the *sequence* of transaction prices provides more information than the *current* stock price to forecast the final payoff. We argue that lacking semi-strong efficiency, in the presence of correlated liquidity trading, first period investors have a private learning channel from the price which provides them with additional information on the future stock price. We also provide a closed form characterization of the equilibrium, emphasizing the role of this private learning channel in generating equilibrium multiplicity.

Other authors find that in the presence of short-term traders multiple equilibria can arise (see, e.g., Spiegel (1998) and Watanabe (2008)). However, in these cases multiplicity arises out of the bootstrap nature of expectations in the steady state equilibrium of an *infinite horizon* model with overlapping generations of two-period lived investors. Spiegel (1998) studies the model with *no* asymmetric information.[3] Watanabe (2008) extends the model of Spiegel (1998)

[2] Multiple, self-fulfilling equilibria can also arise because of participation externalities, as e.g. in Admati and Pfleiderer (1988), and Pagano (1989). In our context multiplicity is due to a purely informational effect.

[3] Our model with no private information is akin to a finite horizon version of Spiegel (1998) and as we show

to account for the possibility that investors have heterogeneous short-lived private information.[4]

Second, the paper is related to the work that studies the influence of Higher Order Expectations (HOEs) on asset prices (see Allen, Morris, and Shin (2006), Bacchetta and van Wincoop (2008), Kondor (2009), and Nimark (2007)). Allen, Morris, and Shin (2006) find, in a model identical to ours but with transient liquidity trading, that when prices are driven by HOEs about fundamentals, they underweight private information (with respect to the optimal statistical weight) and are farther away from fundamentals compared to consensus. We show that in the unique equilibrium that obtains when liquidity trading is transient, investors dampen their response to private information and this result holds. A similar result also holds in the LIE when liquidity trading is persistent. However, along the HIE the price is more strongly tied to fundamentals compared to consensus, and overweights average private information (compared to the optimal statistical weight).[5] Bacchetta and van Wincoop (2006) study the role of HOEs in the FX market. They show that HOEs worsen the signal extraction problem that investors face when observing changes in the exchange rate that originate from trades based on fundamentals information and hedging motives. In our setup this happens in the LIE, whereas in the HIE, investors' strong reaction to private information eases off the signal extraction problem.

Finally, the paper is also related to the literature on limits to arbitrage. In this respect, our multiplicity result is reminiscent of De Long et al. (1990), but in a model with fully rational traders, and a finite horizon. Thus, our paper naturally relates to the strand of this literature that views limits to arbitrage as the analysis of how "non-fundamental demand shocks" impact asset prices in models with rational agents (Gromb and Vayanos (2010), Vayanos and Woolley (2008)). Our contribution in this respect is twofold: first we prove that when such shocks display persistence, they impact in a non-trivial way the information extraction process of rational investors, generating implications for price efficiency and market liquidity. Second, we relate these findings to the literature on return predictability. In fact, along the HIE, we show that momentum arises at short horizons, while at long horizons reversal occurs in any equilibrium. Intuitively, momentum is the result of two forces. On the one hand, with persistence, the impact sign of any anticipated first period order imbalance on second and third period expected returns is the *same*; on the other hand, as we argued above, along the HIE the conditional covariance of returns is positive. Our results are thus in line with the empirical findings on return anomalies that document the existence of positive return autocorrelation at short horizons, and negative autocorrelation at long horizons (Jegadeesh and Titman (1993), and De Bondt and Thaler (1985)). Our model also predicts that momentum is related to a high volume of informational trading, in line with the evidence in Llorente, Michaely, Saar, and Wang (2002).

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in Corollary 6, in this case we obtain a unique equilibrium.

[4]In his case too the analysis concentrates on the steady state equilibrium, which does not make his results directly comparable to ours. Furthermore, in Watanabe (2008) fundamentals information is short lived, whereas in our model it is long lived, which substantially changes the nature of the inference problem faced by first period investors. Relatedly, Dennert (1991) studies an OLG extension of Grossman and Stiglitz (1980), concentrating on the steady state solution. In his setup too private information is short-lived.

[5]In a related paper, we show that a similar conclusion holds in a model with long term investors (see Cespa and Vives (2012)).

The rest of the paper is organized as follows. In the next section we analyze the static benchmark. In the following section, we study the two-period extension and present the multiplicity result, relating it to liquidity traders' persistence. In the following sections we relate our results to the literature on Higher Order Expectations and asset pricing anomalies. The final section summarizes our results and discusses their empirical implications. Most of the proofs are relegated to the appendix.

1 The static benchmark

Consider a one-period stock market where a single risky asset with liquidation value v, and a risk-less asset with unitary return are traded by a continuum of risk-averse, informed investors in the interval $[0,1]$ together with liquidity traders. We assume that $v \sim N(\bar{v}, \tau_v^{-1})$. Investors have CARA preferences (denote with γ the risk-tolerance coefficient) and maximize the expected utility of their wealth: $W_i = (v-p)x_i$.[6] Prior to the opening of the market every informed investor i obtains private information on v, receiving a signal $s_i = v + \epsilon_i$, $\epsilon_i \sim N(0, \tau_\epsilon^{-1})$, and submits a demand schedule (generalized limit order) to the market $X(s_i, p)$ indicating the desired position in the risky asset for each realization of the equilibrium price.[7] Assume that v and ϵ_i are independent for all i, and that error terms are also independent across investors. Liquidity traders submit a random demand u (independent of all other random variables in the model), where $u \sim N(0, \tau_u^{-1})$. Finally, we make the convention that, given v, the average signal $\int_0^1 s_i di$ equals v almost surely (i.e. errors cancel out in the aggregate: $\int_0^1 \epsilon_i di = 0$).[8]

In the above CARA-normal framework, a symmetric rational expectations equilibrium (REE) is a set of trades contingent on the information that investors have, $\{X(s_i, p) \text{ for } i \in [0,1]\}$ and a price functional $P(v,u)$ (measurable in (v,u)), such that investors in $[0,1]$ optimize given their information and the market clears:

$$\int_0^1 x_i di + u = 0.$$

Given the above definition, it is easy to verify that a unique, symmetric equilibrium in linear strategies exists in the class of equilibria with a price functional of the form $P(v,u)$ (see, e.g. Admati (1985), Vives (2008)). The equilibrium strategy of an investor i is given by

$$X(s_i, p) = \frac{a}{\alpha_E}(E[v|s_i, p]] - p),$$

where

$$a = \gamma \tau_\epsilon, \tag{1}$$

denotes the responsiveness to private information, $\tau_i \equiv (\text{Var}[v|s_i, p]])^{-1}$, and $\alpha_E = \tau_\epsilon / \tau_i$ is

[6] We assume, without loss of generality with CARA preferences, that the non-random endowment of informed investors is zero.

[7] The unique equilibrium in linear strategies of this model is symmetric.

[8] See Section 3.1 in the Technical Appendix of Vives (2008) for a justification of the convention.

the optimal statistical (Bayesian) weight to private information. Imposing market clearing the equilibrium price is given by

$$p = \int_0^1 E_i[v]di + \frac{\alpha_E}{a}u \tag{2}$$

$$= E[v|p] + \Lambda E[u|p], \tag{3}$$

where $E[u|p] = a(v - E[v|p]) + u$, and

$$\Lambda = \frac{\text{Var}_i[v]}{\gamma}. \tag{4}$$

Equations (2), and (3) show that the price can be given two alternative representations. According to the first one, the price reflects the consensus opinion investors hold about the liquidation value plus the impact of the demand from liquidity traders (multiplied by the risk-tolerance weighted uncertainty over the liquidation value). Indeed, in a static market owing to CARA and normality, an investor's demand is proportional to the expected gains from trade $E[v|s_i, p] - p$. As the price aggregates all investors' demands, it reflects the consensus opinion $\int_0^1 E_i[v]di$ shocked by the orders of liquidity traders.

According to (3), the anticipated impact of liquidity traders' demand moves the price away from the semi-strong efficient price. Therefore, Λ captures the "inventory" related component of market liquidity.[9] Liquidity traders' demand has an additional impact on the price, through the effect it produces on $E[v|p]$. This is an adverse selection effect which adds to the inventory effect, implying that the (reciprocal of the) liquidity of the market is measured by:

$$\lambda \equiv \frac{\partial p}{\partial u} = \Lambda + (1 - \alpha_E)\frac{a\tau_u}{\tau},$$

where $\tau = 1/\text{Var}[v|p] = \tau_v + a^2\tau_u$.[10]

Finally, note that the private signal in this case only serves to forecast the liquidation value v. In the next section we will argue that due to persistence, liquidity traders' demand impacts the order flow across different trading dates. In this case investors also use their private signals to extrapolate the demand of liquidity traders from the order flow to anticipate the impact this has on future prices. This additional use of private information will be responsible for equilibrium multiplicity.

[9]When risk averse investors accommodate an expectedly positive demand of liquidity traders, they require a compensation against the possibility that the liquidation value is higher than the public expectation (if instead $E[u|p] < 0$, investors require to pay a price lower than $E[v|p]$ to cover the risk that $v < E[v|p]$). Such a compensation is larger, the higher is the uncertainty investors face (captured by Λ) and the wider is their expected exposure to the liquidity traders' shock (their expected inventory, $E[u|p]$).

[10]The adverse selection effect comes from the signal extraction problem dealers face in this market: since $a > 0$, if investors on average have good news they buy the asset, and $E[v|p]$ increases, reflecting this information. However, this effect cannot be told apart from the buying pressure of liquidity traders, which also makes $E[v|p]$ increase.

2 A two-period market with short term investors

Consider now a two-period extension of the market analyzed in the previous section. At date 1 (2), a continuum of short-term investors in the interval $[0, 1]$ enters the market, loads a position in the risky asset which it unwinds in period 2 (3). Investor i has CARA preferences (denote with γ the common risk-tolerance coefficient) and maximizes the expected utility of his short term profit $\pi_{in} = (p_{n+1} - p_n)x_{in}$, $n = 1, 2$ (with $p_0 = \bar{v}$ and $p_3 = v$).[11] The short term horizons of investors can be justified on grounds of incentive reasons related to performance evaluation, or because of difficulties associated with financing long-term investment in the presence of capital market imperfections (see Holmström and Ricart i Costa (1986), and Shleifer and Vishny (1990)). An investor i who enters the market in period 1 receives a signal $s_i = v + \epsilon_i$ which he recalls in the second period, where $\epsilon_i \sim N(0, \tau_\epsilon^{-1})$, v and ϵ_i are independent for all i.[12] We make the convention that, given v, the average signal $\int_0^1 s_i di$ equals v almost surely (i.e., errors cancel out in the aggregate $\int_0^1 \epsilon_i di = 0$).

We restrict attention to equilibria in linear demand functions, $X_1(s_i, p_1) = a_2 s_i - \varphi_1(p_1)$, and $X_2(s_i, p_1, p_2) = a_2 s_i - \varphi_2(p_1, p_2)$, indicating the desired position in the risky asset for each realization of the equilibrium price. The constant a_n denotes the private signal responsiveness, while $\varphi_n(\cdot)$ is a linear function of the equilibrium prices.[13]

The position of liquidity traders is assumed to follow an AR(1) process:

$$\begin{aligned} \theta_1 &= u_1 \\ \theta_2 &= \beta\theta_1 + u_2, \end{aligned} \tag{5}$$

where $\beta \in [0, 1]$ and $\{u_1, u_2\}$ is an i.i.d. normally distributed random process (independent of all other random variables in the model) with $u_n \sim N(0, \tau_u^{-1})$. If $\beta = 1$, $\{\theta_1, \theta_2\}$ follows a random walk and we are in the usual case of independent liquidity trade increments: $u_2 = \theta_2 - \theta_1$ is independent from u_1 (e.g., Kyle (1985), Vives (1995)). If $\beta = 0$, then liquidity trading is i.i.d. across periods (this is the case considered by Allen et al. (2006)).[14]

Let $x_n = \int_0^1 x_{in} di$, $n = 1, 2$. Market clearing in period 1 is given by $x_1 + \theta_1 = 0$, and in period 2 by $x_2 + \theta_2 = 0$. In the second period market clearing involves (i) the reverting position of first period informed investors $-x_1$, (ii) the position of second period informed investors x_2, (iii) a fraction $1 - \beta$ of the first period liquidity traders' position θ_1 (who revert), and (iv) the new generation of liquidity traders with demand u_2. Letting $\Delta x_2 \equiv x_2 - x_1$, $\Delta\theta_2 \equiv \theta_2 - \theta_1 = u_2 - (1 - \beta)\theta_1$, market clearing implies

$$x_2 - x_1 + u_2 - (1 - \beta)\theta_1 = 0 \Leftrightarrow \Delta x_2 + \Delta\theta_2 = 0 \Leftrightarrow x_2 + \beta\theta_1 + u_2 = 0.$$

[11] We assume, without loss of generality, that the non-random endowment of investors is zero.

[12] The model can be extended to the case in which investors receive a new private signal in the second period. However, this complicates the analysis without substantially changing its qualitative results.

[13] The equilibria in linear strategies of this model are symmetric.

[14] See Chordia and Subrahmanyam (2004), Easley et al. (2008), and Hendershott and Seasholes (2009) for empirical evidence on liquidity traders' demand predictability.

Note that while due to short horizons, the *entire* position of first period informed investors reverts at time 2, assuming persistent liquidity trading implies that *only* a fraction $(1-\beta)\theta_1$ of first period liquidity trades reverts (while the complementary fraction is held until the liquidation date). The lower is β, the higher is this fraction (see Table 1).

Trading Date		1	2
Liquidity traders	Holding	$-$	$\beta\theta_1$
	New shock	u_1	u_2
	Position	$\theta_1 = u_1$	$\theta_2 = \beta\theta_1 + u_2$
	Reverting	$-$	$(1-\beta)\theta_1$
Informed investors	Position	x_1	x_2
	Reverting	$-$	x_1

Table 1: The evolution of liquidity trades and informed investors' positions in the two periods. The position of liquidity traders in every period is given by the sum of "Holding" and "New shock." Market clearing at period n requires that the sum of liquidity traders' and informed investors' positions offset each other.

We denote by $p^n = \{p_t\}_{t=1}^n$, and by $E_{in}[Y] = E[Y|s_i, p^n]$, $E_n[Y] = E[Y|p^n]$, $\text{Var}_{in}[Y] = \text{Var}[Y|s_i, p^n]$, and $\text{Var}_n[Y] = \text{Var}[Y|p^n]$, respectively the expectation and variance of the random variable Y formed by a trader conditioning on the private and public information he has at time n, and that obtained conditioning on public information only. The variables τ_n and τ_{in} denote instead the precisions of the investors' forecasts of v based only on public and on public and private information: $\tau_n = (1/\text{Var}_n[v])$, and $\tau_{in} = (1/\text{Var}_{in}[v])$. Letting $\alpha_{E_n} = \tau_\epsilon/\tau_{in}$, we have $E_{in}[v] = \alpha_{E_n} s_{in} + (1-\alpha_{E_n})E_n[v]$.

2.1 Equilibrium pricing and market depth

We start by giving a general description of the equilibrium. The following proposition characterises equilibrium prices:

Proposition 1. *At a linear equilibrium:*

1. *The price is given by*

$$p_n = \alpha_{P_n}\left(v + \frac{\theta_n}{a_n}\right) + (1-\alpha_{P_n})E_n[v], \tag{6}$$

where $\theta_n = u_n + \beta\theta_{n-1}$, and a_n, α_{P_n} denote, respectively, the responsiveness to private information displayed by investors and by the price at period n. We have that $\alpha_{P_2} = \alpha_{E_2} < 1$.

2. *Let* $\Delta a_2 = a_2 - \beta a_1$*,* $z_1 \equiv a_1 v + \theta_1$ *and denote by* $z_2 \equiv \Delta a_2 v + u_2$ *the noisy informational additions about* v *generated in period* 1 *and* 2*. Then* z_1 *is observationally equivalent (o.e.) to* p_1*, and* $\{z_1, z_2\}$ *is o.e. to* $\{p_1, p_2\}$*.*

According to (6), at period n the equilibrium price is a weighted average of the market expectation about the fundamentals v, and the (noisy) average private information held by investors. Rearranging this expression yields

$$p_n - E_n[v] = \frac{\alpha_{P_n}}{a_n}\left(a_n\left(v - E_n[v]\right) + \theta_n\right) \qquad (7)$$
$$= \Lambda_n E_n[\theta_n],$$

where $\Lambda_n \equiv \alpha_{P_n}/a_n$, implying that there is a discrepancy between p_n and $E_n[v]$ which, as in (3), captures a premium which is proportional to the expected stock of liquidity trading that investors accommodate at n:

Corollary 1. *At a linear equilibrium, the price incorporates a premium above the semi-strong efficient price:*

$$p_n = E_n[v] + \Lambda_n E_n[\theta_n], \qquad (8)$$

where $\Lambda_2 = \mathrm{Var}_{i2}[v]/\gamma$*, and*

$$\Lambda_1 = \frac{\mathrm{Var}_{i1}[p_2]}{\gamma} + \beta\Lambda_2. \qquad (9)$$

Comparing (9) with (4) shows that short term trading affects the inventory component of liquidity. In a static market when investors absorb the demand of liquidity traders, they are exposed to the risk coming from the randomness of v. In a dynamic market, short term investors at date 1 face instead the risk due to the randomness of the following period price (at which they unwind). As liquidity trading displays persistence, second period informed investors absorb part of first period liquidity traders' position and this contributes to first period investors' uncertainty over p_2, yielding (9).

