Market Power and Welfare in Asymmetric Divisible Good Auctions*

Carolina Manzano          Xavier Vives
Universitat Rovira i Virgili and CREIP     IESE Business School

January 31, 2020

Abstract

We analyze a divisible good uniform-price auction that features two groups, each with a finite number of identical bidders, who compete in demand schedules. In the linear-quadratic-normal framework, it presents conditions under which the unique equilibrium in linear demands exists and derives novel comparative statics results that highlight the interaction between payoff and information parameters with asymmetric groups. We find that the strategic complementarity in the slopes of traders’ demands is reinforced by inference effects from prices, and display the role of payoff and information asymmetries in explaining deadweight losses. Furthermore, price impact and the deadweight loss may be negatively associated and market integration may reduce welfare. The results are consistent with the available empirical evidence.

KEYWORDS: demand/supply schedule competition, private information, liquidity auctions, Treasury auctions, electricity auctions, market integration.

JEL: D44, D82, G14, E58

*For helpful comments we are grateful to Roberto Burguet, Maryam Farboodi, Vitali Gretschko, Jakub Kastl, Leslie Marx, Meg Meyer, Antonio Miralles, Stephen Morris, Andrea Prat, and Tomasz Sadzik as well as seminar participants at the BGSE Summer Forum, Chicago, Columbia, Como Information Economics Workshop, EARIE, ESSET, Federal Reserve Board, Jornadas de Economía Industrial, Northwestern, Princeton, UPF, Queen Mary Theory Workshop, Stanford, UC Berkeley, UC San Diego, and Wharton. We are also indebted to Jorge Paz for excellent research assistance. Manzano acknowledges the financial support from the Ministry of Economy and Competitiveness ECO 2016-75410-P (AEI/FEDER, UE), and the Generalitat de Catalunya, AGAUR grant 2017SGR770. Vives acknowledges the financial support from the Ministry of Science, Innovation and Universities PGC2018-096325-B-I00 (MCIU/AEI/FEDER, UE) and the Generalitat de Catalunya, AGAUR grants 2014SGR1496 and 2017SGR1244.
1 Introduction

Divisible good auctions are common in many markets, including government bonds, liquidity (refinancing operations), electricity, and emission markets. In those auctions, both market power (price impact) and asymmetries among the participants are important; asymmetries can make price impact relevant even in large markets. However, theoretical work in this area has been hampered by the difficulties of dealing with bidders that are asymmetric, have market power, and are competing in terms of demand or supply schedules in the presence of private information. This paper helps to fill that research gap by analyzing uniform-price auctions in which there are two asymmetric groups of bidders with interdependent values. Our aims are to characterize the equilibrium, derive novel comparative statics results that highlight the interaction between payoff and information parameters with asymmetric agents, perform a welfare analysis (from the standpoint of revenue and deadweight loss), and finally draw implications for policy.

Divisible good auctions are typically populated by heterogeneous participants in a concentrated market, and often we can distinguish a core group of bidders together with a fringe. The former are strong in the sense that they have better information, endure lower transaction costs, and are more oligopolistic (or oligopsonistic) than members of the fringe. As examples, we discuss Treasury and liquidity auctions, in addition to wholesale electricity markets. Uniform-price auctions are often used in such auctions.

Treasury auctions have bidders with significant market shares. That will be the case in most systems featuring a primary dealership, where participation is limited to a fixed number of bidders (this occurs, for example, in 29 out of 39 countries surveyed by Arnone and Iden 2003). Many papers report asymmetries between bidders in Treasury auctions. In particular, primary dealers enjoy an information advantage because they aggregate the information of indirect

---

1See Lopomo et al. (2011) for examples of such auctions.

2A difficulty found in indivisible good auctions is that closed form solutions for the equilibrium in asymmetric auctions are typically not available (even with independent private values). Consequently, the analysis of this type of auctions has posed many challenges and progress in this area has been sporadic (see, e.g., Hafalir and Krishna 2008 and Hubbard and Kirkegaard 2015).

