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SUPERMODULARITY AND SUPERMODULAR GAMES

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Supermodularity and supermodular games

Xavier Vives

The concept of complementarity is well established in economics at least since Edgeworth. The basic idea of complementarity is that the marginal value of an action is increasing in the level of other actions available. The mathematical concept of supermodularity formalizes the idea of complementarity. The theory of monotone comparative statics and supermodular games provides the toolbox to deal with complementarities. This theory, developed by Topkis (1978, 1979), Vives (1985, 1990) and Milgrom and Roberts (1990a), in contrast to classical convex analysis, is based on order and monotonicity properties on lattices. See Topkis (1998), Vives (1999), and Vives (2005) for detailed accounts of the theory and applications. Monotone comparative statics analysis provides conditions under which optimal solutions to optimization problems change monotonically with a parameter. The theory of supermodular games exploits order properties to ensure that the best response of a player to the actions of rivals is increasing in their level. Indeed, this is the characteristic of games of strategic complementarities (the term was coined in Bulow et al. (1983)). The power of the approach is that it clarifies the drivers of comparative statics results and the need of regularity conditions; it allows very general strategy spaces, including indivisibilities and functional spaces such as those arising in dynamic or Bayesian games; it establishes the existence of equilibrium in pure strategies (without requiring quasiconcavity of payoffs, smoothness assumptions, or interior solutions); it allows a global analysis of the equilibrium set when there are multiple equilibria, which has an order structure with largest and smallest elements; and, finally, it finds that those extremal equilibria have strong stability properties and there is an algorithm to compute them.

We will provide first an introduction to the approach and some definitions; move on to the basic monotone comparative statics results; and provide the basic results for supermodular games.

Preliminaries and definitions. A binary relation $\geq$ on a nonempty set X is a *partial order* if $\geq$ is reflexive, transitive, and antisymmetric (a binary relation is antisymmetric if $x \geq y$ and $y \geq x$ implies that $x = y$). A partially ordered set $(S, \geq)$ is *completely ordered* if for x and y in S either $x \geq y$ or $y \geq x$. An upper bound on a subset $A \subset X$ is $z \in X$ such that $z \geq x$ for all $x \in A$. A greatest element of A is an element of A that is also an upper bound on A. Lower bounds and least elements are defined analogously. The greatest and least elements of A, when they exist, are denoted max A and min A, respectively. A *supremum* (resp., *infimum*) of A is a least upper bound (resp., greatest lower bound); it is denoted sup A (resp., inf A). A *lattice* is a partially ordered set $(X, \geq)$ in which any two elements have a supremum and an infimum. Any interval of the real line with the usual order is a lattice since any two points have a supremum and an infimum in the interval. However, the set in $\mathbb{R}^2$ $\{(1,0),(0,1)\}$ is not a lattice with the vector ordering (the usual component-wise ordering), since (1,0) and (0,1) have no joint upper bound in the set. However, if we add the points (0,0) and (1,1) the set becomes a lattice with the vector ordering (see Figure 1 and let x = (0,1) and y = (1,0)). A lattice $(X, \geq)$ is *complete* if every nonempty subset has a supremum and an infimum. Any compact interval of the real line with the usual order, or product of compact intervals with the vector order, is a complete lattice. Open intervals are lattices but they are not complete (e.g. the supremum of the interval (a, b) does not belong to (a, b)). A subset L of the lattice X is a *sublattice* of X if the supremum and infimum of any two elements of L belong also to L. A lattice is always a sublattice of itself, but a lattice need not be a sublattice of a larger lattice. Let $(X, \geq)$ and $(T, \geq)$ be partially ordered sets. A function $f: X \to T$ is *increasing* if, for x, y in X, $x \geq y$ implies that $f(x) \geq f(y)$.