As in the static benchmark, besides the impact of the inventory component Λ_n, with differential information the price impact also reflects an asymmetric information component as the following corollary shows:

Corollary 2. *At a linear equilibrium, the price can be written as a linear combination of the informational innovation* z_n *and the information contained in the previous period price* $\hat{p}_{n-1}$*:*

$$p_n = \lambda_n z_n + (1 - \lambda_n \Delta a_n)\hat{p}_{n-1}, \qquad (10)$$

where λ_n *measures the impact of trades in period* n*:*

$$\lambda_n \equiv \frac{\partial p_n}{\partial u_n} = \alpha_{P_n}\frac{1}{a_n} + (1 - \alpha_{P_n})\frac{\Delta a_n \tau_u}{\tau_n}, \qquad (11)$$

for $n = 1, 2$, *and* $\hat{p}_0$ *and* $\hat{p}_1$, *depending on* $\bar{v}$ *and* z_1, *respectively, are defined in the appendix (see* (A.6) *and* (A.16)*).*

According to (11) the asymmetric information component of liquidity at $n = 2$ is captured by

$$(1 - \alpha_{P_2})\frac{\Delta a_2 \tau_u}{\tau_2}. \tag{12}$$

Differently from the static benchmark, in a dynamic market this effect depends on the β-weighted *net* position of informed investors yielding trading intensity $\Delta a_2 = a_2 - \beta a_1$. Indeed, market clearing implies that:

$$x_1 + \theta_1 = 0, \text{ and } x_2 + \beta\theta_1 + u_2 = 0 \Rightarrow x_2 - \beta x_1 + u_2 = 0.$$

First period investors absorb the position of liquidity traders (θ_1) and carry it over to the second period to unwind a fraction β against the demand of informed investors (x_2) and the new liquidity traders (u_2).[15] As a result, the impact of private information in the second period depends on the *change* in informed investors' position as measured by $\Delta a_2 = a_2 - \beta a_1$. When $a_2 > \beta a_1$, the effect of private information impounded in the price by second period investors prevails over that coming from first period investors. In this case, asymmetric information generates adverse selection in the second period, and *augments* the price impact of trades. That is, an increase in u_2 needs a higher price to be accomodated since there is the risk that the demand comes from informed trading (a buy order from liquidity traders may be confused with "good news" about the asset since the additional information content of the second period price $z_2 \equiv \Delta a_2 v + u_2$ increases with v). This is always the case when $\beta = 0$ in which case the position of first period informed investors does not matter. When $a_2 - \beta a_1 = 0$ there is no net informed trade in period 2. The liquidity premium is only due to an inventory effect. When $a_2 < \beta a_1$, the unloading of the position of first period traders dominates over the demand of second period informed investors. This implies that a *higher* liquidation value, conditional on the price realization, would be met by a *lower* aggregate demand by informed investors.[16] Thus, an increase in u_2 needs now a *lower* price to be accomodated since the risk that the demand comes from informed trading is in fact good news implying *favorable* selection (in this case the increase in u_2 may be confused with a decrease in v since the additional information content of p_2, $z_2 \equiv \Delta a_2 v + u_2$ with $\Delta a_2 < 0$, decreases with v). Differently from the case with adverse selection, with favorable selection asymmetric information can instead *reduce* the price impact of trades. We will see that in equilibrium both possibilities, $\Delta a_2 > 0$ and $\Delta a_2 < 0$, may arise. In fact, in equilibrium we will have that $\Delta a_2 < 0$ if and only if $\lambda_2 < 0$. The possibility of a negative price impact in the second period when $\beta > 0$ is at the root of multiple equilibria.

Note that according to (10) asymmetric information affects the relative weight of the noisy informational innovation z_2 compared to the information reflected in p_1 in the determination

[15] This is consistent with the intuition for the AR(1) process for liquidity trading.

[16] Intuitively, suppose first period informed investors buy more aggressively than second period ones based on private information. When they unwind their position at time 2, this does not signal bad news about the asset.

of p_2. With adverse selection, as in the static benchmark, asymmetric information *magnifies* λ_2. This implies that z_2 receives a larger weight in p_2. With favorable selection, asymmetric information has the power to *lower* λ_2. Thus, in this case z_2 receives a lower weight in p_2, so that the price reflects more the information revealed by the market in period 1.

2.2 The private learning channel from prices

In the first period investors use their private signal on the fundamentals to anticipate the second period price, insofar as the latter is related to v. However, as argued above, due to liquidity trading persistence when $\beta > 0$, p_2 (which depends on v, θ_2 and p_1) also reflects the demand of first period liquidity traders. This leads investors to use their private information also to infer θ_1 from p_1. Using (6) and (7):

$$\begin{aligned} E_{i1}[p_2] &= E_{i1}\left[\alpha_{P_2}\left(v + \frac{\theta_2}{a_2}\right) + (1-\alpha_{P_2})E_2[v]\right] \\ &= \left(\alpha_{P_2} + (1-\alpha_{P_2})\frac{(\Delta a_2)^2\tau_u}{\tau_2}\right)E_{i1}[v] + (1-\alpha_{P_2})\frac{\tau_1}{\tau_2}E_1[v] + \beta\Lambda_2 E_{i1}[\theta_1] \\ &= \lambda_2\Delta a_2 E_{i1}[v] + (1-\lambda_2\Delta a_2)\hat{p}_1, \end{aligned} \tag{13}$$

where $\hat{p}_1$ depends on p_1 and is defined in the Appendix (see (A.6)). According to (13), when $\beta\Lambda_2 > 0$ the private signal serves two purposes: it allows to predict the impact of fundamentals on p_2 and it creates a private learning channel from the first period price (as in Amador and Weill (2010) and Manzano and Vives (2011)) that allows investors to recover information on θ_1 from the observation of p_1 (which depends on v and θ_1) to predict the impact of θ_2 on p_2. Note that the precision of the inference investors can make about θ_1 is inversely related to the average response to private information. Indeed, as one can verify $\text{Var}_{i1}[\theta_1] = a_1^2/\tau_{i1}$ is increasing in a_1. This is so because the more (less) aggressively investors respond to private information, the less (more) the first period price is driven by liquidity trading. As we will argue this implies that when in equilibrium a_1 is large (small), liquidity trading has a small (large) impact on the second period price which ends up being very (poorly) informative about the fundamentals and poorly (very) informative about liquidity trading.

It is worth noting that the individual assessment of θ_1, $E_{i1}[\theta_1] = a_1(v - E_{i1}[v]) + \theta_1$, is decreasing in $E_{i1}[v]$. For a given p_1 a higher assessment of the fundamentals $E_{i1}[v]$ goes together with a lower assessment of liquidity trading $E_{i1}[\theta_1]$. The consequence of this fact is that an increase in β will push towards a lower response to private information ($E_{i1}[v]$) to forecast p_2 since from (13) the total weight on $E_{i1}[v]$ in $E_{i1}[p_2]$ is

$$\left(\alpha_{P_2} + (1-\alpha_{P_2})\frac{(\Delta a_2)^2\tau_u}{\tau_2}\right) - \beta\Lambda_2 a_1 = \lambda_2\Delta a_2. \tag{14}$$

The first part of the weight corresponds to the usual response to private information because of market making (α_{P_2}) and because of speculation on fundamentals ($(1-\alpha_{P_2})(\Delta a_2)^2\tau_u/\tau_2$). The second part ($\beta\Lambda_2$) corresponds to the private learning channel from prices which detracts from the weight to private information, the more so when β grows. In equilibrium we will have always $\lambda_2 \Delta a_2 > 0$.

2.3 The best response function of informed investors

It is instructive to think of the equilibrium in terms of the best response function which determines the signal responsiveness of a first period investor given the average responsiveness of his peers. Owing to CARA and normality, an investor in the first period trades according to

$$X_1(s_i, p_1) = \gamma \frac{E_{i1}[p_2] - p_1}{\text{Var}_{i1}[p_2]}. \tag{15}$$

An investor's best response obtains as the ratio between the risk-adjusted weight to private information in the investor's private forecast $\gamma E_{i1}[p_2]$, and the investor's uncertainty over the liquidation price $\text{Var}_{i1}[p_2]$:[17]

$$\psi(a_1) = \zeta a_2, \text{ where } \zeta \equiv \frac{\gamma \Delta a_2 \tau_u}{1 + \gamma \Delta a_2 \tau_u} = \frac{1}{1 + \left(\lambda_2 \Lambda_2^{-1} - 1\right)^{-1}}. \tag{16}$$

Note that $\zeta > (<)1$ if and only if $\lambda_2 < (>)0$. The existence of a private learning channel from the price has two effects on the first period best reply mapping. First, it makes it discontinuous. Second, it can add an element of strategic complementarity to the best response. To see this, note that inspection of (16) yields that for $a_1 = \hat{a}_1 \equiv \beta^{-1}(a_2 + (\gamma \tau_u)^{-1})$ the best response is discontinuous: $\lim_{a_1 \to \hat{a}_1^-} \psi(a_1) = -\infty$, and $\lim_{a_1 \to \hat{a}_1^+} \psi(a_1) = +\infty$. Indeed, when $\lambda_2 \Lambda_2^{-1} = 1 + \gamma \Delta a_2 \tau_u = 0$ an individual investor in period 1 would like to take an unbounded position since the favorable selection effect exactly offsets the inventory risk effect, implying that he does not face any price risk: $\text{Var}_{i1}[p_2] = 0$. This discontinuity implies that the best response has two branches and two equilibria appear (see Figure 1).

Using (14) we can decompose the best response in the following way (see Appendix B for details):

$$\psi(a_1) = \quad = \underbrace{\frac{\gamma \alpha_{E_1}}{\text{Var}_{i1}[p_2]} \times \frac{(\Delta a_2)^2 \tau_u}{\tau_2}}_{1} + \underbrace{\frac{\gamma \alpha_{E_1}}{\text{Var}_{i1}[p_2]} \times \Lambda_2 a_2 \frac{\tau_1}{\tau_2}}_{2} + \underbrace{\left(-\frac{\gamma \alpha_{E_1}}{\text{Var}_{i1}[p_2]} \times \beta \Lambda_2 a_1\right)}_{3},$$

where $\alpha_{E_1} = \tau_\epsilon / \tau_{i1}$. In the last line of the above expression,

- Term 1 captures the response to private signals that reflects the anticipated impact of the fundamentals information arriving at time 2 (i.e., how z_2 affects $E_2[v]$).
- Term 2 captures the response to private signals that reflects the anticipated impact that the innovation in liquidity trading has on the second period price.[18]

[17] See (A.12).

[18] That is, how u_2 affects $E_2[\theta_2]$. While at date 1 an investor cannot predict u_2, he can predict how the market in period 2 will react to u_2, since this is recorded by $E_2[\theta_2] = a_2(v - E_2[v]) + \theta_2$.

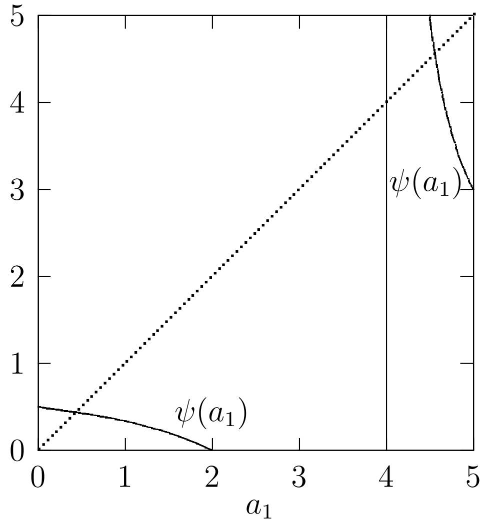


Figure 1: Equilibrium determination and stability. The figure displays the best reply function $\psi(a_1)$ (solid line) and the 45-degree line a_1 (dotted line). Equilibria obtain at the points where the two intersect. The vertical line (drawn at the point $\hat{a}_1 = 4$) shows the value of a_1 for which the best reply mapping is discontinuous. Parameters' values are as follows: $\tau_v = \tau_u = \tau_\epsilon = \gamma = 1$, and $\beta = .5$. For these values the equilibria are $a_1^* = 0.438$ and $a_1^{**} = 4.561$. Inspection of the equilibria shows that $|\psi'(a_1^*)| < 1$, while $|\psi'(a_1^{**})| > 1$.

- Term 3 captures the response to private signals that reflects the anticipated impact of first period liquidity trading on the second period price (i.e., how θ_1 affects $E_2[\theta_2]$). This reflects investors' private learning channel from the first period price and arises as long as $\beta > 0$.

In Figure 2 we display the behavior of the three terms (parameters' values are the same of Figure 1). For $a_1 < \hat{a}_1$ we have that term 1 is non monotone (first decreasing, then increasing) in a_1, term 2 is increasing in a_1, and term 3 is negative and decreasing in a_1. For $a_1 = \hat{a}_1$ all three terms diverge (the first and second to $+\infty$, the third to $-\infty$). For $a_1 > \hat{a}_1$ terms 1 and 2 are decreasing in a_1 while term 3 is increasing in a_1 (that is it grows towards 0). This illustrates that the private learning channel introduces an element of strategic complementarity to the best response function of informed investors in period 1 in the second branch.

2.4 Equilibrium analysis

In this section we charaterize the equilibria of the market.

Proposition 2. *Linear equilibria always exist. In equilibrium, $a_2 = \gamma\tau_\epsilon$, and a_1 is implicitly defined by the equation $\phi(a_1) \equiv a_1(1 + \gamma\tau_u\Delta a_2) - \gamma a_2\Delta a_2\tau_u = 0$. If $\beta \in (0, 1]$:*

1. *There are two equilibria a_1^* (LIE) and a_1^{**} (HIE), where $a_1^* < \gamma\tau_\epsilon < a_1^{**}$ (see* (A.18), *and* (A.19), *in the appendix for explicit expressions).*

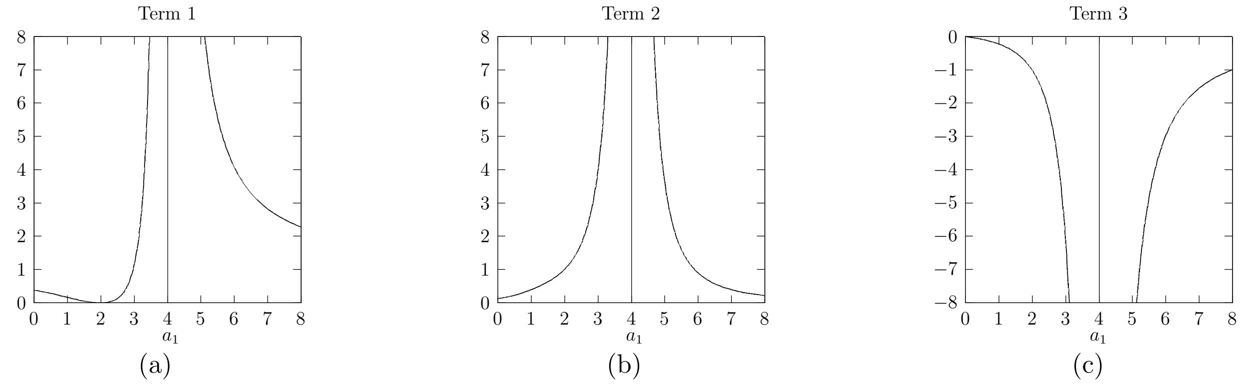


Figure 2: The figure plots the three terms that determine an investor's response to private information when $\beta > 0$. Parameter values are as in Figure 1.

2. *We have that* $a_2 - \beta a_1^* > 0$, *and* $\lambda_2^* > 0$, *while* $a_2 - \beta a_1^{**} < 0$, *and* $\lambda_2^{**} < 0$. *Furthermore,* $|\lambda_2^{**}| < \lambda_2^*$, *and prices are more informative along the HIE:* $\tau_n^{**} > \tau_n^*$.

If $\beta = 0$, *the equilibrium is unique:*

$$a_1 = \lim_{\beta \to 0} a_1^* = \frac{\gamma a_2^2 \tau_u}{1 + \gamma a_2 \tau_u}. \tag{17}$$

Second period informed investors always react to their private signal as in a static model: $a_2 = \gamma \tau_\epsilon$. This is so since the horizon ends at $n = 3$. The problem for first period investors is more complex since their reaction to private information depends on two forces. First, a_1 depends on p_2's reaction to fundamentals information or liquidity trading. Second, when $\beta > 0$, a_1 also depends on the information about θ_1 that investors read from p_1, to forecast p_2. That is their private learning channel from the price. As shown by (14) both these forces are reflected in λ_2. Note, however, that the choice of a_1 in turn impinges on the information about v or liquidity trading that is reflected in p_1 and p_2 (again via λ_2). Thus, and as long as liquidity trading is persistent, the optimal response to private information in the first period is the result of a rational expectations loop which relates a_1 both directly and indirectly (through the private learning channel) to p_2.