3In US Treasury auctions, which are uniform-price auctions since 1998, the top five bidders typically purchase close to half of US Treasury issues (see Malvey and Archibald 1998). Primary dealers underwent a substantial reduction going from 46 in 1998 to 23 presently. Those account for a very substantial portion of volume (from 69% to 88% of tendered quantities in the sample of Hortaçsu et al. (2018) for the years 2009-2013). Indirect bidders place their bids through the primary dealers and other direct bidders tender from 6 to 13%.

bidders and face lower transaction costs.\textsuperscript{5} According to Hortaçsu et al. (2018), primary dealers systematically bid lower prices than the other participants in the auction, not because they have a lower valuation of the securities, but because they exercise market power.\textsuperscript{6} There is evidence that market participants in these auctions can be divided into two distinct groups, which differ in terms of transaction costs and quality of information, and that there is bid synchronization among bidders of a certain group.\textsuperscript{7}

Bindseil et al. (2009) and Cassola et al. (2013) find that the heterogeneity of bidders in liquidity auctions is relevant. Cassola et al. (2013) analyze the evolution of bidding data from the European Central Bank’s weekly refinancing operations before and during the early part of the financial crisis. The authors show that the effects of the 2007 subprime market crisis were heterogeneous among European banks, and they conclude that the significant shift in bidding behavior after 9 August 2007 may reflect a change in the cost of short-term funding on the interbank market and/or a strategic response to other bidders. In particular, the authors find that one third of bidders experienced no change in their costs of short-term funds from alternative sources; this means that their altered bidding behavior was mainly strategic: bids were increased as a response to the higher bids of rivals.\textsuperscript{8}

Concentration is also high in markets such as wholesale electricity. A number of empirical studies have concluded that sellers have exercised significant market power in wholesale electricity markets.\textsuperscript{9} Most wholesale electricity markets prefer using a uniform-price auction to using a pay-as-bid auction (Cramton and Stoft 2006, 2007). In several of these markets (e.g., California, Australia), generating companies bid to sell power and wholesale customers (sometimes the same in integrated companies) bid to buy power. In such markets, asymmetries are also prevalent. For example, some generators of wholesale electricity rely heavily on nuclear technology, which has flat marginal costs, whereas others rely mostly on fuel technologies, which have

\textsuperscript{5}For evidence from Canadian Treasury auctions, see Hortaçsu and Kastl (2012); for a theoretical model see Boyarchenko et al. (2015).

\textsuperscript{6}Experimental work has found also substantial demand reduction in uniform-price auctions (see e.g. Kagel and Levin 2001; Engelbrecht-Wiggans et al. 2006).


\textsuperscript{8}Bidder asymmetry has also been found in procurement markets, including school milk (Porter and Zona 1999; Pesendorfer 2000) and public works (Bajari 1998).

\textsuperscript{9}See, e.g., Green and Newbery (1992), Wolfram (1998), Borenstein et al. (2002), and Joskow and Kahn (2002).
steep marginal costs. Holmberg and Wolak (2015) argue that, in wholesale electricity markets, information on suppliers’ production costs is asymmetric. For example, a source of incomplete information about costs are plant outages, while oil price shocks provide a common component of cost uncertainty.\footnote{For evidence on the effect of cost heterogeneity on bidding in wholesale electricity markets, see Crawford et al. (2007) and Bustos-Salvagno (2015).}

Our paper makes progress within the linear-Gaussian family of models by incorporating bidders’ asymmetries with regard to payoffs and information. We model a uniform-price auction where asymmetric strategic bidders compete in terms of demand schedules for an inelastic supply (we can easily accommodate supply schedule competition for an inelastic demand as well as a double auction). We consider a model in which the equilibrium is privately revealing, that is, where the signal received by a trader and the price are a sufficient statistic for the trader. This allows us to focus the analysis on the inefficiencies derived from private information and market power, with no information externality present. Our modelling allows us to disentangle the price impact from the inference effects of traders, who have market power and private information, and that use price-contingent strategies.