Supermodular functions. A function $g: X \to \mathbb{R}$ on a lattice X is *supermodular* if, for all x, y in X, $g(\inf(x,y)) + g(\sup(x,y)) \geq g(x) + g(y)$. It is *strictly supermodular* if the inequality is strict for all pairs x, y in X that neither $x \geq y$ nor $y \geq x$ holds. A function f is (*strictly*) *submodular* if $-f$ is (strictly) supermodular; a function f is (*strictly*) *log-supermodular* if $\log f$ is (strictly) supermodular. Let X be a lattice and T a partially ordered set. The function $g: X \times T \to R$ has (*strictly*) *increasing differences* in (x, t) if

$g(x',t)-g(x,t)$ is (strictly) increasing in t for $x'>x$ or, equivalently, if $g(x,t')-g(x,t)$ is (strictly) increasing in x for $t'>t$. Decreasing differences are defined analogously.

Supermodularity is a stronger property than increasing differences: if T is also a lattice and if g is (strictly) supermodular on $X\times T$, then g has (strictly) increasing differences in (x,t). However, the two concepts coincide on the product of completely ordered sets: in such case a function is supermodular if and only if it has increasing differences in any pair of variables. Both concepts formalize the idea of complementarity: increasing one variable raises the return to increase another variable. For example, the Leontieff utility function $U(x)=\min\{a_1x_1,...,a_nx_n\}$ with $a_i\geq 0$ for all i is supermodular on $\mathbb{R}^n$. The complementarity idea can be made transparent by thinking of the rectangle in $\mathbb{R}^n$ with vertices {min(x,y), y, max(x,y), x} and rewriting the definition of supermodularity as g(max(x,y)) - g(x) ≥ g(y) - g(min(x,y)). Consider, for example, points in $\mathbb{R}^2$ $x = (x_1, x_2)$ and $y = (y_1, y_2)$ with the usual order. Then going from $\min(x, y) = (x_1, y_2)$ to y, for given y_2, increases the payoff less than going from x to $\max(x, y) = (y_1, x_2)$, for given $x_2 \geq y_2$ (See Figure 1).

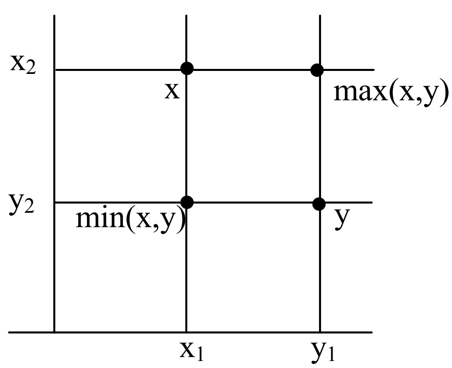


Figure 1

If X is a convex subset of $\mathbb{R}^n$ and if $g:X\to R$ is twice-continuously differentiable, then g has increasing differences in (x_i,x_j) if and only if $\partial^2 g(x)/\partial x_i\partial x_j\geq 0$ for all x and $i\neq j$. For decreasing differences (or submodularity) we would have $\partial^2 g(x)/\partial x_i\partial x_j\leq 0$. This characterization has a direct counterpart with the concept of (weak) cost complementarities if g is a cost function and $x \geq 0$ the production vector. If $\partial^2 g(x)/\partial x_i \partial x_j > 0$ for all x and i ≠ j, then g is strictly supermodular. The differential characterization of supermodularity can be motivated by the figure as before. As an example consider assortative matching when types x and y in $[0,1]$ produce f(x,y) when matched and nothing otherwise. If $\partial^2 f/\partial x \partial y > 0$ then in a core allocation matching is positively assortative, that is, matched partners are identical (Becker (1973), see Shimer and Smith (2000) for a dynamic model with search where it is required also that $\log \partial f/\partial x$ and $\log \partial^2 f/\partial x \partial y$ are supermodular).

Positive linear combinations and pointwise limits preserve the complementarity properties (supermodularity/increasing differences) of a family of functions $g_n : X \times T \rightarrow \mathbb{R}$. Supermodularity is also preserved under integration. This has important consequences for comparative statics under uncertainty and games of incomplete information (see Vives (1990) and Athey (2001)). Supermodularity is preserved as well under the maximization operation. Supermodularity is unrelated to convexity, concavity or returns to scale. Indeed, any real-valued function on a completely ordered set (say the reals) is both supermodular and submodular. This fact also makes clear that supermodularity in Euclidean spaces, in contrast to concavity or convexity, has no connection with continuity or differentiability properties. Note also that if g is a twice-continuously differentiable function, supermodularity only puts restrictions on the cross partials of g while the other concepts impose restrictions also on the diagonal of the matrix of second derivatives.