Suppose that the first period investors anticipate a market with positive (and relatively strong) price impact in period 2 ($\lambda_2 > 0$). In this case p_2 reflects more the informational innovation z_2, which increases investors' uncertainty over the liquidation price. Thus, they react little to their private signal, p_1 is not very informative about v and is dominated by θ_1. As a consequence, the private learning channel leads them to deduct more from the weight to private information, since they need less the private signal to estimate θ_1 from p_1 (see panel (c) in Figure 1, at the left of $\hat{a}_1$). The result is that the effect of private information impounded in the price by second period investors prevails over that coming from first period investors, $a_2 > \beta a_1^*$, and $\lambda_2^* > 0$ is validated. This equilibrium has the traditional features of noisy REE with risk averse market making, in that adverse selection magnifies the second period price impact of trades. In this equilibrium (LIE), a_1^* is low, and the second period market is thin with $\lambda_2^* > 0$.

Suppose now that the first period investors anticipate a market with negative (and relatively weak) price impact in period 2 ($\lambda_2 < 0$). In this case p_2 reflects less the informational innovation z_2, which reduces investors' uncertainty over the liquidation price. Thus, they react more to their private signal, p_1 is less informative about θ_1 and is dominated by v. As a consequence, the private learning channel leads them to deduct less from the weight to private information, since they need less to estimate θ_1 from p_1 (see panel (c) in Figure 1, at the right of $\hat{a}_1$).[19] As a result, the effect of private information impounded in the price by first period investors prevails over that coming from second period investors, $a_2 < \beta a_1^{**}$ inducing *favorable* selection and a mild price impact: $|\lambda_2^{**}|$ is small. This equilibrium has opposite features compared to the standard REE, in that favorable selection yields a negative but small price impact $\lambda_2^{**} < 0$ (that is, an increase in u_2 is accomodated by lowering the price p_2).[20] In this equilibrium (HIE), a_1^{**} is high and the second period market is deep with $\lambda_2^{**} < 0$. Figure 3 illustrates the self-fulfilling loops leading to the two different types of equilibria.

When $\beta = 0$, liquidity trading is transient, and first period investors cannot use their information to forecast its impact on p_2. This implies necessarily $\lambda_2 > 0$, and eliminates the private learning channel from the first period price, yielding a unique equilibrium.[21]

As the persistence in liquidity trading (β) increases with β in both equilibria first period informed investors speculate less aggressively on their private information. This is so since if persistence increases (i.e. β grows), the first period price is more informative about θ_2, reinforcing the private learning channel from p_1 and detracting from the weight to private information. However, in both equilibria we have that βa_1 increases with β, since the direct effect of an increase in β prevails.

Corollary 3. *At equilibrium for all* $\beta \in (0,1)$, $\partial a_1/\partial\beta < 0$ *and* $\partial\left(\beta a_1\right)/\partial\beta > 0$

Proof. The equation that yields the first period responsiveness to private information can be written as:

$$\phi(a_1) \equiv \lambda_2 \tau_{i2} a_1 - \gamma \Delta a_2 \tau_u \tau_\epsilon = 0.$$

The result then follows immediately, since from implicit differentiation of the above with respect

[19] The different behavior displayed by term 3 in the two branches of the best response captures the different nature of the private learning channel in the two equilibria. In the LIE, the higher is the average response to private information, the lower the weight each investor puts on private information (used as a private learning channel). This is because in this equilibrium investors want to learn θ_1, and their private signal is not useful for this purpose. Conversely, in the HIE, the higher is the average response to private information, the higher is the weight each investor puts on his private signal used as a private learning channel. This is because in this equilibrium investors want to learn v.

[20] The private learning channel from prices provides a reinforcing mechanism in this equilibrium. This is so since βa_1 is increasing in β (the degree of persistence of liquidity trading).

[21] As we show in the proof of the proposition, $\lim_{\beta\to 0} a_1^{**} = +\infty$.

to β:

$$\frac{\partial a_1}{\partial \beta} = -\frac{\gamma \tau_u a_1 (a_2 - a_1)}{(1 + \gamma \tau_u \Delta a_2) + \gamma \beta \tau_u (a_2 - a_1)} < 0$$
$$\frac{\partial (\beta a_1)}{\partial \beta} = \frac{(1 + \gamma \tau_u \Delta a_2) a_1}{1 + \gamma \tau_u \Delta a_2 + \gamma \beta \tau_u (a_2 - a_1)} > 0,$$

independently of the equilibrium that arises; in the LIE we have $\beta a_1 < a_1 < a_2 \equiv \gamma \tau_\epsilon$, and in the HIE $a_1 > a_2/\beta > a_2$ and $1 + \gamma \tau_u \Delta a_2 < 0$. □

Stability

In this section we use the best response (16) to perform a stability analysis of the equilibria. We start by computing the slope of the best response, obtaining

$$\psi'(a_1) = -\frac{\gamma \beta a_2 \tau_u}{(1 + \gamma \Delta a_2 \tau_u)^2} < 0, \tag{18}$$

for all $\beta > 0$ and is negative in both branches. The decisions on the weight assigned to private information in the first period are strategic substitutes. This is the outcome of the interaction of the usual Grossman-Stiglitz type forces for strategic susbtitutability in the use of information together with the influence of the private learning channel from prices.

To analyze the stability of equilibrium, consider the following argument. Assume that the market is at an equilibrium point $\bar{a}_1$, so that $\bar{a}_1 = \psi(\bar{a}_1)$. Suppose now that a small perturbation to $\bar{a}_1$ occurs. As a consequence, first period investors modify their weights to private information so that the aggregate weight becomes $\bar{a}_1' = \psi(\bar{a}_1')$. If the market goes back to the original $\bar{a}_1$ according to the best reply dynamics with the best response function $\psi(\cdot)$, then the equilibrium is stable. Otherwise it is unstable. In a stable (unstable) equilibrium if investors other than i put a lower weight on their signals then the price is noisier and investor i reacts by putting a larger (lower) weight on his signal and tending to restore (diminish more) price informativeness. Formally, we have the following definition:

Definition 1 (Stability)**.** *An equilibrium is stable (unstable) if and only if its corresponding value for a_1 is a stable (unstable) fixed point for the best response function $\psi(\cdot)$ (i.e., if and only if its corresponding value for a_1 satisfies $|\psi'(a_1) < 1|$).*

We obtain the following.

Corollary 4. *The LIE (HIE) is stable (unstable) with respect to the best response dynamics:*

$$|\psi'(a_1^{**})| > 1 > |\psi'(a_1^{*})| \,. \tag{19}$$

The above result implies that the HIE is unstable according to the best reply dynamics (see Figure 1) while the LIE is stable. Strategic substitutability is much stronger in the second

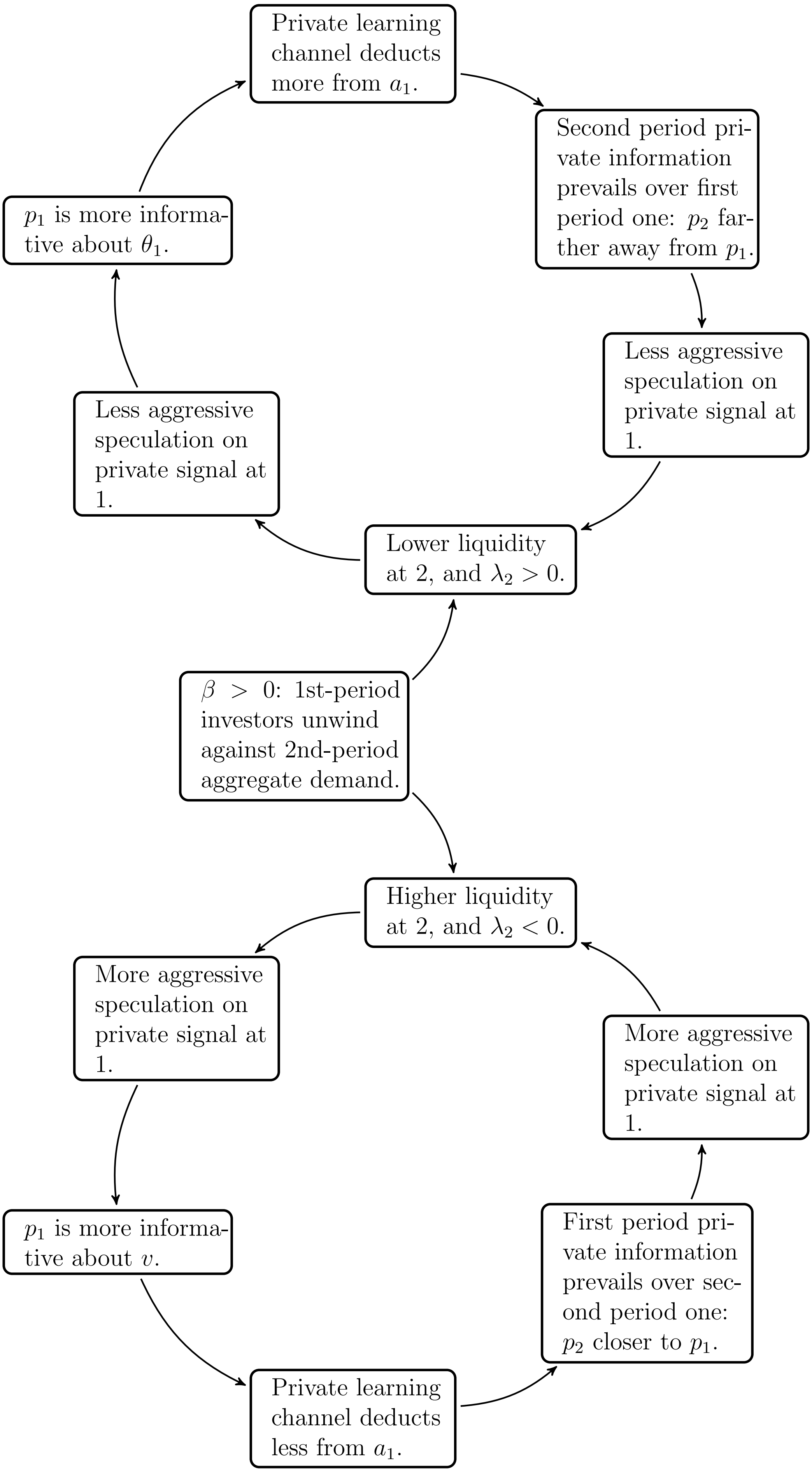


Figure 3: Multiple equilibria with self-fulfilling liquidity.

branch of the best response, leading to instability (see Appendix B for an explanation of the degree of strategic substitutability in terms of the decomposition of the traditional and private learning channel effects on the best response along the lines of (14)).

Equilibrium strategies

The next result characterizes investors' strategies:

Corollary 5. *At a linear equilibrium, the strategies of an informed investor are given by*

$$X_1(s_i, p_1) = \frac{a_1}{\alpha_{E_1}}(E_{i1}[v] - p_1) + \frac{\alpha_{P_1} - \alpha_{E_1}}{\alpha_{E_1}} E_1[\theta_1] \tag{20}$$

$$X_2(s_i, p_1, p_2) = \frac{a_2}{\alpha_{E_2}}(E_{i2}[v] - p_2). \tag{21}$$

When $\beta \in (0, 1]$ *and (i)* $a_1 = a_1^*$, $\alpha_{P_1} < \alpha_{E_1}$ *, and when (ii)* $a_1 = a_1^{**}$, $\alpha_{P_1} > \alpha_{E_1}$. *For* $\beta = 0$, $\alpha_{P_1} < \alpha_{E_1}$.

According to (21), in the second period an investor acts like in a static market. In the first period, instead, he loads his position anticipating the second period price, and scaling it down according to the uncertainty he faces on p_2, as shown by (15). In this case, his strategy can be expressed as the sum of two components (see (20)). The first component captures the investor's activity based on his *private* estimation of the difference between the fundamentals and the equilibrium price. This may be seen as akin to "long-term" speculative trading, aimed at taking advantage of the investor's superior information on the liquidation value of the asset, since p_2 is correlated with v. The second component captures the investor's activity based on the extraction of order flow, i.e. *public*, information. This trading is instead aimed at timing the market by exploiting short-run movements in the asset price determined by the evolution of the future aggregate demand. Indeed, using the expressions in Corollaries 1, 2:

$$\mathrm{Cov}_1[v - p_2, p_2 - p_1] = -\frac{\lambda_2}{\gamma \tau_{i1} \tau_u} > 0, \tag{22}$$

and due to Proposition 2, the sign of this expression depends on the equilibrium that arises. In particular, along the HIE (LIE), $\lambda_2 < (>)0$, implying that based on public information the investor expects that returns display momentum (reversal). As a consequence, when observing

$$E_1[\theta_1] = a_1(v - E_1[v]) + \theta_1 > 0,$$

the investor infers that this realization is more driven by fundamentals information (liquidity trading) and goes long (short) in the asset, "chasing the trend" ("making" the market), anticipating that second period investors will bid the price up (down) when he unwinds his position.

Remark 1. The model can be extended to encompass the possibility that investors receive additional private signals at each trading round. The analysis becomes more complicated, without affecting the qualitative results. That is, in this case too we can show that two equilibria with the stated properties arise as long as $\beta > 0$.

Discussion of equilibrium multiplicity

Multiple equilibria with a finite horizon market arise because of the combination of short-term privately informed investors, risk averse market making, and persistent liquidity trading. Under these conditions traders use their private information to speculate on short-run prices changes.

The equilibrium is unique whenever the private learning channel disappears. This happens with:

- Short-term risk averse investors and transient liquidity trading (case $\beta = 0$).
- Short-term risk averse investors and uninformed risk neutral market makers (Vives (1995)).
- Short-term uninformed (or symmetrically informed) risk averse investors (case $\tau_\epsilon = 0$ with $\beta > 0$, see below).

In all those cases the feedback loop responsible for equilibrium multiplicity dies out. We have already explained the case $\beta = 0$ with short-term investors. In this case the private learning channel from prices vanishes ($\beta\Lambda_2 = 0$). The same happens when there is a risk neutral competitive market making sector since then $\Lambda_2 = 0$ (and $\alpha_{P_2} = 0$), and the depth of the market is only determined by adverse selection considerations. In this case $E_{i1}[p_2] = E_{i1}[E_2[v]]$ since $p_2 = E_2[v] = E[v|p_2]$, and the trading intensity of informed traders is increasing with time, implying $a_2 > a_1$ (Vives (1995)).[22]

It is easy to see that absent private information, equilibrium multiplicity disappears even when $\beta > 0$. This can be seen as an extreme situation in which the private learning channel disappears, as in this case prices are invertible in the demand of liquidity traders, and the model is akin to Grossman and Miller (1988):

Corollary 6. *When $\tau_\epsilon = 0$, for all $\beta \in [0, 1]$ there exists a unique equilibrium where*

$$p_n = \bar{v} + \Lambda_n \theta_n \tag{23}$$

$$X_n(p_n) = -\Lambda_n^{-1}(p_n - \bar{v}), \tag{24}$$

$\Lambda_2 = \text{Var}[v]/\gamma$, *and*

$$\Lambda_1 = \frac{\text{Var}_1[p_2]}{\gamma} + \beta\Lambda_2.$$

According to (24), informed investors always take the other side of the order flow, buying the asset at a discount when $\theta_n < 0$, and selling it at a premium otherwise. For a given realization of the innovation in liquidity traders' demand u_n, the larger is Λ_n, the larger is the adjustment

[22] The intensity of trade of first period informed investors (a_1) depends positively on the precision of prices in period 2 in the estimation of the fundamental value. This is so since short-term speculators have information about v but cannot hold the asset until it is liquidated. The more informative is p_2 about v the better first period investors can predict p_2 because they receive private signals about v.

in the price investors require in order to absorb it. Therefore, Λ_n proxies for the liquidity of the market.

Finally, with short-term investors and an infinite horizon (overlapping generations) multiple equilibria arise even with symmetric information (Spiegel (1998)). The reason is the usual bootstrap expectations arising in infinite horizons.[23]

With long-term traders, short-run price speculation based on private information, and potential multiplicity of equilibrium, also arises if there is residual uncertainty. We analyze in the next section the effect of residual uncertainty when investors have a short horizon first and then we compare the results with the case where they have a long horizon.

3 Robustness

In this section we perform four robustness exercises. First, we allow for the possibility that investors receive private signals of a different precision across periods. Next, we extend our model to encompass the possibility that residual uncertainty affects the asset payoff. Then, we study the implications of introducing a public signal in the second period. Finally, we show that a very similar pattern of equilibrium multiplicity also arises in a model with long term traders who face residual uncertainty on the final payoff.

3.1 Private signals with differential precisions

Along the HIE investors speculate very aggressively on their signals because they anticipate that the information they contribute to impound in the price at date 1 is also what mainly drives the price at the liquidation date. To see this more clearly, consider a slight variation of the model we have analyzed so far. Assume that the precision of the private signals investors receive in the two periods differ: $\tau_{\epsilon_1} \neq \tau_{\epsilon_2}$. Then, it is possible to show that at equilibrium $a_2 = \gamma\tau_{\epsilon_2}$, while the first period responsiveness obtains as a fixed point of the mapping

$$\psi(a_1) = \gamma^2 \frac{\Delta a_2 \tau_u \tau_{i2} \tau_{\epsilon_1}}{(1 + \gamma\tau_u \Delta a_2)(\tau_2 + \tau_{\epsilon_1})}.$$

Suppose now that investors' private information in the second period is infinitely noisy: $\tau_{\epsilon_2} = 0$. Then, $a_2 = 0$, and a_1 obtains as a fixed point of the mapping

$$\psi(a_1) = \gamma^2 \frac{-\beta a_1 \tau_u \tau_2 \tau_{\epsilon_1}}{(1 - \gamma\tau_u \beta a_1)(\tau_2 + \tau_{\epsilon_1})}.$$

When $\beta = 0$, the unique equilibrium calls for $a_1 = a_2 = 0$. Thus, in this case the lack of new fundamentals information in the second period discourages investors in the first period from using their signals. As a result prices only reflect the demand of liquidity traders, and $\lambda_2 = 1/\gamma\tau_v$. Suppose now that $\beta > 0$. Then, there are two equilibria which correspond to the

[23]Indeed, it is possible to show that in a two-period version of Spiegel (1998), there exists a unique equilibrium in linear strategies.