Bidders may differ in their valuations, transaction costs, and/or the precision of their private information. With an empirical basis, we reduce heterogeneity to two groups; within each group, agents are identical and receive the same signal.\footnote{In case one group does not receive a signal we reproduce the information structure in Grossman and Stiglitz (1980) of uninformed and informed traders.} This information structure is consistent with the abovementioned empirical evidence in Hamao and Jegadeesh (1998) and Cao and Lu (2004), which tends to suggest the presence of a group with very correlated signals and high precision, and another with low correlation and poor or uninformative signals. We seek to identify the conditions under which there exists a linear equilibrium with symmetric treatment of agents in the same group (i.e., we are looking for equilibria such that demand functions are both linear and identical among individuals of the same type). After showing that any such equilibrium must be unique, we derive comparative statics results.

We identify two basic forces that drive the comparative statics of a parameter change: a basic strategic effect of strategic complementarity in the slopes of demands submitted by traders, which is present with complete information (e.g., Back and Zender 1993) and an inference effect, when there is incomplete information and learning from the price, which tends to reinforce the first effect. Our contribution is to characterize novel comparative statics across groups and identify the co-movements of payoff and information parameters (e.g., in a crisis situation) that magnify the impact of parameter changes.

More specifically, our analysis establishes that the number of group members, the transac-
tion costs, the extent to which bidders’ valuations are correlated, and the precision of private information affect the sensitivity of traders’ demands to private information and prices. For example, we find that a decrease in the expected valuation of one group raises the demand of the other group or that when valuations are more correlated, all groups react less to the private signal and to the price. Furthermore, if the transaction costs or the noise in the signal of a group increase, then the traders of the other group respond by diminishing their reaction to private information and submitting steeper demand schedules.\textsuperscript{12} This result is consistent with the empirical findings of Cassola et al. (2013) in European post-crisis liquidity auctions. It is key that increases in transaction costs, correlation of values, and noise in the signals, all descriptive of a crisis situation, impinge in the direction of steepening demand schedules and illiquidity. We also find that if a group of traders is stronger in the sense that its private information is more precise, its transaction costs are lower, and it is more oligopolistic, then the members of that group react more (than the bidders of the other group) to the private signal and to the price, and have more price impact. This result is consistent with the evidence in Hortaçsu et al. (2018) that primary dealers exercise market power.

When there is both a small and a large group of bidders, then the former (oligopsonistic) group has more price impact and yet, even the latter (large) group does not behave competitively, since it retains some price impact due to incomplete information, whenever there is learning from prices. However, the equilibrium under imperfect competition converges to a price-taking equilibrium in the limit as the number of traders of both groups becomes large.

Finally, we provide a welfare analysis. First, we calculate the optimal quantity to offer for the auctioneer who wants to maximize revenue and bound the expected revenue of the auction between the revenues of auctions involving extremal yet symmetric groups. Second, we characterize the deadweight loss at the equilibrium and show how a subsidy scheme may induce an efficient allocation. We find that if one group is stronger (as previously defined), then it should garner a higher per capita subsidy rate; the reason is that traders in the stronger group will behave more strategically and so must be compensated more to become competitive. The paper also underscores how bidder heterogeneity (in terms of information, preferences, or group size documented in previous work) may increase deadweight losses. In particular, when the strong group values the asset at least as much as the weak group, the deadweight loss increases with the quantity auctioned and also with the degree of payoff asymmetries. We also find that price impact need not be positively correlated with deadweight losses under asymmetry as is usually implicitly assumed in applied work. Furthermore, we provide conditions under which market integration increases or decreases welfare. Market integration is always welfare

\textsuperscript{12}A "steeper demand schedule" should be interpreted, as usual in economics, as a higher slope of inverse demand.
improving if bidders behave competitively or if the bidder groups are symmetric. However, the result may not hold if bidders have market power, the amount auctioned is large, and the groups are asymmetric. In such a case, gains from trade of integration may be overwhelmed by the inefficiency generated by group asymmetries and price impact.

Our work is related to the literature on divisible good auctions. Results in symmetric pure common value models have been obtained by Wilson (1979), Back and Zender (1993), and Wang and Zender (2002), among others. Kastl (2011) extends the Wilson model to consider discrete bids in an independent values context. This model is extended in Hortaçsu and Kastl (2012) and Hortaçsu et al. (2018).