<u>Monotone comparative statics.</u> Let X be a compact rectangle in Euclidean space and let T be a partially ordered set. Let $g : X \times T \rightarrow \mathbb{R}$ be a function that (a) is supermodular and continuous on X for each $t \in T$ and (b) has increasing differences in (x,t). Let $\varphi(t) = \arg\max_{x \in X} g(x,t)$. Then (Topkis (1978)):

1) $\varphi(t)$ is a non-empty compact sublattice for all t;

2) φ is increasing in the sense that, for $t' > t$ and for $x' \in \varphi(t')$ and $x \in \varphi(t)$, we have $\sup(x', x) \in \varphi(t')$ and $\inf(x', x) \in \varphi(t)$; and

3) $t \mapsto \sup \varphi(t)$ and $t \mapsto \inf \varphi(t)$ are increasing selections of φ.

Several remarks are in order: (i) The continuity requirement of g can be relaxed. In more general spaces the requirement is for X to be a complete lattice and for g to fulfill an appropriate order continuity property. (ii) If g has strictly increasing differences in (x, t), then all selections of φ are increasing. (iii) If solutions are interior, and $\partial g / \partial x_i$ is strictly increasing in t for some i, then all selections of φ are strictly increasing (Edlin and Shannon (1998)). (iv) Milgrom and Shanon (1994) relax the complementarity conditions to ordinal complementarity conditions (quasisupermodularity and a single crossing property), and develop necessary and sufficient conditions for monotone comparative statics.

Let us illustrate the result when $T \subset \mathbb{R}$ and g is twice-continuously differentiable on $X \times T$. Suppose first that $X \subset \mathbb{R}$, *g* is strictly quasiconcave in x (with $\partial g / \partial x = 0$ implying that $\partial^2 g / (\partial x)^2 < 0$), and that the solution to the maximization problem $\varphi(t)$ is interior. Then, using the implicit function theorem on the interior solution, for which $\partial g(\varphi(t), t) / \partial x = 0$, we obtain that φ is continuously differentiable and $\varphi' = -\frac{\partial^2 g}{\partial x \partial t} / \frac{\partial^2 g}{(\partial x)^2}$. Obviously, sign φ' = sign $\frac{\partial^2 g}{\partial x \partial t}$. The solution is increasing (decreasing) in t if there are increasing, $\frac{\partial^2 g}{\partial x \partial t} \geq 0$, (decreasing, $\frac{\partial^2 g}{\partial x \partial t} \leq 0$) differences. The monotone comparative statics result asserts that the solution $\varphi(t)$ will be monotone increasing in t even if g is not strictly quasiconcave in x, in which case $\varphi(t)$ need not be a singleton or convex-valued, provided that $\frac{\partial^2 g}{\partial x \partial t}$ does not change sign. For example, consider a single-product monopolist with revenue function R(x) and cost function C(x, t), where x is the output of the firm and t a cost efficiency parameter. We have therefore, g(x, t) = R(x) - C(x, t). If $C(\cdot)$ is smooth and $\frac{\partial^2 C}{\partial x \partial t} \leq 0$, an increase in t reduces marginal costs. Then if $R(\cdot)$ is continuous the comparative static result applies, and the largest $\bar{\varphi}(t)$ and the smallest $\underline{\varphi}(t)$ monopoly outputs are increasing in t. If $\frac{\partial^2 C}{\partial x \partial t} < 0$, then all selections of the set of monopoly outputs are increasing in t. It is worth noting that the comparative statics result is obtained with no concavity assumption on the profit of the firm.