LIE and the HIE: in one equilibrium $a_1 = a_2 = 0$, so that once again $\lambda_2 = 1/\gamma\tau_v > 0$; in the other equilibrium $a_2 = 0$, and $a_1 > 1/\gamma\beta\tau_u > 0$, so that $\lambda_2 = (1 - \gamma\tau_u\beta a_1)/\gamma\tau_2 < 0$.[24] In this latter equilibrium first period investors speculate on private information, *precisely* because when $\beta > 0$ they anticipate that the second period price will reflect that information.

3.2 The effect of residual uncertainty and price crashes

In this section we perform a robustness exercise and assume that investors face residual uncertainty over the final liquidation value. Therefore, we model the final payoff as $\hat{v} = v + \delta$, where $\delta \sim N(0, \tau_\delta^{-1})$ is a random term orthogonal to all the random variables in the market, and about which no investor is informed. The addition of the random term δ allows to study the effect of an increase in the residual uncertainty that surrounds investors' environment in periods of heightened turbulence, and shows that a price crash can occur within our framework.

With residual uncertainty, the expressions for prices and investors' strategies do not change (that is, expressions (8), (20), and (21) hold). However, the equilibrium obtains as the solution of a system of two cubic equations and is therefore no longer closed form solvable.

Proposition 3. *When investors face residual uncertainty over the final payoff, there always exists a linear equilibrium, where a_1 and a_2 obtain as a solution to the following system of cubic equations:*

$$\phi_2(a_1, a_2) \equiv a_2(1 + \kappa) - \gamma\tau_\epsilon = 0 \tag{25}$$

$$\phi_1(a_1, a_2) \equiv a_1(1 + \kappa + \gamma\Delta a_2\tau_u) - \gamma a_2\Delta a_2\tau_u(1 + \kappa) = 0, \tag{26}$$

where $\kappa \equiv \tau_{i2}/\tau_\delta$.

Studying (25) shows that for any a_1 there exists a unique real solution to $\phi_2(\cdot) = 0$, which simplifies the numerical analysis.[25] Let $a_2^*(a_1)$ be the unique real solution to (25), then the first period responsiveness to private information obtains as a fixed point of the following map:

$$\psi_1(a_1, a_2^*(a_1)) \equiv \gamma\frac{a_2\Delta a_2\tau_u(1 + \kappa)}{1 + \kappa + \gamma\Delta a_2\tau_u}. \tag{27}$$

Numerical analysis of the solution to the above fixed point shows that for low values of $1/\tau_\delta$ there are typically three equilibria, which can be ranked in terms of responsiveness to private information: $a_1^* < a_1^{**} < a_1^{***}$, and second period liquidity (see Figure 4).

Depending on parameters' values equilibria can be stable or unstable. In Figure 4, panel (a) we show an example in which both the low and the intermediate liquidity equilibria are

[24] In the latter case the equilibrium arises as a solution of the cubic $\phi(a_1) = (1 - \beta\gamma\tau_u a_1)(\tau_2 + \tau_{\epsilon_1}) + \gamma^2\beta\tau_{\epsilon_1}\tau_u\tau_2 = 0$. It is immediate that a real root requires $a_1 > 1/\gamma\beta\tau_u$. Furthermore, analysis of the equation discriminant shows that such root is unique.

[25] This is immediate, since owing to (25), at equilibrium $a_2 > 0$, which implies that we can restrict attention to the positive orthant. Then, $\phi_2(a_1, 0) = -\gamma\tau_\delta\tau_\epsilon < 0$, and $(\partial\phi_2/\partial a_2) = 3a_2^2\tau_u - 4\beta a_1 a_2\tau_u + \tau_{i1} + (\beta a_1)^2\tau_u + \tau_\delta$, which is a quadratic with discriminant $\Delta = 4\tau_u((\beta a_1)^2\tau_u - 3(\tau_{i1} + \tau_\delta)) < 0$, and thus is always positive. Therefore, for any a_1, the second period optimal responsiveness to private information is uniquely determined.

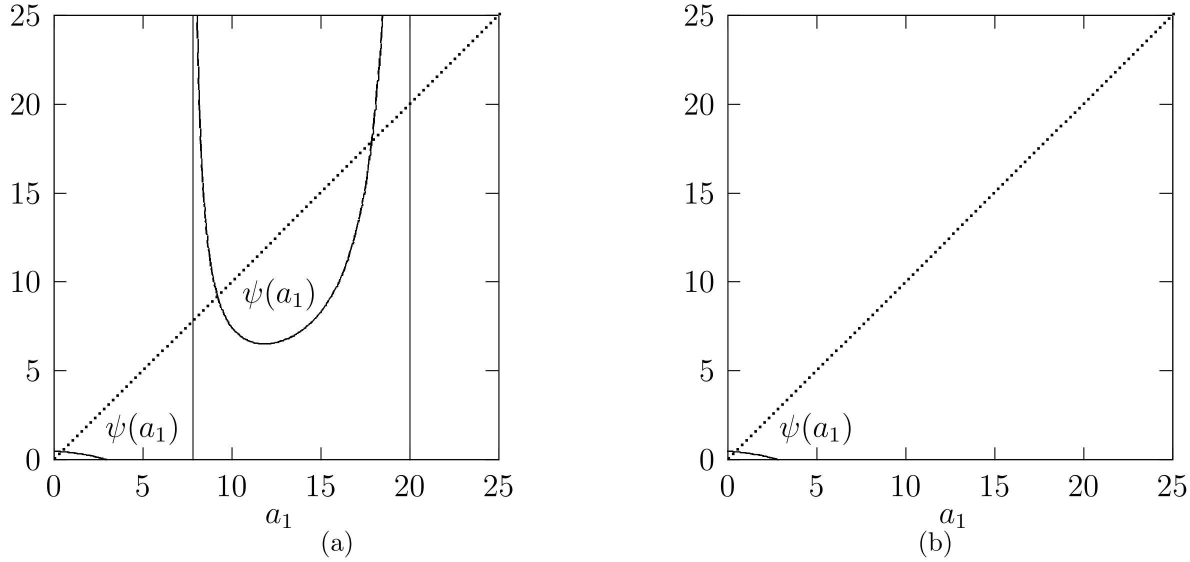


Figure 4: Equilibrium determination and stability with residual uncertainty. The figure displays the best reply function $\psi(a_1)$ (solid line) and the 45-degree line a_1 (dotted line). Equilibria obtain at the points where the two intersect. The vertical lines in panel (a) show the values of a_1 for which the best reply mapping is discontinuous. Parameters' values are as follows: $\tau_v = \tau_u = \tau_\epsilon = \gamma = 1$, $\beta = .3$, $1/\tau_\delta = 1/90$ in panel (a), and $1/\tau_\delta = 1/60$ in panel (b). For these values the equilibria in panel (a) are $(a_1^*, a_2(a_1^*)) = (0.447, 0.968)$, $(a_1^{**}, a_2(a_1^{**})) = (9.928, 0.494)$, and $(a_1^{***}, a_2(a_1^{***})) = (17.88, 0.205)$. Numerical evaluation of the slope of the reaction function yields: $\psi'(a_1^*, a_2(a_1^*)) = -0.088$, $\psi'(a_1^{**}, a_2(a_1^{**})) = -0.781$, and $\psi'(a_1^{***}, a_2(a_1^{***})) = 10.789$. In panel (b) we have a unique equilibrium with $(a_1^*, a_2(a_1^*)) = (0.439, 0.954)$, and $\psi'(a_1^*, a_2(a_1^*)) = -0.088$.

stable according to the best reply dynamics. As residual uncertainty increases (τ_δ^{-1} grows larger), only a_1^* survives. Intuitively, a higher residual uncertainty, increases κ, and lowers second period investors' response to private signals. Thus, first period investors anticipate that the second period price is less related to the fundamentals, which lowers their reliance on private information. As a consequence, the first period price reflects less v and more θ_1, further lowering first period investors' reliance on private information. Therefore, even though a high liquidity equilibrium can be made stable, it is bound to disappear in periods of heavy market turbulence (when τ_δ^{-1} increases).

Based on the above results, we now show that an increase in residual uncertainty may yield a price crash.[26] To see this, let's assume that $\bar{u} < 0$, so that informed investors in period 1 expect to hold a positive amount of the asset. In appendix C we show that this assumption implies

$$E[p_1] \equiv \bar{p}_1 = \bar{v} + \Lambda_1 \bar{u}. \tag{28}$$

Indeed, when $\bar{u} < 0$ first period investors anticipate absorbing a positive supply of the asset at equilibrium and thus require a compensation on the price they pay which lowers the expected

[26] Other authors have investigated price crashes in markets with asymmetric information. See Gennotte and Leland (1990), Romer (1993), and Barlevy and Veronesi (2003).

price below the unconditional expectation of the payoff the more the higher is Λ_1: $E[p_1] < \bar{v}$. Now, assume the same set of parameters of Figure 4, and $\bar{v} = -\bar{u} = 1$. Start with $1/\tau_\delta = 1/90$, so that there are three equilibria, which imply three expected price levels at respectively

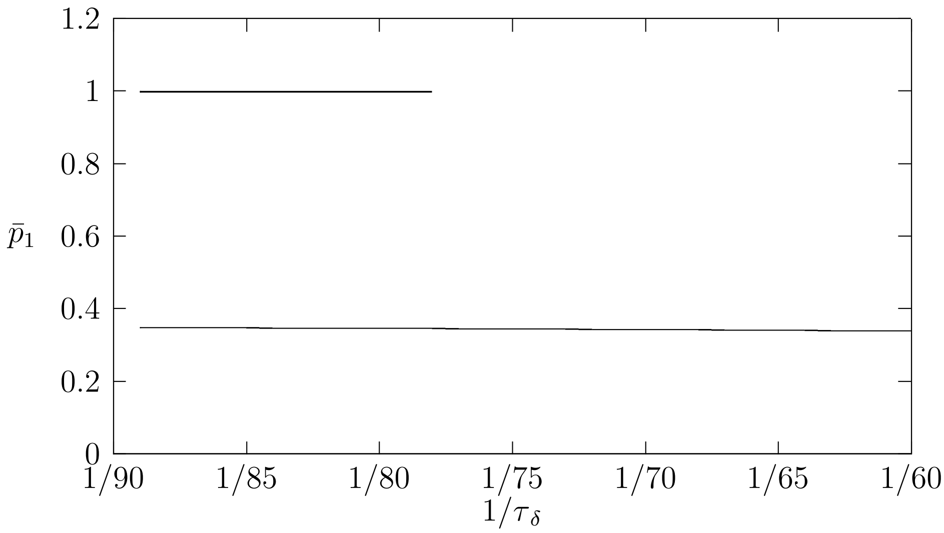
$$\bar{p}_1^* = 0.3479,\ \bar{p}_1^{**} = 0.9966,\ \bar{p}_1^{***} = 0.9991, \tag{29}$$

reflecting the fact that in the high liquidity equilibrium (the one with three stars) the uncertainty on p_2 is very small, and thus the expected price is very high since investors demand a small compensation to absorb the expected selling pressure from liquidity traders. Conversely, in the low liquidity equilibrium (one star) the opposite occurs. Now if $1/\tau_\delta = 1/60$ we know that the number of equilibria narrows down to 1 and

$$\bar{p}_1^* = 0.337991.$$

If we let $1/\tau_\delta$ increase gradually, we obtain Figure 5. According to the figure, for low values of the residual uncertainty parameter, three equilibria arise with expected prices that rank from farther away to very close to $\bar{v} = 1$, respectively for the low, intermediate and high liquidity equilibrium (expected prices in the latter two equilibria are very close to each other, as testified by the values in (29)). Suppose that investors have coordinated on the equilibrium with intermediate liquidity and that the market is suddenly shocked by an increase in residual uncertainty. As a result the equilibrium set becomes a singleton, and the expected price crashes to a much lower level, even though the fundamentals of the economy have not changed.

Figure 5: Example of a price crash. For $1/\tau_\delta < 1/78$, there are three equilibria, which narrow down to one when $1/\tau_\delta \geq 1/78$. Parameters' values are as follows: $\gamma = \tau_\epsilon = \tau_v = \tau_u = 1$, $\beta = .3$, $\bar{v} = -\bar{u} = 1$, and $1/\tau_\delta \in \{1/90, 1/89, \ldots, 1/60\}$.



3.3 The effect of a public signal

In section 2.1 we have shown that when differentially informed risk averse investors have a short term investment horizon, persistence in liquidity trading yields two equilibria with strikingly different informational properties. As we showed in Corollary 4, however the HIE is unstable. Intuitively, the reason is that in the HIE the liquidation price differs very marginally from the price at which (first period) investors load their positions. On the one hand, this generates a huge reduction in first period investors' uncertainty, which leads them to escalate their response to private information; on the other hand as p_1 reflects such information, investors shy away from using their private information to predict p_2. As a result, the best response (18) displays strong strategic substitutabilities, which make the equilibrium unstable. This suggests that when fundamental information is impounded in the price only by privately informed investors, the equilibrium price is *bound to be* poorly informationally efficient, in the sense that it reflects more noise than fundamentals. Thus, taking away part of the control of the information flow from investors may possibly be beneficial to the stability properties of the HIE. In this section we explore this idea, assuming that investors in the second period observe a public signal on the fundamentals.

More concretely, suppose that in the second period all investors observe

$$s = v + \eta,$$

with $\eta \sim N(0, \tau_\eta^{-1})$ and orthogonal to all the random variables in the model. Then, we obtain the following result

Proposition 4. *When a public signal $s = v + \eta$ is distributed in the second period, there always exists a linear equilibrium where $a_2 = \gamma\tau_\epsilon$ and a_1 obtains as a fixed point of the following function:*

$$\psi(a_1) = \gamma \frac{(\lambda_2 \Delta a_2 + \tau_\eta/\tau_{i2})\alpha_{E_1}}{(\lambda_2 \Delta a_2 + \tau_\eta/\tau_{i2})^2/\tau_{i1} + \lambda_2^2/\tau_u + \tau_\eta/\tau_{i2}^2}. \tag{30}$$

Comparing (30) with (16) shows that the existence of a public signal in the second period eliminates the discontinuity in the first period best response. Indeed, even if first period investors where to completely eliminate the impact of news from the second period price (that is trading in a way that $\lambda_2 = 0$), the public signal would still make the price responsive to new fundamentals information, feeding back in first period investors' uncertainty about p_2. As a result, the best response is continuous in a_1. Furthermore, numerical simulations show that in this case three equilibria, which can be ranked in terms of first period responsiveness, can arise. The two extreme equilibria correspond to the LIE and HIE, and interestingly the latter equilibrium can be stable.

Thus, we can conclude that a public signal has the effect of lessening the strategic substitutability effect in the response to private information of first period investors. As a result, it anchors the price more firmly to the fundamentals.

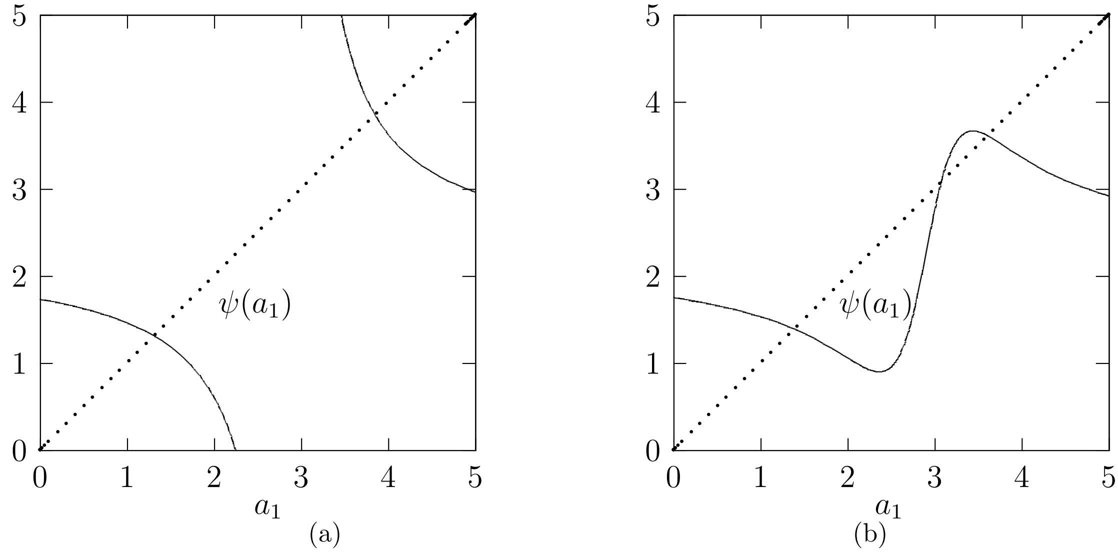


Figure 6: Equilibrium determination and stability with a public signal in the second period. The figure displays the best reply function $\psi(a_1)$ (solid line) and the 45-degree line a_1 (dotted line). Equilibria obtain at the points where the two intersect. Parameter values are as follows: $\tau_v = \tau_u = \beta = 1$, $\tau_\epsilon = \gamma = 1.5$. In panel (a) we assume $\tau_\eta = 0$ and two equilibria arise where $a_1^* = 1.314$, $a_1^{**} = 3.853$, $\psi'(a_1^*) = -.584$, and $\psi'(a_1^{**}) = -1.712$. In panel (b) we assume $\tau_\eta = 0.3$ and obtain three equilibria where $a_1^* = 1.392$, $a_1^{**} = 3.074$, $a_1^{***} = 3.615$, $\psi'(a_1^*) = -.432$, $\psi'(a_1^{**}) = 3.886$, and $\psi'(a_1^{***}) = -.539$.