Results in interdependent values models with symmetric bidders are obtained by Vives (2011, 2014) and Ausubel et al. (2014), for example. Vives (2011), while focusing on the tractable family of linear-Gaussian models, shows how increased correlation in traders’ valuations increases the price impact of those traders. Bergemann et al. (2015) generalize the information structure in Vives (2011) while retaining the assumption of symmetry. Rostek and Weretka (2012, 2015) partially relax that assumption and replace it with a weaker “equicommonality” assumption on the matrix correlation among the agents’ values. Du and Zhu (2017a) consider a dynamic auction model with ex post equilibria. For the case of complete information, progress has been made in divisible good auction models by characterizing linear supply function equilibria (e.g., Klemperer and Meyer 1989; Akgün 2004; Anderson and Hu 2008). Kyle (1989) incorporates incomplete information considering a Gaussian model of a divisible good double auction in which some bidders are privately informed and others are uninformed. Andreyanov and Sadzik (2017) study the design of robust exchange mechanisms in a two-type model similar to the one we present here.

To sum up, the two closest papers to ours are Vives (2011) and Kyle (1989). The novelty

---

13Wilson (1979) compares a uniform-price auction for a divisible good with an auction in which the good is treated as an indivisible good; he finds that the price can be significantly lower if bidders are allowed to submit bid schedules rather than a single bid price. That work is extended by Back and Zender (1993), who compare a uniform-price auction with a discriminatory auction. These authors demonstrate the existence of equilibria in which the seller’s revenue in a uniform-price auction can be much lower than the revenue obtained in a discriminatory auction. According to Wang and Zender (2002), if supply is uncertain and bidders are risk averse, then there may exist equilibria of a uniform-price auction that yield higher expected revenue than that from a discriminatory auction.

14This assumption states that the sum of correlations in each column of this matrix (or, equivalently, in each row) is the same and that the variances of all traders’ values are also the same. Unlike our model, Rostek and Weretka’s (2012) model maintains the symmetry assumption as regards transaction costs and the precision of private signals. The equilibrium they derive is therefore still symmetric because all traders use identical strategies.

of our paper with respect to Vives (2011) is that in our model we allow asymmetries among bidders, and with respect to Kyle (1989), that we consider interdependent values instead of a common value setup with non-optimizing liquidity traders in a double auction.

Despite the importance of bidder asymmetry, results in multi-unit auctions have been difficult to obtain. As a consequence, most papers that deal with this issue focus on auctions for a single item. In sealed-bid, first-price, single-unit auctions, an equilibrium exists under quite general conditions (Lebrun 1996; Maskin and Riley 2000a; Athey 2001; Reny and Zamir 2004). Uniqueness is explored in Lebrun (1999), and Maskin and Riley (2003). Maskin and Riley (2000b) study asymmetric auctions, and Cantillon (2008) shows that the seller’s expected revenue declines as bidders become less symmetric. On the multi-unit auction front, progress in establishing the existence of monotone equilibria has been made by McAdams (2003, 2006); those papers address uniform-price auctions characterized by multi-unit demand, interdependent values and independent types.\(^\text{16}\) Reny (2011) stipulates more general existence conditions that allow for infinite-dimensional type and action spaces; these conditions apply to uniform-price, multi-unit auctions with weakly risk-averse bidders and interdependent values (and where bids are restricted to a finite grid).

The rest of our paper is organized as follows. Section 2 outlines the model. Section 3 characterizes the equilibrium, analyzes its existence and uniqueness, and derives comparative statics results. We develop the welfare analysis in Section 4 and address the case of an oligopsony with a competitive fringe in Section 5. Section 6 concludes. Proofs are gathered in Appendix A and supplementary material in Appendix B.