Suppose now that $X \subset \mathbb{R}^k$. If g is strictly concave in x (with the Jacobian of $\left(\frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_k}\right)$ with respect to x, H_X, negative definite) and the solution to the optimization problem $\varphi(t) = (\varphi_1(t), \ldots, \varphi_k(t))$ is interior, then $\varphi(t)$ is continuously differentiable, and $\left(\frac{\partial \varphi_1}{\partial t}, \ldots, \frac{\partial \varphi_k}{\partial t}\right) = -H_x^{-1}\left(\frac{\partial^2 g}{\partial x_1 \partial t}, \ldots, \frac{\partial^2 g}{\partial x_k \partial t}\right)$. If the off-diagonal elements of H_X are nonnegative $\frac{\partial^2 g}{\partial x_i \partial x_j} \geq 0, j \neq i$, then all the elements of $-H_x^{-1}$ are nonnegative and the diagonal elements are positive (McKenzie (1959)). A sufficient condition for $\frac{\partial \varphi_i}{\partial t} \geq 0$ for all i is that $\frac{\partial^2 g}{\partial x_i \partial t} \geq 0$ for all i (the statement also holds with strict inequalities). As before, even if H_X is not negative definite, the assumptions that $\frac{\partial^2 g}{\partial x_i \partial x_j} \geq 0, j \neq i$, and that $\frac{\partial^2 g}{\partial x_i \partial t} \geq 0$ imply that the solution set $\varphi(t)$ has the monotonicity properties stated in the monotone comparative statics result. Note that when X is multidimensional, the restriction that g be supermodular on X, ensuring that for any components i and j an increase in the variable x_j raises the marginal return of variable x_i, when coupled with increasing differences on $X \times T$ is needed to guarantee the monotonicity of the solution For example, consider a multiproduct monopolist. If the revenue function $R(\cdot)$ is continuous and supermodular on X, the cost function $C(\cdot)$ continuous and submodular on X, and $C(\cdot)$ displays decreasing differences in (x, t), the comparative static result follows. That is, the largest $\bar{\varphi}(t)$ and the smallest $\underline{\varphi}(t)$ monopoly output vectors are increasing in t. In the differentiable case, $\frac{\partial^2 R}{\partial x_i \partial x_j} \geq 0$ and $\frac{\partial^2 C}{\partial x_i \partial x_j} \leq 0$, for all i ≠j, and $\frac{\partial^2 C}{\partial x_i \partial t} \leq 0$ for all i. The result hinges on revenue and cost complementarities among outputs, and the impact of the efficiency parameter on marginal costs, and not concavity of profits.

Consider a team problem. Suppose that n persons share a common objective $g(x_1,..., x_n, t)$ where the action of player i x_i is in the rectangle $X_i \subset \mathbb{R}^{k_i}$ for each i and t is a payoff relevant parameter. If g is supermodular on $X = \prod_{i=1}^{n} X_i$ and has strictly increasing differences in (x, t), then any optimal solution is increasing in the level of the parameter. For example, the optimal production g(x, t) of the firm (seen as a team problem) is increasing in the level of information technology t (that raises the marginal productivity of any worker of the firm).

Supermodular games. Consider the game (A_i, π_i; $i \in N$) where for each i = 1, …, n in the set of players N, A_i is the strategy set, a subset of Euclidean space, and π_i the payoff of the player (defined on the cross product of the strategy spaces of the players A). Let $a_i \in A_i$ and $a_{-i} \in \prod_{j \neq i} A_j$ (i.e. denote by a_{-i} the strategy profile (a_1, …, a_n) except the ith element). The strategy profiles are endowed with the usual component-wise order. We will say that the game (A_i, π_i; $i \in N$) is (*strictly*) *supermodular* if for each i, A_i is a compact rectangle of Euclidean space, π_i is continuous and (i) supermodular in a_i for fixed a_{-i} and (ii) displays (strictly) increasing differences in (a_i, a_{-i}). We will say that the game (A_i, π_i; $i \in N$) is *smooth* (*strictly*) *supermodular* if furthermore π_i (a_i, a_{-i}) is twice continuously differentiable with (i) $\partial^2\pi_i/\partial a_{ih}\partial a_{ik} \geq 0$ for all $k \neq h$, and (ii) $\partial^2\pi_i/\partial a_{ih}\partial a_{jk} \geq (>) 0$ for all $j \neq i$ and for all h and k, where a_{ih} denotes the hth component of the strategy a_i of player i. Condition (i) is the strategic complementarity property in own strategies a_i. Condition (ii) is the strategic complementarity property in rivals' strategies a_{-i} .