3.4 Long-term investors

Consider the market with residual uncertainty but now investors have a long horizon and maximize the expected utility of final wealth. In this case multiple equilibria are also possible and the reasons are similar to those of the case with short-term investors.

A long-term investor in the first period speculates on short-term returns and takes into account the hedging possibilities of second period trading. The equilibrium strategy of investor i in the first period is in fact a linear combination of $(E_{i1}[p_2] - p_1)$ and $E_{i1}[x_{i2}]$ (Cespa and Vives (2012)).[27] Were traders not to expect a change in prices, then their optimal period 1 position would be like the one of a static market, and the risk of holding such a position would only be due to the unpredictability of the liquidation value. [28] If a change in prices is expected, traders optimally exploit short-run price differences. Two factors add to the risk of their period 1 position, as traders suffer from the partial unpredictability of the price change, and from the impossibility of determining exactly their future position. However, the opportunity to trade again in the future also grants a hedge against potentially adverse price movements. This, in equilibrium, yields a risk-reduction which when there is *no residual uncertainty* exactly offsets the price risk conditional on private information. As a consequence, *with no residual*

[27] It is of the form $x_{i1} = \Gamma_2^1(E_{i1}[p_2] - p_1) + \Gamma_2^2 E_{i1}[x_{i2}]$ where Γ_2^1 and Γ_2^2 are equilibrium parameters and $E_{i1}[x_{i2}] = \Lambda_2^{-1}(1 - \lambda_2 \Delta a_2)(E_{i1}[v] - \hat{p}_1)$.

[28] Intuitively, if given today's information the asset price is not expected to change, no new private information is expected to arrive to the market and the model collapses to one in which traders hold for two periods the risky asset. Their position, then, naturally coincides with the one they would hold in a static market.

uncertainty, traders' strategies have a static nature in their response to private information.[29] Still investors may speculate on price differences but only for market making purposes to profit from the mean reversion of liquidity trading. With residual uncertainty strategies are truly dynamic and informed investors speculate on short-term price movements based on their private information.

We have that in equilibrium the responsiveness to private information (when informed traders do not receive a new signal in the second period as in our base model, see Cespa and Vives (2012)) is given by:

$$a_1 = \frac{\gamma\tau_\epsilon(1+\gamma\tau_u\Delta a_2)}{1+\kappa+\gamma\tau_u\Delta a_2},$$
$$a_2 = \frac{\gamma\tau_\epsilon}{1+\kappa}.$$

When $\kappa = 0$ then $a_1 = a_2 = \gamma\tau_\epsilon$. With long-term investors, and under the assumptions of the model, the feed-back loop that generates multiplicity is broken because the optimal strategy of an informed trader is static (buy-and-hold): in the first period informed traders receive their private signal, take a position and then in the second period there is no informed trading, the informed traders just make the market absorbing the demands of liquidity traders.[30]

When $\kappa > 0$, $a_1 = \rho\gamma\tau_\epsilon/(1+\kappa)$ with $\rho > 1$ at any equilibrium. The endogenous parameter ρ captures the deviation from the long term private signal responsiveness due to the presence of residual uncertainty. Thus, prior to the last trading round, investors react to their private signals more aggressively than if the liquidation value were to be realized in the next period. Indeed, while residual uncertainty makes investors less confident about their signals, the presence of an additional trading round increases the opportunities to adjust suboptimal positions prior to liquidation. This, in turn, boosts investors' reaction to private information. Residual uncertainty implies that informed investors speculate on short-term price movements based on their private information. This makes possible multiple equilibria. Indeed, faced with uncertain impending liquidation a long-term trader in period 2 is not going to use much his private signal. This makes the trader behave in the first period more like a short-term trader since he will try to unwind his first period holdings in the market at time 2, and carry little of that inventory to the liquidation date. In this case the liquidity of the second period market becomes much more important to determine the trader's reaction to private information in the first period and multiple self-fulfilling expectational loops are possible as in the case with short-term traders. Again the possibility of multiple equilibria is linked to having a negative price impact in the second period ($\lambda_2 < 0$) due to a large response to private information in the first period (gen-

[29]In fact, when $\beta = 1$ we have that $x_{i1} = E[x_{i2}|s_i, p_1]$. When $\beta < 1$ traders speculate also on price changes based on public information.

[30]If informed traders were to receive a second signal in the second period then there would be informed trading in this period but still the strategies would be static and price impact would still be positive. The reason is that the trading intensity in the second period will always be larger than the one in the first period, $a_2 = \gamma(\tau_{\epsilon_2} + \tau_{\epsilon_1}) > a_1 = \gamma\tau_\epsilon$, because private information about the liquidation value accumulates over time. With long-term traders we can not have negative price impacts when the joint information of traders reveals the liquidation value.

erating $\Delta a_2 < 0$). For example, three equilibra arise with $\tau_\delta = 200, \tau_\epsilon = \tau_v = \tau_u = \gamma = 1$ and $\beta = .2$ and only in the low a_1 equilibrium we have $\Delta a_2 > 0$ and stability. In general three equilibria are obtained for high β and high τ_δ. Multiple equilibria may arise also when there is a common shock in the private signal (Grundy and McNichols (1989)).

In summary, with long-term risk averse investors and either residual uncertainty (He and Wang (1995), and Cespa and Vives (2012)), or a common shock in the private signals (Grundy and McNichols (1989)) there may be multiple equilibria. We may have situations then with a negative price impact in the second period. This arises because in those cases informed traders have incentives to use their private information to speculate on short-term price movements and long-term traders may behave as short-term ones.

4 Average expectations and reliance on public information

In this section we use our model to investigate the claim that when investors have a short horizon, prices reflect the latter HOEs about fundamentals and are farther away from the final payoff compared to average expectations (Allen, Morris, and Shin (2006)). We show here that as with liquidity trading persistence investors use their private information *also* to infer the demand of liquidity traders from the first period order flow, the first period price is driven by investors' HOEs about fundamentals *and* by their average expectations about liquidity trading. This, in turn, has implications for price reliance on public information. The consensus opinion about the fundamentals at time n is denoted by $\bar{E}_n[v] \equiv \int_0^1 E_{in}[v]di$. Note that $\bar{E}_n[v] = \alpha_{E_n} v + (1 - \alpha_{E_n})E_n[v]$.

Starting from the second period, and imposing market clearing yields

$$\int_0^1 X_2(s_i, p_1, p_2)di + \theta_2 = 0. \tag{31}$$

Due to CARA and normality, we have

$$X_2(s_i, p_1, p_2) = \gamma \frac{E_{i2}[v] - p_2}{\text{Var}_{i2}[v]}.$$

Replacing the above in (31) and solving for the equilibrium price we obtain

$$p_2 = \bar{E}_2[v] + \Lambda_2 \theta_2.$$

Similarly, in the first period, imposing market clearing yields:

$$\int_0^1 X_1(s_i, p_1)di + \theta_1 = 0,$$

and solving for the equilibrium price we obtain

$$p_1 = \bar{E}_1[p_2] + \frac{\mathrm{Var}_{i1}[p_2]}{\gamma}\theta_1. \tag{32}$$

Substituting the above obtained expression for p_2 in (32) yields

$$\begin{aligned} p_1 &= \bar{E}_1\left[\bar{E}_2[v] + \Lambda_2\theta_2\right] + \frac{\mathrm{Var}_{i1}[p_2]}{\gamma}\theta_1 \\ &= \bar{E}_1\left[\bar{E}_2[v]\right] + \beta\Lambda_2\bar{E}_1\left[\theta_1\right] + \frac{\mathrm{Var}_{i1}[p_2]}{\gamma}\theta_1. \end{aligned} \tag{33}$$

According to (33), there are three terms that form the first period price: investors' second order average expectations over the liquidation value ($\bar{E}_1[\bar{E}_2[v]]$), the risk-adjusted impact of the first period stock of liquidity trades (θ_1), and investors' average expectations over first period liquidity trades ($\bar{E}_1[\theta_1]$). The latter arises since p_2 depends on θ_2, which in turn is correlated with θ_1 when $\beta > 0$. Thus, investors in period 1 are interested in estimating θ_1.

Expression (33) implies that due to persistence in liquidity trading, the weight placed by the price on investors' average information is the sum of two terms: the first term captures the impact of HOEs on v, the second term reflects the impact of investors' average expectations over θ_1. Computing

$$\begin{aligned} \bar{E}_1\left[\bar{E}_2[v]\right] &= \bar{\alpha}_{E_1}v + (1-\bar{\alpha}_{E_1})\,E_1[v] \\ \bar{E}_1[\theta_1] &= a_1(1-\alpha_{E_1})(v - E_1[v]) + \theta_1, \end{aligned}$$

where

$$\bar{\alpha}_{E_1} = \alpha_{E_1}\left(1 - \frac{\tau_1}{\tau_2}(1-\alpha_{E_2})\right).$$

Given (33), this implies that the *total* weight the price places on average private information is given by

$$\alpha_{P_1} = \bar{\alpha}_{E_1} + \beta\Lambda_2 a_1(1-\alpha_{E_1}).$$

Note that $\forall\beta$, $\bar{\alpha}_{E_1} < \alpha_{E_1}$. Thus, when liquidity trading is transient ($\beta = 0$) the first period price places a larger weight on public information than the optimal statistical weight. This finding is in line with Morris and Shin (2002), and Allen, Morris, and Shin (2006). The latter prove that with heterogeneous information, prices reflect investors' HOEs about the final payoff. In this case, the law of iterated expectations does not hold, and investors' forecasts overweight public information because these anticipate the average market opinion knowing that this also depends on the public information observed by other investors. The price is then systematically farther away from fundamentals compared to investors' consensus.

However, when liquidity trading is persistent, the price also reflects investors' average expectations about the impact that the demand of first period liquidity traders has on the second period price. Thus, an additional term adds to $\bar{\alpha}_{E_1}$ which for

$$a_1 > \frac{\alpha_{E_1}}{\beta\Lambda_2(1-\alpha_{E_1})},$$

can increase the weight placed on average private information *above* the optimal statistical weight. Due to Corollary 5 we then have

Corollary 7. *At equilibrium,*

1. *When* $\beta \in (0,1]$, *if*

$$a_1 = \begin{cases} a_1^*, & \textit{then} \quad \alpha_{P_1} < \alpha_{E_1}, \quad \textit{and} \quad \text{Cov}[p_1, v] < \text{Cov}[\bar{E}_1[v], v] \\ a_1^{**}, & \textit{then} \quad \alpha_{P_1} > \alpha_{E_1}, \quad \textit{and} \quad \text{Cov}[p_1, v] > \text{Cov}[\bar{E}_1[v], v]. \end{cases}$$

2. *When* $\beta = 0$, $\alpha_{P_1} < \alpha_{E_1}$ *and* $\text{Cov}[p_1, v] < \text{Cov}[\bar{E}_1[v], v]$.

With persistent liquidity trading, along the HIE, investors escalate their response to private information. In this case the extra weight that adds to $\bar{\alpha}_{E_1}$ is high enough to draw the price closer to fundamentals compared to consensus. In view of the results obtained in Section 2.4 this equilibrium is, however, unstable. Along the equilibrium with low liquidity the price is farther away from fundamentals compared to consensus. This equilibrium, which shares the same properties of the one found by Allen, Morris, and Shin (2006), is instead stable.

Remark 2. We can generalize the model, adding more trading dates. This has the effect of widening the set of equilibria, and generating a richer dynamic. For example, it can happen that prices are in one period closer and in the other period farther away from fundamentals compared to consensus. However, in our numerical simulations it is always possible to find an equilibrium path in which prices are *always* farther away from the fundamentals compared to consensus, and one in which the opposite occurs.

5 Market quality and asset pricing implications

In this section we investigate the implications of our analysis for market quality and asset pricing. In particular, we first analyze the expected losses of liquidity traders across the two equilibria. Next, we show that liquidity trading persistence can generate positive autocovariance of returns, without the need to impose heterogenous beliefs (as in Banerjee, Kaniel, and Kremer (2009)) or to assume that investors' preferences display a behavioral bias (as in, e.g., Daniel, Hirshleifer, and Subrahmanyam (1998)). We then look at the expected volume of informational trading and, consistently with the evidence presented in Llorente, Michaely, Saar, and Wang (2002), we find that in our setup a high volume of informational trading predicts momentum. As we argue below, these results are consistent with a large body of evidence that points at the existence of patterns in the autocorrelation of returns.

5.1 The losses of liquidity traders

The previous sections have established that when liquidity traders' demand is persistent, short term investors exploit a private learning channel from the first period price to learn the demand of liquidity traders. This generates multiple equilibria with strikingly different features. Along the HIE, investors speculate on price continuation, whereas along the LIE they speculate on reversal. We find that it is precisely the activity of using *private* information to infer the demand of liquidity trading from *public* information that, via the private learning channel, can generate an equilibrium with high informational efficiency, and a thick market. However, such equilibrium turns out to be unstable (in the absence of residual uncertainty, with respect to the best reply dynamics) and even if stable is likely to disappear when market conditions deteriorate and private information is a poor guide to investment decisions.

Our results also cast a different light on order anticipation. Order anticipators are commonly interpreted as "parasitic traders," who profit from the exploitation of other traders' orders without contributing to the informational efficiency of prices, nor improving market liquidity. Our model clarifies that order anticipation can enhance market quality, via the effect of the private learning channel from prices on investors' use of private information.

Consider the impact on the expected losses of liquidity traders. As argued above, with persistence, a fraction β of first period liquidity traders hold their position until the event date, while the remaining unwind at date 2. This implies that first period liquidity traders' expected profits are given by

$$\begin{aligned}\Pi_{\theta_1} &\equiv E[\beta\theta_1(v-p_1)+(1-\beta)\theta_1(p_2-p_1)] \\ &= -\left(\beta\lambda_1+(1-\beta)\lambda_2\Delta a_2\left(\frac{\tau_{i1}-\tau_v}{a_1\tau_{i1}}\right)\right)\tau_u^{-1}<0. \end{aligned} \tag{34}$$

It is easy to see that along the HIE $\lim_{\beta\to 0}\Pi_{\theta_1}=0$, since in this case the first period signal responsiveness diverges at $\beta=0$ (see Proposition 2). On the contrary, along the LIE

$$\lim_{\beta\to 0}\Pi_{\theta_1}=\Pi_{\theta_1}\big|_{a_1=\frac{\gamma a_2^2\tau_u}{1+\gamma a_2\tau_u},\beta=0}<0.$$

In general, plotting (34) along the two equilibria that arise with $\beta>0$ one obtains Figure 7.

Depending on parameter values the two plots intersect or not, but the bottomline is that for β small, liquidity traders' expected losses are always smaller along the HIE. Thus, if liquidity traders could coordinate on a specific equilibrium, for small persistence, they would choose the HIE. Adding the expected losses of second period liquidity traders reinforces this conclusion without substantially qualitatively changing the analysis. Indeed, in the HIE, second period liquidity traders make positive profits since there is negative price impact (see Proposition 2). It is also worth remarking that if liquidity traders were able to choose when to trade, they would choose the second period if convinced that the HIE obtains. However, this would be a risky bet since the HIE is unstable.

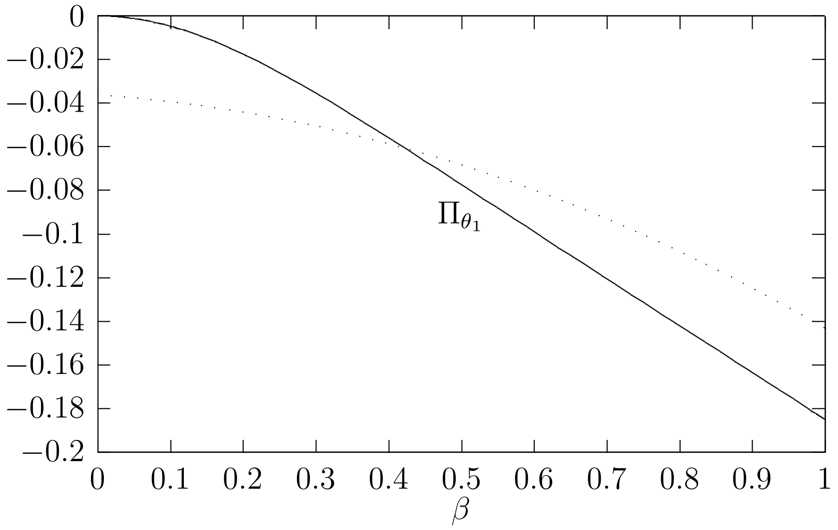


Figure 7: Expected profit along the LIE (dotted line) and along the HIE (continuous line) as a function of β. Other parameter values are $\tau_v = 10$, $\tau_u = 1$, $\tau_\epsilon = 1$, $\gamma = 1$.

5.2 Momentum and reversal

We start by computing the return autocovariance at different horizons:

Corollary 8 (Autocovariance of returns)**.** *At equilibrium:*

1. *For all* $\beta \in [0, 1]$*,* $\mathrm{Cov}[p_2 - p_1, p_1 - \bar{v}] < 0$*.*
2. *For* $\beta \in (0, 1]$*,* $\mathrm{Cov}[v - p_2, p_1 - \bar{v}] < 0$ *. For* $\beta = 0$*,* $\mathrm{Cov}[v - p_2, p_1 - \bar{v}] = 0$*.*
3. *For* $\beta \in (0, 1]$*, along the HIE* $\mathrm{Cov}[v - p_2, p_2 - p_1] > 0$*. Along the LIE, for* $\tau_v < \hat{\tau}_v$*, there exists a value* $\hat{\beta}$ *such that for all* $\beta > \hat{\beta}$*,* $\mathrm{Cov}[v - p_2, p_2 - p_1] > 0$ *(the expression for* $\hat{\tau}_v$ *is given in the appendix, see equation* (A.39)*). If* $\beta = 0$*,* $\mathrm{Cov}[v - p_2, p_2 - p_1] < 0$*.*

According to the above result, along the HIE, momentum occurs at short horizons (close to the end of the trading horizon), whereas at a longer horizon, returns display reversal.[31] This is in line with the empirical findings on return anomalies that document the existence of positive return autocorrelation at short horizons (ranging from six to twelve months, see Jegadeesh and Titman (1993)), and negative autocorrelation at long horizons (from three to five years, see De Bondt and Thaler (1985)).

The first two results derive from the fact that a given estimated first period imbalance, $E_1[\theta_1]$, has an opposite effect on $p_1 - \bar{v}$, and $p_2 - p_1$, $v - p_2$.[32] For the third result, a covariance

[31] Numerical simulations show that in a model with three periods, in the equilibrium with high liquidity, both $\mathrm{Cov}[v - p_3, p_3 - p_2]$ and $\mathrm{Cov}[p_3 - p_2, p_2 - p_1]$ are positive.

[32] As one can verify $\mathrm{Cov}[v - p_2, p_1 - \bar{v}] = \mathrm{Cov}[E_1[v - p_2], E_1[p_1 - \bar{v}]] = -\beta\Lambda_2\mathrm{Cov}[E_1[\theta_1], p_1] < 0$, and $\mathrm{Cov}[p_2 - p_1, p_1 - \bar{v}] = \mathrm{Cov}[E_1[p_2 - p_1], E_1[p_1 - \bar{v}] = (\beta\Lambda_2 - \Lambda_1)\mathrm{Cov}[E_1[\theta_1], p_1] < 0$.

decomposition (and the normality of returns) yields:

$$\begin{aligned}\mathrm{Cov}[v-p_2,p_2-p_1] &= \mathrm{Cov}\left[E_1[v-p_2],E_1[p_2-p_1]\right]+\mathrm{Cov}_1[v-p_2,p_2-p_1]\\ &= \beta\frac{\mathrm{Var}_{i1}[p_2]}{\gamma}\mathrm{Var}[E_1[\theta_1]]+\left(-\frac{\lambda_2}{\gamma\tau_{i1}\tau_u}\right),\end{aligned} \tag{35}$$

implying that the short-term interim returns' autocovariance can be decomposed in two terms. The first term captures the returns' covariation due to the fact that both $E_1[v-p_2]$ and $E_1[p_2-p_1]$ vary with p_1. The second term captures the returns' covariation due to the fact that for each p_1 both second and third period returns jointly vary around their corresponding conditional expectations. All else equal, with persistence the anticipated impact of the first period imbalance has the same sign on both the second and third period expected returns, so that the first term is always positive when $\beta > 0$.[33]

For the second term, factoring out the impact of first period information, the joint covariation of returns around their expectations could be driven either by liquidity trading or by fundamentals information. In the HIE, as prices are close to fundamentals, the second effect predominates and returns positively covary around their means. Conversely, in the LIE, prices are more driven by liquidity trades, so that returns tend to covary around their means in opposite directions.

Equation (35) shows that optimal investment behavior in our model departs in a substantial way from the one of an outside observer that relies on the sign of the *unconditional* return covariance to trade. Indeed, an investor in our model engages in momentum trading only for a stock that displays a very strong positive autocovariance. Equivalently, he may adopt a contrarian strategy even when an outside observer would see $\mathrm{Cov}[v-p_2,p_2-p_1] > 0$. This is because, as argued in Corollary 5, informed investors base their decision to chase the trend or act as contrarians on the sign of $\mathrm{Cov}_1[v-p_2,p_2-p_1]$.

It is interesting to relate our result on momentum with Daniel, Hirshleifer, and Subrahmanyam (1998) who assume that *overconfident* investors underestimate the dispersion of the error term affecting their signals and "overreact" to private information. This, in turn, generates long term reversal and, in the presence of confirming public information which due to *biased self attribution* boosts investors' confidence, also lead to short term positive return autocorrelation. This pattern of overreaction, continuation, and correction is likely to affect stocks which are more difficult to value (e.g., growth stocks). In such a context, momentum is thus a symptom of mispricing and hence related to prices wandering away from fundamentals (conversely, reversal is identified with price corrections). In our model, along the HIE, investors rationally react more strongly to their private signals compared to the static benchmark, in contrast to the overreaction effect of the behavioral literature.[34] However, this heavy reaction

[33] If p_1 decreases compared to p_2, both $p_2 - p_1$ and $v - p_2$ are expected to increase, given that the selling pressure could come from liquidity traders which have a persistent supply. Thus, liquidity trades persistence offsets the mean reversion effect due to first period short-term investors' unwinding at date 2. In fact, in a model with no private information as the one considered in Corollary 6, momentum can also arise, provided liquidity trading is sufficiently persistent, that is when $\beta > \gamma^2\tau_v\tau_u$.

[34] Indeed, the static solution calls for $a_1 = \gamma\tau_\epsilon$ (see, e.g, Admati (1985), or Vives (2008)), and it is easy to

to private information leads to stronger information impounding and to prices that track better the fundamentals (see Proposition 2). Momentum at short horizons in this case is therefore associated with a rapid convergence of the price to the full information value. To illustrate this fact, in Figure 8 we plot the mean price paths along the LIE (thick line), the HIE (thin line), and along the "static" equilibrium, that is the one that would obtain if investors reacted to information as if they were in a static market (dotted line). From the plot it is apparent that in the HIE the price displays a faster adjustment to the full information value than in the LIE (and the static equilibrium). This shows that the occurrence of momentum is not at odds with price (informational) efficiency.

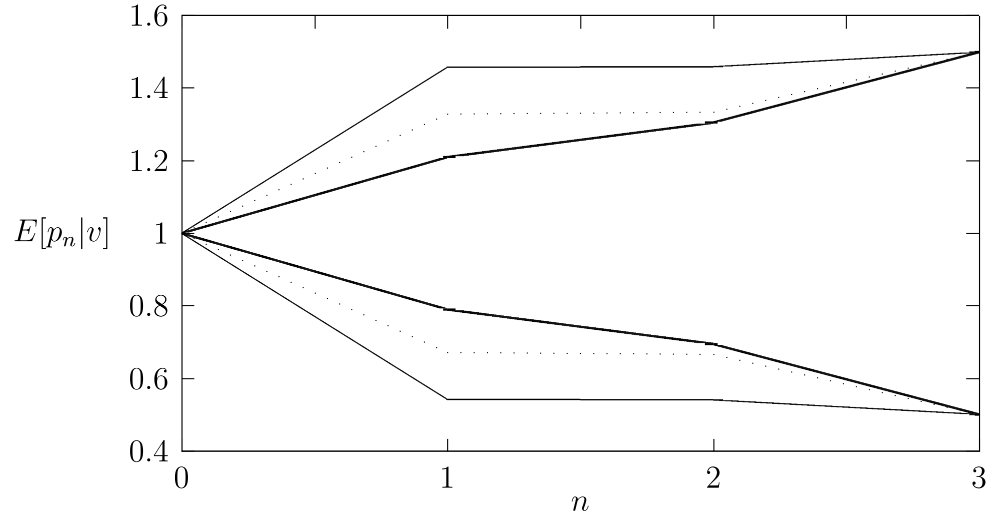


Figure 8: Mean price paths along the LIE (thick line), the HIE (thin line), and assuming that first period investors react to private information as if they were in a static market (i.e., setting $a_1 = \gamma\tau_\epsilon$). Parameters' values are as follows: $\tau_v = \tau_\epsilon = \tau_u = \gamma = 1$, $\bar{v} = 1$, $\beta = .9$ and $v \in \{1.5, .5\}$.

As stated in the corollary, momentum can also occur along the LIE, provided that investors are sufficiently uncertain about the liquidation value prior to trading (that is, τ_v is low) and that liquidity trading is sufficiently persistent (β high). In that equilibrium, investors respond less to private information, information impounding is staggered, and prices adjust more slowly to the full information value (see Figure 5). However, if sufficiently persistent, liquidity trading exerts a continuous price pressure which can eventually outweigh the former effect. Therefore, along this equilibrium momentum arises with slow convergence to the full information value, implying that the occurrence of a positive autocorrelation at short horizons *per se* does not allow to unconditionally identify the informational properties of prices.

Finally, at long horizons, the effect of private information on the correlation of returns washes out and the only driver of the autocovariance is the persistence in liquidity trading, which generates reversal.

verify that $0 < a_1^* < \gamma\tau_\epsilon < a_1^{**}$. In Daniel, Hirshleifer, and Subrahmanyam (1998) overconfident investors overweight private information in relation to the prior.

5.3 Expected volume and return predictability

We now turn our attention to the implications of our results for the expected volume of informational trading and the predictability of returns along the two equilibria. We show that the expected volume of informational trading is high (low) along the HIE (LIE). This implies that a high volume of informational trading predicts momentum, in line with the evidence presented by Llorente, Michaely, Saar, and Wang (2002). However, as we have argued in the previous section, also along the LIE momentum can occur, provided liquidity trading displays sufficiently strong persistence (and the ex-ante uncertainty about the liquidation value is sufficiently high). This implies that a low volume of informational trading can also predict continuation. In this case, though, momentum is a signal of slow price convergence to the liquidation value. In sum, momentum is compatible with both a high and a low volume of informational trading, but the implications that return continuation has for price informativeness are markedly different in the two situations.

We start by defining the volume of informational trading as the expected traded volume in the market with heterogeneous information net of the expected volume that obtains in the market with no private information analyzed in Corollary 6. This yields:[35]

$$\begin{aligned} V_1 &\equiv \int_0^1 E\left[|X_1(s_i,p_1)|\right] di - \int_0^1 E\left[|X_1(p_1)|\right] di \\ &= \int_0^1 \sqrt{\frac{2}{\pi}\mathrm{Var}\left[X_1(s_i,p_1)\right]}di - \int_0^1 \sqrt{\frac{2}{\pi}\mathrm{Var}\left[X_1(p_1)\right]}di \\ &= \sqrt{\frac{2}{\pi}}\left(\sqrt{a_1^2\tau_\epsilon^{-1}+\tau_u^{-1}} - \sqrt{\tau_u^{-1}}\right), \end{aligned} \tag{36}$$

and

$$\begin{aligned} V_2 &\equiv \int_0^1 E\left[|X_2(s_i,p_1,p_2) - X_1(s_1,p_1)|\right] di - \int_0^1 E\left[|X_2(p_2) - X_1(p_1)|\right] di \\ &= \int_0^1 \sqrt{\frac{2}{\pi}\mathrm{Var}\left[X_2(s_i,p_1,p_2) - X_1(s_i,p_1)\right]}di - \int_0^1 \sqrt{\frac{2}{\pi}\mathrm{Var}\left[X_2(p_2) - X_1(p_1)\right]}di \\ &= \sqrt{\frac{2}{\pi}}\left(\sqrt{(a_1^2+a_2^2)\tau_\epsilon^{-1} + (1+(\beta-1)^2)\tau_u^{-1}} - \sqrt{(1+(\beta-1)^2)\tau_u^{-1}}\right). \end{aligned} \tag{37}$$

We measure the total volume of informational trading with $V_1 + V_2$, and obtain

Corollary 9 (Expected volume of informational trading)**.** *At equilibrium, for all $\beta \in (0,1]$ the expected volume of informational trading is higher along the HIE. When $\beta = 0$ only the equilibrium with a low volume of informational trading survives.*

Proof. Rearranging the expressions for investors' strategies obtained in Corollary 5 yields $x_{in} = a_n\epsilon_{in} - \theta_n$, for $n = 1, 2$. Owing to the fact that for a normally distributed random

[35] This is consistent with He and Wang (1995).

variable Y we have

$$E[|Y|] = \sqrt{\frac{2}{\pi}\text{Var}[Y]},$$

which implies (see (36), and (37)), that $V_1 + V_2$ is an increasing function of a_1. Recall that while $a_2 = \gamma\tau_\epsilon$, in the first period the response to private information is higher along the HIE: $a_1^{**} > a_1^*$, and the result follows. Finally, from Proposition 2 when $\beta = 0$,

$$a_1 = \frac{\gamma a_2^2 \tau_u}{1 + \gamma a_2 \tau_u} < a_1^{**}.$$

□

The intuition for the above result is straightforward: as along the HIE investors step up the response to their signals, the position change due to private information is higher along such equilibrium.

Taking together Corollaries 8 and 9 imply that a high volume of informational trading in the second period predicts return continuation, no matter what the persistence in liquidity trading is. A low volume of informational trading, on the other hand, can also be associated with momentum, provided liquidity trading is sufficiently persistent.

6 Conclusions

When liquidity traders' positions are positively correlated across trading dates, investors exploit a private learning channel to infer the demand of liquidity traders from the price at which they load their positions, and anticipate the price at which they unwind them. We show that this effect generates multiple equilibria which can be ranked in terms of investors' responsiveness to private information, liquidity, and informational efficiency. Three ingredients are necessary for multiplicity: risk averse privately informed investors, short run price speculation, and persistent liquidity trading.

Our analysis clarifies the role of HOEs in asset pricing. With liquidity trading persistence, prices are driven by average expectations about *fundamentals* and *liquidity trading*. This, in contrast to the beauty contest results of Allen, Morris, and Shin (2006), can draw prices either systematically farther away from or closer to fundamentals compared to investors' consensus (respectively, along the LIE and the HIE). We show that only the LIE is stable.

Our paper also provides an alternative interpretation for empirically documented regularities on the patterns of return autocorrelation. As we have argued, at long horizons returns display reversal. However, return correlation at short horizons depends on the equilibrium that prevails in the market. In the HIE, investors escalate their response to private information and momentum arises. Conversely, in the LIE investors scale down their response to private signals and, when liquidity trading is not very persistent, returns tend to revert. While this offers an explanation for returns' predictability which departs from behavioral assumptions, our analysis also makes the empirical prediction that both a high or a low volume of informational trading can predict momentum. In the former case, this is a signal that prices rely poorly on public information and accurately reflect fundamentals starting from the earlier stages of the trading process. In this case momentum at short horizons proxies for a rapid price convergence to the full information value. In the latter case, instead, prices heavily rely on public information and offer a poor signal of fundamentals. In this case, therefore, momentum proxies for a continuing, liquidity-driven, price pressure.

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A Appendix

PROOF OF PROPOSITION 1

Consider a candidate linear (symmetric) equilibrium and let $z_1 \equiv a_1 v + \theta_1$ be the "informational content" of the first period order flow and simlarly $z_2 \equiv \Delta a_2 v + u_2$ where $\Delta a_2 \equiv a_2 - \beta a_1$ for the second period. Then, it is easy to see that p_1 is observationally equivalent (o.e.) to z_1 and that the sequence $\{z_1, z_2\}$ is o.e. to $\{p_1, p_2\}$. Consider a candidate linear (symmetric) equilibrium $x_{i1} = a_1 s_i - \varphi_1(p_1)$, $x_{i2} = a_2 s_i - \varphi_2(p_1, p_2)$, where $\varphi_n(\cdot)$ is a linear function. Letting $x_n \equiv \int_0^1 x_{in} di$, and imposing market clearing in the first period implies (due to our convention):

$$x_1 + \theta_1 = 0 \Leftrightarrow a_1 v + \theta_1 = \varphi_1(p_1). \tag{A.1}$$

In the second period the market clearing condition is

$$\begin{aligned} x_2 + \beta\theta_1 + u_2 = 0 &\Leftrightarrow x_2 - \beta x_1 + u_2 = 0 \\ &\Leftrightarrow a_2 v - \varphi_2(p_1, p_2) - \beta(a_1 v - \varphi_1(p_1)) + u_2 = 0 \\ &\Leftrightarrow \Delta a_2 v + u_2 = \varphi_2(p_1, p_2) - \beta\varphi_1(p_1), \end{aligned} \tag{A.2}$$

where in the second line we use (A.1). From (A.1) and (A.2) it is easy to see that z_1 is o.e. to p_1 and that $\{z_1, z_2\}$ is o.e. to $\{p_1, p_2\}$. It thus follows that $E_n[v] = \tau_n^{-1}(\tau_v + \tau_u \sum_{t=1}^n \Delta a_t z_t)$, $\text{Var}_n[v] = (\tau_v + \tau_u \sum_{t=1}^n (\Delta a_t)^2)^{-1}$, $E_{in}[v] = \tau_{in}^{-1}(\tau_n E_n[v] + \tau_\epsilon s_i)$, and $\text{Var}_{in}[v] = (\tau_n + \tau_\epsilon)^{-1} \equiv \tau_{in}^{-1}$.