In a more general formulation strategy spaces need only be complete lattices and this includes functional spaces such as those arising in dynamic or incomplete information games. The complementarity conditions can be weakened to define an “ordinal supermodular” game (see Milgrom and Shannon (1994)). Furthermore, the application of the theory can be extended by considering increasing transformations of the payoff (which do not change the equilibrium set of the game). For example, we will say that the game is log-supermodular if π_i is nonnegative and log π_i fulfils conditions (i) and (ii). This is the case of a Bertrand oligopoly with differentiated substitutable products, where each firm produces a different variety and marginal costs are constant, whenever the own-price elasticity of demand for firm i is decreasing in the prices of rivals, as with constant elasticity, logit, or constant expenditure demand systems.

In the duopoly case (n = 2) the case of strategic substitutability can also be covered. Indeed, suppose that there is strategic complementarity or supermodularity, in own strategies ($\partial^2\pi_i / \partial a_{ih} \partial a_{ik} \geq 0$ for all $k \neq h$, in the smooth version) and strategic substitutability in rivals' strategies or decreasing differences in (a_i, a_{-i}) ($\partial^2\pi_i / \partial a_{ih} \partial a_{jk} \leq 0$ for all $j \neq i$ and for all h and k, in the smooth version). Then the game obtained by reversing the order in the strategy space of one of the players, say player 2, is supermodular (Vives (1990)). Cournot competition with substitutable products displays typically strategic substitutability between the (output) strategies of the firms.

In a supermodular game best responses are monotone increasing even when π_i is not quasiconcave in a_i. Indeed, in a supermodular game each player has a largest, $\overline{\Psi}_i(a_{-i}) = \sup \Psi_i(a_{-i})$, and a smallest, $\underline{\Psi}_i(a_{-i}) = \inf \Psi_i(a_{-i})$, best reply, and they are increasing in the strategies of the other players. Let $\overline{\Psi} = (\overline{\Psi}_1, \ldots, \overline{\Psi}_n)$ and $\underline{\Psi} = (\underline{\Psi}_1, \ldots, \underline{\Psi}_n)$ denote the extremal best reply maps.

<u>Result 1</u>. In a supermodular game there always *exist* a largest $\bar{a} = \sup \{a \in A: \overline{\Psi}(a) \geq a\}$ and a smallest $\underline{a} = \inf \{a \in A: \underline{\Psi}(a) \leq a\}$ equilibrium. (Topkis (1979).)

The result is shown applying Tarski's fixed point theorem to the extremal selections of the best-reply map, $\overline{\Psi}$ and $\underline{\Psi}$, which are monotone because of the strategic complementarity assumptions. Tarski's theorem (1955) states that if A is a complete lattice (e.g. a compact rectangle in Euclidean space) and $f : A \rightarrow A$ an increasing function then f has a largest $\sup\{a \in A : f(a) \geq a\}$ and a smallest $\inf\{a \in A : a \geq f(a)\}$ fixed point. There is no reliance on quasiconcave payoffs and convex strategy sets to deliver convex-valued best replies as required when showing existence using Kakutani's fixed point theorem. The equilibrium set can be shown also to be a complete lattice (Vives (1990), Zhou (1994)).The result proves useful in a variety of circumstances to get around the existence problem highlighted by Roberts and Sonnenschein (1976). This is the case, for example, of the (log–)supermodular Bertrand oligopoly with differentiated substitutable products.

Result 2. In a *symmetric* supermodular game (i.e. a game with payoffs and strategy sets exchangeable against permutations of the players) the extremal equilibria $\overline{a}$ and $\underline{a}$ are symmetric and, if strategy spaces are completely ordered and the game is strictly supermodular, then *only* symmetric equilibria exist. (See Vives (1999).)

The result is useful to show uniqueness since if there is a unique symmetric equilibrium then the equilibrium is unique. For example, in a symmetric version of the Bertrand oligopoly model with constant elasticity of demand and constant marginal costs, it is easy to check that there exists a unique symmetric equilibrium. Since the game is (strictly) log-supermodular, we can conclude that the equilibrium is unique. The existence result of symmetric equilibria is related to the classical results of McManus (1962, 1964) and Roberts and Sonnenschein (1976).