To prove our argument, we proceed by backwards induction. In the last trading period traders act as in a static model and owing to CARA and normality we have

$$X_2(s_{i1}, z^2) = \gamma \frac{E_{i2}[v] - p_2}{\text{Var}_{i2}[v]}, \tag{A.3}$$

and

$$\begin{aligned} p_2 &= \bar{E}_2[v] + \frac{\text{Var}_{i2}[v]}{\gamma} \\ &= \alpha_{P_2}\left(v + \frac{\theta_2}{a_2}\right) + (1 - \alpha_{P_2})\, E_2[v], \end{aligned} \tag{A.4}$$

where $a_2 = \gamma\tau_\epsilon$, and $\alpha_{P_2} = \alpha_{E_2}$. Rearranging (A.4) we obtain

$$\begin{aligned} p_2 &= \frac{\alpha_{P_2}}{a_2}(a_2 v - \beta a_1 v + \beta a_1 v + \theta_2) + (1 - \alpha_{P_2}) E_2[v] \\ &= \left(\underbrace{\frac{\alpha_{P_2}}{a_2} + (1 - \alpha_{P_2})\frac{\Delta a_2 \tau_u}{\tau_2}}_{\lambda_2}\right) z_2 + (1 - \lambda_2 \Delta a_2)\hat{p}_1, \end{aligned} \tag{A.5}$$

where

$$\hat{p}_1 \equiv \frac{\gamma\tau_1 E_1[v] + \beta z_1}{\gamma\tau_1 + \beta a_1}, \tag{A.6}$$

which provides an alternative expression for p_2 which separates the impact on second period "news" from the information contained in the first period order flow.

In the first period owing to CARA and normality, an agent i trades according to

$$X_1(s_{i1}, z_1) = \gamma \frac{E_{i1}[p_2] - p_1}{\text{Var}_{i1}[p_2]}, \tag{A.7}$$

where, using (A.5),

$$E_{i1}[p_2] = \lambda_2 \Delta a_2 E_{i1}[v] + (1 - \lambda_2 \Delta a_2)\hat{p}_1 \tag{A.8}$$

$$\text{Var}_{i1}[p_2] = \lambda_2^2 \left(\frac{\tau_{i2}}{\tau_{i1}\tau_u} \right). \tag{A.9}$$

Replacing (A.8) and (A.9) in (A.7) yields

$$X_1(s_{i1}, z_1) = \frac{\gamma \Delta a_2 \tau_{i1} \tau_u}{\lambda_2 \tau_{i2}} (E_{i1}[v] - \hat{p}_1) + \frac{\gamma \tau_{i1} \tau_u}{\lambda_2^2 \tau_{i2}} (\hat{p}_1 - p_1).$$

Imposing market clearing and identifying equilibrium coefficients yields

$$p_1 = \alpha_{P_1} \left(v + \frac{\theta_1}{a_1} \right) + (1 - \alpha_{P_1}) E_1[v], \tag{A.10}$$

where

$$\alpha_{P_1} \equiv \alpha_{E_1} \left(1 + \frac{(\beta\rho - 1)\tau_1}{\tau_{i2}} \right), \tag{A.11}$$

a_1 obtains as a fixed point of the mapping

$$\begin{aligned} \psi(a_1) &= \gamma \frac{\lambda_2 \Delta a_2 \alpha_{E_1}}{\text{Var}_{i1}[p_2]} \\ &= \gamma \frac{a_2 \Delta a_2 \tau_u}{1 + \gamma \Delta a_2 \tau_u}, \end{aligned} \tag{A.12}$$

and $\rho \equiv a_1/(\gamma\tau_\epsilon)$. Finally, note that using (A.11) and (A.12) and rearranging the expression for the first period strategy yields

$$X_1(s_{i1}, z_1) = \frac{a_1}{\alpha_{E_1}} (E_{i1}[v] - p_1) + \frac{\alpha_{P_1} - \alpha_{E_1}}{\alpha_{E_1}} \frac{a_1}{\alpha_{P_1}} (p_1 - E_1[v]).$$

□

Proof of Corollary 1

In the second period, rearranging (A.4), $p_2 = E_2[v] + \Lambda_2 E_2[\theta_2]$, where $\Lambda_2 = \text{Var}_{i2}[v]/\gamma$. In the first period due to short term horizons, we have

$$
\begin{aligned}
p_1 &= \bar{E}_1[p_2] + \frac{\mathrm{Var}_{i1}[p_2]}{\gamma}\theta_1 \\
&= \bar{E}_1\left[\bar{E}_2[v]\right] + \frac{\mathrm{Var}_{i2}[v]}{\gamma}\beta\bar{E}_1\left[\theta_1\right] + \frac{\mathrm{Var}_{i1}[p_2]}{\gamma}\theta_1 \\
&= \alpha_{P_1}v + (1-\alpha_{P_1})E_1[v] + \left(\beta\frac{\mathrm{Var}_{i2}[v+\delta]}{\gamma} + \frac{\mathrm{Var}_{i1}[p_2]}{\gamma}\right)\theta_1,
\end{aligned}
\tag{A.13}
$$

where

$$
\alpha_{P_1} = \bar{\alpha}_{E_1} + \frac{\mathrm{Var}_{i2}[v]}{\gamma}\beta a_1(1-\alpha_{E_1}),
$$

and

$$
\bar{\alpha}_{E_1} = \alpha_{E_1}\left(1 - \frac{\tau_1}{\tau_2}(1-\alpha_{E_2})\right).
$$

Now, we know that at a linear equilibrium

$$
p_1 = \alpha_{P_1}\left(v + \frac{\theta_1}{a_1}\right) + (1-\alpha_{P_1})E_1[v]. \tag{A.14}
$$

Comparing (A.13) and (A.14), we then see that an alternative expression for a_1 is the following:

$$
a_1 = \gamma\frac{\alpha_{P_1}}{\mathrm{Var}_{i1}[p_2] + \beta\mathrm{Var}_{i2}[v]}.
$$

Given the definition of the inventory component market depth, from the last equation we conclude that

$$
\Lambda_2 = \frac{\mathrm{Var}_{i1}[p_2] + \beta\mathrm{Var}_{i2}[v]}{\gamma}. \tag{A.15}
$$

□

Proof of Corollary 2

For the second period price, see (A.5). For the first period price, we rearrange (A.10) to obtain

$$
p_1 = \left(\underbrace{\frac{\alpha_{P_1}}{a_1} + (1-\alpha_{P_1})\frac{a_1\tau_u}{\tau_1}}_{\lambda_1}\right)z_1 + (1-\alpha_{P_1})\frac{\tau_v}{\tau_1}\bar{v}. \tag{A.16}
$$

□

Proof of Proposition 2

For any $\beta \in [0,1]$, in the second period an equilibrium must satisfy $a_2 = \gamma\tau_\epsilon$. In the first period, using (A.12), an equilibrium must satisfy

$$\begin{aligned}\phi_1(a_1, a_2) &\equiv a_1\lambda_2(\tau_2+\tau_\epsilon) - \gamma\tau_\epsilon\Delta a_2\tau_u \\ &= a_1(1+\gamma\tau_u\Delta a_2) - \gamma^2\tau_\epsilon\Delta a_2\tau_u = 0. \end{aligned} \tag{A.17}$$

The above equation is a quadratic in a_1 which for any $a_2 > 0$ and $\beta > 0$ possesses two positive, real solutions:

$$a_1^* = \frac{1+\gamma\tau_u a_2(1+\beta) - \sqrt{(1+\gamma\tau_u a_2(1+\beta))^2 - 4\beta(\gamma\tau_u a_2)^2}}{2\beta\gamma\tau_u} \tag{A.18}$$

$$a_1^{**} = \frac{1+\gamma\tau_u a_2(1+\beta) + \sqrt{(1+\gamma\tau_u a_2(1+\beta))^2 - 4\beta(\gamma\tau_u a_2)^2}}{2\beta\gamma\tau_u}, \tag{A.19}$$

with $a_1^{**} > a_1^*$. This proves that for $\beta > 0$ there are two linear equilibria.

Inspection of the above expressions for a_1 shows that $\beta a_1^* < a_2$, while $\beta a_1^{**} > a_2$. This implies that $\beta\rho > 1$ for $a_1 = a_1^{**}$ and $\beta\rho < 1$ otherwise. Thus, using (A.11), we obtain $\alpha_{P_1}(a_1^*, a_2) < \alpha_{E_1}(a_1^*, a_2)$, and $\alpha_{P_1}(a_1^{**}, a_2) > \alpha_{E_1}(a_1^{**}, a_2)$. The result for λ_2 follows from substituting (A.18) and (A.19) in λ_2. To see that prices are more informative along the HIE note that in the first period $\text{Var}[v|z_1]^{-1} = \tau_1 = \tau_v + a_1^2\tau_u$. In the second period, the price along the HIE is more informative than along the LIE if and only if

$$\frac{(1+\beta^2+\gamma a_2\tau_u((1-\beta^2)+\beta(1+\beta^2)))\sqrt{(1+\gamma a_2\tau_u(1+\beta))^2 - 4\beta(\gamma a_2\tau_u)^2}}{\gamma^2\beta^2\tau_u} > 0,$$

which is always true.

When $\beta \to 0$, along the HIE we have

$$\lim_{\beta\to 0} \frac{1+\gamma\tau_u(a_2+\beta\gamma\tau_\epsilon) + \sqrt{1+\gamma\tau_u(2(a_2+\beta\gamma\tau_\epsilon)+\gamma\tau_u(a_2-\beta\gamma\tau_\epsilon)^2)}}{2\beta\gamma\tau_u} = \infty,$$

while along the LIE, using l'Hospital's rule,

$$\lim_{\beta\to 0} \frac{1+\gamma\tau_u(a_2+\beta\gamma\tau_\epsilon) - \sqrt{1+\gamma\tau_u(2(a_2+\beta\gamma\tau_\epsilon)+\gamma\tau_u(a_2-\beta\gamma\tau_\epsilon)^2)}}{2\beta\gamma\tau_u} = \frac{\gamma a_2^2\tau_u}{1+\gamma a_2\tau_u}.$$

From (A.11) it then follows that in this case $\alpha_{P_1} < \alpha_{E_1}$. Finally, defining

$$a_{10}^* = \frac{\gamma a_2^2\tau_u}{1+\gamma a_2\tau_u},$$

and taking the limit of λ_2 as $\beta \to 0$ when $a_1 = a_1^*$ yields

$$\lim_{\beta\to 0}\lambda_2(a_1^*, a_2) = \frac{1+\gamma\tau_u a_2}{\gamma(\tau_v+(a_{10}^*)^2\tau_u + a_2^2\tau_u+\tau_\epsilon)} > 0,$$

whereas $\lim_{\beta\to 0}\lambda_2(a_1^{**}, a_2) = 0$. □

Proof of Corollary 4

Starting from the LIE, we need to verify that $|\psi'(a_1^*)| < 1$, or that when $a_1 = a_1^*$,

$$\gamma\beta a_2\tau_u < (1 + \gamma\tau_u\Delta a_2)^2.$$

Substituting (A.18) on R.H.S. of the above inequality and rearranging yields

$$|\psi'(a_1^*)| < 1 \Leftrightarrow -2(1+a_2\gamma\tau_u(1-\beta))(1+a_2\gamma\tau_u(1-\beta)+\sqrt{(1+\gamma\tau_u a_2(1+\beta))^2 - 4\beta(\gamma\tau_u a_2)^2}) < 0,$$

which is always satisfied. For the HIE, we need instead to verify that $|\psi'(a_1^{**})| > 1$, or that when $a_1 = a_1^{**}$,

$$\gamma\beta a_2\tau_u > (1 + \gamma\tau_u\Delta a_2)^2.$$

Substituting (A.19) on R.H.S. of the above inequality and rearranging yields

$$|\psi'(a_1^{**})| > 1 \Leftrightarrow 2(1+a_2\gamma\tau_u(1-\beta))(-(1+a_2\gamma\tau_u(1-\beta))+\sqrt{(1+\gamma\tau_u a_2(1+\beta))^2 - 4\beta(\gamma\tau_u a_2)^2}) > 0,$$

which is always satisfied, since the first factor in the product on the R.H.S. of the above expression is positive, while manipulating the second factor shows that

$$\sqrt{(1+\gamma\tau_u a_2(1+\beta))^2 - 4\beta(\gamma\tau_u a_2)^2} > (1 + a_2\gamma\tau_u(1-\beta)) \Leftrightarrow 4a_2\beta\gamma\tau_u > 0.$$

□

Proof of Corollary 5

See the proof of Proposition 1.

□

Proof of Corollary 6

Assume that in a linear equilibrium $x_n = -\phi_n(p_n)$, with $\phi_n(\cdot)$ a linear function of p_n. This implies that the market clearing equation at time n reads as follows:

$$x_n + \theta_n = 0 \Leftrightarrow -\phi_n(p_n) + \theta_n = 0.$$

Hence, at equilibrium p_n is observationally equivalent to θ_n, i.e. at time n investors *know* the realisation of the noise stock θ_n. To solve for the equilibrium we proceed by backward induction and start from the second trading round, where due to CARA and normality, we have

$$\begin{aligned} X_2(p_2) &= \gamma\frac{E_2[v] - p_2}{\text{Var}_2[v]} \\ &= -\Lambda_2^{-1}(p_2 - \bar{v}), \end{aligned} \tag{A.20}$$

with

$$\Lambda_2 \equiv \frac{1}{\gamma\tau_v}, \tag{A.21}$$

and

$$p_2 = \bar{v} + \Lambda_2\theta_2. \tag{A.22}$$

In the first period, we have

$$\begin{aligned} X_1(p_1) &= \gamma\frac{E_1[p_2] - p_1}{\text{Var}_1[p_2]} \\ &= -\Lambda_1^{-1}(p_1 - \bar{v}), \end{aligned} \tag{A.23}$$

with

$$\Lambda_1 \equiv \frac{1 + \gamma\beta\Lambda_2^{-1}\tau_u}{\gamma\Lambda_2^{-2}\tau_u}, \tag{A.24}$$

and

$$p_1 = \bar{v} + \Lambda_1\theta_1. \tag{A.25}$$

□

PROOF OF PROPOSITION 3

With residual uncertainty, in the second period investors trade according to

$$X_2(s_i, z^2) = \gamma\frac{E_{i2}[v] - p_2}{\text{Var}_{i2}[v + \delta]}, \tag{A.26}$$

which implies that at equilibrium

$$a_2 = \frac{\gamma\tau_\epsilon}{1 + \kappa}, \tag{A.27}$$

$$p_2 = \lambda_2 z_2 + (1 - \lambda_2\Delta a_2)\hat{p}_1,$$

where

$$\lambda_2 = \frac{1 + \kappa + \gamma\tau_u\Delta a_2}{\gamma\tau_{i2}} \tag{A.28}$$

$$\hat{p}_1 = \frac{\gamma\tau_1 E_1[v] + \beta(1 + \kappa)z_1}{\gamma\tau_1 + \beta a_1(1 + \kappa)}, \tag{A.29}$$

with $\kappa \equiv \tau_{i2}/\tau_\delta$. In the first period, we then have

$$X_1(s_i, z_1) = \gamma\frac{E_{i1}[p_2] - p_1}{\text{Var}_{i1}[p_2]},$$

where $E_{i1}[p_2] = \lambda_2\Delta a_2 E_{i1}[v] + (1 - \lambda_2\Delta a_2)\hat{p}_1$, and $\text{Var}_{i1}[p_2] = \lambda_2^2\text{Var}_{i1}[z_2]$. Identifying the first period signal responsiveness yields

$$a_1 = \gamma\frac{\Delta a_2^2\tau_u(1 + \kappa)}{1 + \kappa + \gamma\Delta a_2\tau_u(1 - \beta(1 + \kappa))}. \tag{A.30}$$

The equilibrium obtains as a solution to the system (A.27)-(A.30) which corresponds to (25)-(26). □

Proof of Proposition 4

Suppose that second period investors observe a public signal $s = v + \eta$, where $\eta \sim N(0, \tau_\eta^{-1})$ and orthogonal to all the random variables in the model. Then, a second period investor i conditions his position on $\{z_1, z_2, s_{i2}, s\}$. Therefore,

$$E_{i2}[v] = \frac{\tau_2 E_2[v] + \tau_{\epsilon_2} s_{i2} + \tau_\eta s}{\tau_{i2}} \tag{A.31a}$$

$$\text{Var}_{i2}[v] = \frac{1}{\tau_2 + \tau_{\epsilon_2} + \tau_\eta} \equiv \frac{1}{\tau_{i2}}. \tag{A.31b}$$

If we replace the above in the strategy of an informed investor in the second period, we get

$$\begin{aligned} X_2(s_{i2}, p_2, p_2, s) &= \gamma \frac{E_{i2}[v] - p_2}{\text{Var}_{i2}[v]} \\ &= \gamma(\tau_2 E_2[v] + \tau_{\epsilon_2} s_{i2} + \tau_\eta s - \tau_{i2} p_2). \end{aligned} \tag{A.32}$$

Hence, at equilibrium we have $a_2 = \gamma \tau_{\epsilon_2}$, and

$$\gamma \tau_2 E_2[v] + a_2 v + \gamma \tau_\eta s - \gamma \tau_{i2} p_2 + \theta_2 = 0,$$

which implies that

$$p_2 = \lambda_2 z_2 + \frac{\gamma \tau_1 E_1[v] + \beta z_1}{\gamma \tau_{i2}} + \frac{\tau_\eta}{\tau_{i2}} s,$$

where

$$\lambda_2 = \frac{1 + \gamma \tau_u \Delta a_2}{\gamma \tau_{i2}},$$

as in the baseline case, except that now we use (A.31b) to compute the forecast precision. According to the above expression for the price, at time 1 an investor has to forecast only

$$\lambda_2 z_2 + \frac{\tau_\eta}{\tau_{i2}} s.$$

Therefore, we have that the best response is given by

$$\psi(a_1) = \gamma \frac{(\lambda_2 \Delta a_2 + \tau_\eta / \tau_{i2}) \alpha_{E_1}}{(\lambda_2 \Delta a_2 + \tau_\eta / \tau_{i2})^2 / \tau_{i1} + \lambda_2^2 / \tau_u + \tau_\eta / \tau_{i2}^2}.$$

For an equilibrium to exist we need to find a solution to the quintic equation $a_1 = \psi(a_1)$. We can immediately rule out negative roots, since if $a_1 < 0$, $\Delta a_2 > 0$, which yields a contradiction. Therefore, we can have at most 5 equilibria with $a_1 > 0$. □

Proof of Corollary 7

Using (6) the covariance between p_1 and v is given by

$$\text{Cov}[v, p_1] = \alpha_{P_1}\frac{1}{\tau_v} + (1-\alpha_{P_1})\left(\frac{1}{\tau_v} - \frac{1}{\tau_1}\right), \tag{A.33}$$

and carrying out a similar computation for the first period consensus opinion

$$\text{Cov}\left[\bar{E}_1[v], v\right] = \alpha_{E_1}\frac{1}{\tau_v} + (1-\alpha_{E_1})\left(\frac{1}{\tau_v} - \frac{1}{\tau_1}\right). \tag{A.34}$$

We can now subtract (A.34) from (A.33) and obtain

$$\text{Cov}\left[p_1 - \bar{E}_1[v], v\right] = \frac{\alpha_{P_1} - \alpha_{E_1}}{\tau_1}, \tag{A.35}$$

implying that the price at time 1 over relies on public information (compared to the optimal statistical weight) if and only if the covariance between the price and the fundamentals falls short of that between the consensus opinion and the fundamentals. □

Proof of Corollary 8

To compute $\text{Cov}[p_2 - p_1, p_1 - \bar{v}]$ we rearrange expression (6) in the paper to obtain

$$\begin{aligned} p_2 - p_1 &= \lambda_2 u_2 + \lambda_2 \Delta a_2 \left((1-\alpha_{E_1})(v - E_1[v]) - \frac{\alpha_{E_1}}{a_1}\theta_1\right) \\ &= \lambda_2 u_2 + \lambda_2 \Delta a_2 \left(\frac{\tau_v}{\tau_{i1}}(v - \bar{v}) - \frac{\tau_{i1} - \tau_v}{a_1 \tau_{i1}} u_1\right). \end{aligned} \tag{A.36}$$

Next, using expression (8) we obtain

$$\text{Cov}[p_2 - p_1, p_1 - \bar{v}] = -\frac{\lambda_1 \lambda_2 \Delta a_2 \tau_\epsilon}{a_1 \tau_u \tau_{i1}} < 0, \forall \beta \in (0, 1].$$

Computing the limit for $\beta \to 0$ of the above expression yields different results depending on whether we concentrate on the LIE or the HIE. Indeed,

$$\lim_{\beta \to 0} \text{Cov}[p_2 - p_1, p_1 - \bar{v}]\big|_{a_1 = a_1^*} = -\lambda_1 \frac{(1 + \gamma \tau_u a_2)^2 \tau_\epsilon}{\gamma a_2 \tau_u^2 \tau_{i1}} < 0$$

$$\lim_{\beta \to 0} \text{Cov}[p_2 - p_1, p_1 - \bar{v}]\big|_{a_1 = a_1^{**}} = 0.$$

To compute the expression for $\text{Cov}[v - p_2, p_1 - \bar{v}]$ we use (A.37) and (8), and obtain

$$\text{Cov}[v - p_2, p_1 - \bar{v}] = -\frac{\beta \lambda_1}{\gamma \tau_{i2} \tau_u} < 0 \text{ for all } \beta > 0.$$

In this case the taking the limit of the above covariance for $\beta \to 0$ yields the same result across the two equilibria that arise:

$$\lim_{\beta\to 0} \text{Cov}[v - p_2, p_1 - \bar{v}]\big|_{a_1=a_1^*} = \lim_{\beta\to 0} \text{Cov}[v - p_2, p_1 - \bar{v}]\big|_{a_1=a_1^{**}} = 0.$$

To compute $\text{Cov}[v - p_2, p_2 - p_1]$ we use (A.36) and using again expression (8) we get

$$\begin{aligned} v - p_2 &= (1 - \alpha_{E_2})(v - E_2[v]) - \frac{\alpha_{P_2}}{a_2}\theta_2 \\ &= \frac{\tau_v(v - \bar{v})}{\tau_{i2}} - \frac{\beta + \gamma\tau_u a_1}{\gamma\tau_{i2}}u_1 - \frac{1 + \gamma\tau_u \Delta a_2}{\gamma\tau_{i2}}u_2. \end{aligned} \tag{A.37}$$

Using (A.36) and (A.37) we can now compute the autocovariance of returns and get

$$\begin{aligned} \text{Cov}[v - p_2, p_2 - p_1] &= \lambda_2 \left(\frac{\Delta a_2 \tau_v}{\tau_{i1}\tau_{i2}} - \frac{1 + \gamma\tau_u \Delta a_2}{\gamma\tau_{i2}\tau_u} + \Delta a_2 \frac{\tau_{i1} - \tau_v}{a_1\tau_{i1}} \frac{\beta + \gamma\tau_u a_1}{\gamma\tau_{i2}\tau_u} \right) \\ &= -\frac{\lambda_2}{\gamma\tau_{i2}\tau_u} \left(1 - \beta\Delta a_2 \frac{\tau_{i1} - \tau_v}{a_1\tau_{i1}} \right). \end{aligned} \tag{A.38}$$

Looking at (A.38) we can immediately say that along the HIE there is momentum. This is true because in that equilibrium $\lambda_2 < 0$ and $\Delta a_2 < 0$. Along the LIE momentum can occur, *depending on the persistence of liquidity trades.* To see this, note that since in this equilibrium $\lambda_2 > 0$ and $\Delta a_2 > 0$, from (A.38) momentum needs

$$1 - \beta\Delta a_2 \frac{\tau_{i1} - \tau_v}{a_1\tau_{i1}} < 0,$$

which can be rearranged as an (implicit) condition on the magnitude of β:

$$\frac{a_1\tau_{i1}}{\Delta a_2(\tau_{i1} - \tau_v)} < \beta < 1.$$

If $\beta = 0$, the above condition is never satisfied. Indeed, in this case there exists a unique equilibrium in which $\Delta a_2 = a_2 > 0$. Therefore, when $\beta = 0$ returns always display reversal. If $\beta = 1$, the condition is satisfied if

$$a_1\tau_v + a_1(\tau_\epsilon + a_1^2\tau_u) < \Delta a_2\tau_u(\tau_\epsilon + a_1^2\tau_u).$$

Isolating τ_v in the above expression yields:

$$\tau_v < \hat{\tau}_v \equiv \frac{(\Delta a_2 - a_1)(\tau_\epsilon + a_1^2\tau_u)}{a_1}, \tag{A.39}$$

which, since a_1 does not depend on τ_v (see (A.18)), gives an explicit upper bound on τ_v. Hence, if $\tau_v < \hat{\tau}_v$, there exists a $\hat{\beta}$ such that for all $\beta \geq \hat{\beta}$ momentum occurs between the second and third period returns along the LIE. □

B Appendix

In this appendix we decompose the best response function to analyse the effects that impinge on a first period investor's responsiveness to private information. We start by using (8) to compute an investor i's expectation of p_2:

$$\begin{aligned}
E_{i1}[p_2] &= E_{i1}\left[E_2[v] + \Lambda_2 E_2[\theta_2]\right] \qquad \text{(B.1)}\\
&= E_{i1}[E_2[v]] + \Lambda_2 E_{i1}[E_2[\theta_2]]\\
&= E_{i1}\left[\frac{\Delta a_2 \tau_u}{\tau_2} z_2 + \frac{\tau_1}{\tau_2} E_1[v]\right] + \Lambda_2 E_{i1}[a_2(v - E_2[v]) + \theta_2]\\
&= \frac{(\Delta a_2)^2 \tau_u}{\tau_2} E_{i1}[v] + \frac{\tau_1}{\tau_2} E_1[v] + \Lambda_2 a_2 (E_{i1}[v] - E_{i1}[E_2[v]]) + \beta \Lambda_2 E_{i1}[\theta_1]\\
&= \frac{(\Delta a_2)^2 \tau_u}{\tau_2} E_{i1}[v] + \frac{\tau_1}{\tau_2} E_1[v] + \Lambda_2 a_2 \left(E_{i1}[v] - \frac{(\Delta a_2)^2 \tau_u}{\tau_2} E_{i1}[v] - \frac{\tau_1}{\tau_2} E_1[v]\right)\\
&\quad + \beta \Lambda_2 (a_1 (v - E_{i1}[v]) + \theta_1).
\end{aligned}$$

If we collect the terms that multiply $E_{i1}[v]$ in the last row we have

$$\left(\frac{(\Delta a_2)^2 \tau_u}{\tau_2} + \Lambda_2 a_2 \frac{\tau_1}{\tau_2} - \beta \Lambda_2 a_1\right),$$

which implies that the weight an investor places on private information in the first period is given by

$$\underbrace{\frac{\gamma \alpha_{E_1}}{\mathrm{Var}_{i1}[p_2]} \times \frac{(\Delta a_2)^2 \tau_u}{\tau_2}}_{1} + \underbrace{\frac{\gamma \alpha_{E_1}}{\mathrm{Var}_{i1}[p_2]} \times \Lambda_2 a_2 \frac{\tau_1}{\tau_2}}_{2} + \underbrace{\left(-\frac{\gamma \alpha_{E_1}}{\mathrm{Var}_{i1}[p_2]} \times \beta \Lambda_2 a_1\right)}_{3},$$

where $\alpha_{E_1} = \tau_\epsilon / \tau_{i1}$ denotes the optimal statistical weight to private information in the first period. The above expression shows that an investor's response to private information can be decomposed in three terms:

- Term 1 captures the response to private signals that reflects the anticipated impact of the fundamentals information arriving at time 2 (i.e., how z_2 affects $E_2[v]$).
- Term 2 captures the response to private signals that reflects the anticipated impact that the innovation in liquidity trading has on the second period price (i.e., how u_2 affects $E_2[\theta_2]$; note that while at date 1 an investor cannot predict u_2, he can predict how the market in period 2 will react to u_2, since this is recorded by $E_2[\theta_2] = a_2(v - E_2[v]) + \theta_2$).
- Term 3 captures the response to private signals that reflects the anticipated impact of first period liquidity trading on the second period price (i.e., how θ_1 affects $E_2[\theta_2]$). This reflects investors' private learning channel from the first period price.

Now, the three terms above behave differently as a_1 increases depending on the value of β. For $\beta = 0$, term 3 disappears while term 1 decreases in a_1, reflecting the fact that the more aggressively investors respond to private information in the first period, the more the second and first period prices reflect the fundamentals. This, in turn, leads investors to lower their reliance on private information, consistently with the standard, strategic substitutability effect between private and public information of the Grossman-Stiglitz setup. Term 2, instead, increases in a_1, reflecting the fact that the more aggressively investors respond to private information, the closer is the second period price to the fundamentals, and the lower is the uncertainty faced by second period investors and the compensation they require for accommodating the demand of liquidity traders in the second period. This latter effect runs counter to the one coming from term 1 but is not strong enough to offset it. As a result, $\varphi'(a_1^*) < 0$, $a_1^* < \gamma\tau_\epsilon$ in this case (see Figure 9, parameters' values are the same of Figure 1, except for β which in this case is null).

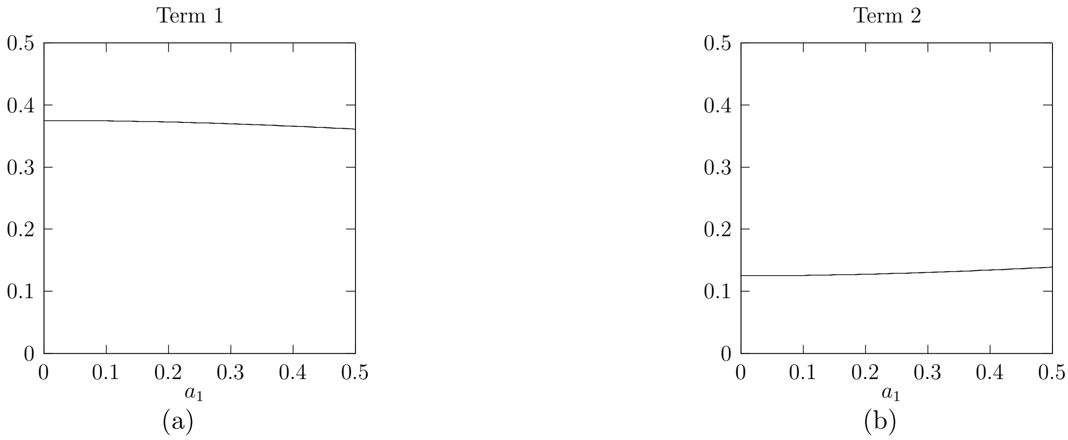

Figure 9: The figure plots the two terms that determine an investor's response to private information when $\beta = 0$. Parameter values are as in Figure 1, except for β that is set to 0.

When $\beta > 0$ the private learning channel from p_1 affects the response to private information in period 1. This, in turn, implies that positive selection can occur in the second period *and* that $\text{Var}_{i1}[p_2]$ can be null.

The implication of this latter effect is to create two regions in the space of solutions to the equation $a_1 = \varphi(a_1)$: $[0, \hat{a}_1)$ and $(\hat{a}_1, +\infty)$, where $\hat{a}_1$ is such that $\text{Var}_{i1}[p_2]$ is null for $a_1 = \hat{a}_1$. Points to the left of $\hat{a}_1$ yield the solution a_1^* and points to the right of $\hat{a}_1$ yield the solution a_1^{**}. We display the behavior of the three terms in Figure 2 (parameters' values are the same of Figure 1). For $a_1 < \hat{a}_1$ we have that term 1 is non monotone (first decreasing, then increasing) in a_1, term 2 is increasing in a_1, and term 3 is negative and decreasing in a_1. For $a_1 = \hat{a}_1$ all three terms diverge (the first and second to $+\infty$, the third to $-\infty$). For $a_1 > \hat{a}_1$ terms 1 and 2 are decreasing in a_1 while term 3 is increasing in a_1 (that is it grows towards 0).

C Appendix

In this appendix we provide the expression for the equilibrium price when liquidity traders' expected demand is non null. Thus, suppose that $u_n \sim N(\bar{u}, \tau_u^{-1})$, with $\bar{u} > 0$. Then, it is easy to that that $\{z_1, z_2\}$ is observationally equivalent to $\{p_1, p_2\}$, while

$$a_1^{-1}(z_1 - \bar{u}) \equiv v + a_1^{-1}(u_1 - \bar{u})|v \sim N(v, a_1^{-2}\tau_u^{-1})$$
$$(\Delta a_2)^{-1}(z_2 - \bar{u}) \equiv v + (\Delta a_2)^{-1}(u_2 - \bar{u})|v \sim N(v, (\Delta a_2)^{-2}\tau_u^{-1}).$$

Therefore, nothing changes for the precisions in the projection expressions while

$$\begin{aligned} E_{in}[v] &= \frac{\overbrace{\tau_v \bar{v} + \tau_u \sum_{t=1}^{n} \Delta a_t (z_t - \bar{u})}^{\tau_2 E_2[v]} + \tau_\epsilon s_{in}}{\tau_{in}} \\ &= \frac{\tau_2 E_2[v] + \tau_\epsilon s_{in}}{\tau_{in}}. \end{aligned}$$

As a result, everything works as in the model with $\bar{u} = 0$, except that now there is a premium above $\bar{v}$ in the expression for the expected price, that is

$$E[p_n] = \bar{v} + \Lambda_n E[\theta_n]. \tag{C.1}$$