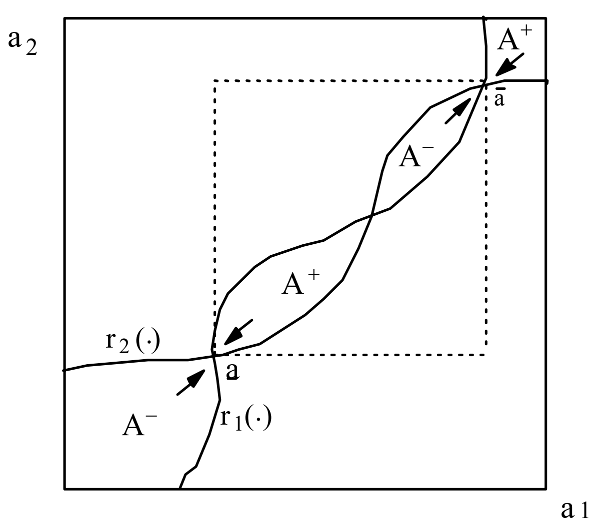
Result 3. In a supermodular game if there are positive spillovers (i.e. the payoff to a player is increasing in the strategies of the other players) then the largest (smallest) equilibrium point is the Pareto best (worst) equilibrium. (Milgrom and Roberts (1990a), Vives (1990).)

Indeed, in many games with strategic complementarities equilibria can be Pareto ranked. In the Bertrand oligopoly example the equilibrium with higher prices is Pareto dominant for the firms. This has proved particularly useful in applications in macroeconomics (e.g. Cooper and John (1988)) and finance (e.g. Diamond and Dybvig (1983)).

<u>Result 4.</u> In a supermodular game:

(i) Best-reply dynamics approach the interval $[\underline{a}, \bar{a}]$ defined by the smallest and the largest equilibrium points of the game. Therefore, if the equilibrium is unique it is globally stable. Starting at any point A^+ (A^-) in the intersection of the upper (lower) contour sets of the largest (smallest) best replies of the players, best-reply dynamics lead monotonically downwards (upwards) to an equilibrium (Vives (1990, 1999)). This provides an iterative procedure to find the largest (smallest) equilibrium (Topkis (1979) starting at sup A (inf A). (See Figure 2.)

(ii) The extremal equilibria correspond to the largest and smallest serially undominated strategies (Milgrom and Roberts (1990a)). Therefore, if the equilibrium is unique, the game is dominance solvable. Rationalizable (Bernheim (1984), Pearce (1984)) or mixed strategy outcomes must lie in the interval $[\underline{a}, \bar{a}]$.



Best reply dynamics in a supermodular game
(with best reply functions $r_1(\cdot), r_2(\cdot)$)

Figure 2

In the Bertrand oligopoly example with linear, constant elasticity, or logit demands, the equilibrium is unique and therefore it is globally stable, and the game is dominance solvable. In the team example it is clear that an optimal solution will be a Nash equilibrium of the game among team members. If the equilibrium is unique then best reply dynamics among team members will converge to the optimal solution. This need not be the case if there are multiple equilibria. (See Milgrom and Roberts (1990b) for an application to the theory of the firm.)

Result 5. Consider a supermodular game with parameterized payoffs $\pi_i (a_i, a_{-i}; t)$ with t in a partially ordered set T. If $\pi_i (a_i, a_{-i}; t)$ has increasing differences in (a_i, t) (in the smooth version $\partial^2\pi_i / \partial a_{ih}\partial t \geq 0$ for all h and i), then the largest and smallest equilibrium points increase with an increase in t, and starting from any equilibrium, best reply dynamics lead to a (weakly) larger equilibrium following the parameter change. The latter result can be extended to adaptive dynamics, which include fictitious play and gradient dynamics. (See Lippman et al. (1987), and Sobel (1988) for early results; and Milgrom and Roberts (1990a), Milgrom and Shannon (1994), and Vives (1999) for extensions.) It is worth noting that continuous equilibrium selections that do not increase monotonically with t predict unstable equilibria (Echenique (2002)). The result yields immediately that an increase in an excise tax in a (log-)supermodular Bertrand oligopoly raises prices at an extremal equilibrium.

The basic intuition for the comparative statics result is that an increase in the parameter increases the actions for one player, for given actions or rivals, and this reinforces the desire of all other players to increase their actions because of strategic complementarity. This initiates a mutually reinforcing process that leads to larger equilibrium actions. This is a typical positive feedback in games of strategic complementarities. In this class of games, unambiguous monotone comparative statics obtain if we concentrate on stable equilibria. We can understand this as a multidimensional version of Samuelson's (1947) Correspondence Principle, which was obtained with standard calculus methods applied to interior and stable one-dimensional models.

A patent race. Consider n firms engaged in a memoryless patent race that have access to the same R&D technology. The winner of the patent obtains the prize V and losers obtain nothing. The (instantaneous) probability of innovating is given by $h(x)$ if a firm spends x continuously, where h is a smooth function with $h(0)=0$, $h'>0$, $\lim_{x\to\infty} h'(x)=0$, and $h'(0)=\infty$. It is assumed also that h is concave but a region of increasing returns for small x may be allowed. If no patent is obtained the (normalized) profit of a firm is zero. The expected discounted profits (at rate r) of firm i investing x_i if rival $j \neq i$ invests x_j is given by

$$\pi_i = \frac{h(x_i)V - x_i}{h(x_i) + \Sigma_{j\neq i} h(x_j) + r}.$$

Lee and Wilde (1980) restrict attention to symmetric Nash equilibria of the game and show that, under a uniqueness and stability condition at a symmetric equilibrium x^* expenditure intensity increases with n. The classical approach requires assumptions to ensure a unique and stable symmetric equilibrium and cannot rule out the existence of asymmetric equilibria. Suppose that there are potentially multiple symmetric equilibria and that going from n to n+1 new equilibria appear. What comparative static result can we infer then? Using the lattice approach we obtain a more general comparative statics result that allows for the presence of multiple symmetric equilibria (Vives (1999, Exercise 2.20, and 2005, Section 5.2). Let $h(0)=0$ with h strictly increasing in $\left[0,\overline{x}\right]$, with $h(x)V - x < 0$ for $x \geq \overline{x} > 0$. Under the assumptions the game is strictly log-supermodular and from Result 2 only symmetric equilibria exist. Let $x_i = x$ and $x_j = y$ for $j \neq i$. Then $\log \pi_i$ has (strictly) increasing differences in (x, n) for all y $(y>0)$, and, according to Result 5, the expenditure intensity x^* at extremal equilibria is increasing in n. Furthermore, starting at any equilibrium, an increase in n will raise research expenditure with out-of-equilibrium adjustment according to best-reply dynamics. This will be so even if new equilibria appear, or some disappear, as a result of increasing n. Finally, if h is smooth with $h' > 0$ and $h'(0) = \infty$, then $\partial \log \pi_i / \partial x_i$ is strictly increasing in n and (at extremal equilibria) x^* is *strictly* increasing in n. This follows because, under our assumptions, equilibria are interior and must fulfill the first-order conditions.

The results can be applied to dynamic and incomplete information games, which have complex strategy spaces. For example, in an incomplete information game, if for given types of players the ex post game is supermodular, then the Bayesian game is also supermodular and therefore there exist Bayesian equilibria in pure strategies (Vives (1990)). If, furthermore, payoffs to any player have increasing differences between the actions of the player and types, and higher types believe that other players are also of a higher type (according to first-order stochastic dominance), then extremal equilibria of the Bayesian game are monotone increasing in types (Van Zandt and Vives (2006)). This defines a class of monotone supermodular games. An example is provided by global games (introduced by Carlsson and Van Damme (1993) and developed by Morris and Shin (2002) and others with the aim of equilibrium selection. Global games are games of incomplete information with type space determined by each player observing a noisy private signal of the underlying state. The result is obtained applying iterated elimination of strictly dominated strategies. From the perspective of monotone supermodular games we know that extremal equilibria are the outcome of iterated elimination of strictly dominated strategies, that they are monotone in type (and therefore in binary action games there is no loss of generality in restricting attention to threshold strategies), and the conditions put to pin down a unique equilibrium in the global game amount to a lessening of the strength of strategic complementarities (see Section 7.2 in Vives (2005)).

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