2 The model

Traders, of whom there are a finite number, face an inelastic supply for a risky asset. Let \( Q \) denote the aggregate quantity supplied in the market. In this market, there are buyers of two types: type 1 and type 2. Suppose that there are \( n_i \) traders of type \( i, i = 1, 2 \). In that case, if the asset’s price is \( p \), then the profits of a representative type-\( i \) trader who buys \( x_i \) units of the asset are given by

\[
\pi_i = (\theta_i - p) x_i - \lambda_i x_i^2 / 2, \quad x_i \in \mathbb{R}.
\]

So, for any trader of type \( i \), the marginal benefit of buying \( x_i \) units of the asset is \( \theta_i - \lambda_i x_i \), where \( \theta_i \) denotes the valuation of the asset and \( \lambda_i > 0 \) reflects an adjustment for transaction

\(^{16}\text{McAdams (2006) uses a discrete bid space and atomless types to show that, with risk neutral bidders, monotone equilibria exist. His demonstration is based on checking that the single-crossing condition used by Athey (2001) for the single-object case extends to multi-unit auctions.}\)
costs or opportunity costs (or a proxy for risk aversion). Traders maximize expected profits and submit demand schedules, after which an auctioneer selects a price that clears the market.\footnote{The case of supply schedule competition for inelastic demand is easily accommodated by considering negative demands ($x_i < 0$) and a negative inelastic supply ($Q < 0$). In this case, a producer of type $i$ has a quadratic production cost $-\theta_i x_i + \lambda_i x_i^2$.}

We assume that $\theta_i$ is normally distributed with mean $\bar{\theta}_i$ and variance $\sigma^2_{\theta_i}$, $i = 1, 2$. The random variables $\theta_1$ and $\theta_2$ may be correlated, with correlation coefficient $\rho \in [0, 1]$. Therefore, $\text{cov}(\theta_1, \theta_2) = \rho \sigma^2_{\theta_i}$.\footnote{The value of $\rho$ will depend of the type of security. In this sense, Bindseil et al. (2009) argue that the common value component is less important in a central bank repo auction than in a T-bill auction.} All type-$i$ traders receive the same noisy signal $s_i = \theta_i + \varepsilon_i$, where $\varepsilon_i$ is normally distributed with null mean and variance $\sigma^2_{\varepsilon_i}$. Error terms in the signals are uncorrelated across groups ($\text{cov}(\varepsilon_1, \varepsilon_2) = 0$) and are also uncorrelated with valuations of the asset ($\text{cov}(\varepsilon_i, \theta_j) = 0$, $i, j = 1, 2$). In what follows, let $\tilde{\sigma}^2_{\varepsilon_i} \equiv \sigma^2_{\varepsilon_i}/\sigma^2_{\theta_i}$, $i = 1, 2$.

In our model, two traders of distinct types may differ in several respects:

- different willingness to possess the asset ($\theta_1 \neq \theta_2$),
- different transaction costs ($\lambda_1 \neq \lambda_2$), or
- different levels of precision of private information ($\sigma^2_{\varepsilon_1} \neq \sigma^2_{\varepsilon_2}$).

Applications of this model are Treasury auctions and liquidity auctions. For Treasury auctions, $\theta_i$ is the private value of the securities to a bidder of type $i$; that value incorporates not only the resale value but also idiosyncratic elements as different liquidity or portfolio immunization needs of bidders in the two groups. Financial intermediaries may assign different values to the Treasury instruments according to their use as collateral. In particular, primary dealers may attach a value to a bond beyond resale value to be used as collateral in operations with the Fed. For liquidity auctions, $\theta_i$ is the price (or interest rate) that group $i$ commands in the secondary interbank market (which is over-the-counter). Here $\lambda_i$ reflects the structure of a counterparty’s pool of collateral in a repo auction. A bidder bank prefers to offer illiquid collateral to the central bank in exchange for funds; as allotments increase, however, the bidder must offer more liquid types of collateral, which have a higher opportunity cost. This yields a declining marginal valuation.\footnote{See Ewerhart et al. (2010).}

### 3 Equilibrium

Denote by $X_i$ the strategy of a type-$i$ bidder, $i = 1, 2$, which is a mapping from the signal space to the space of demand functions. Thus, $X_i(s_i, \cdot)$ is the demand function of a type-$i$ bidder...
that corresponds to a given signal \( s_i \). Given her signal \( s_i \), each bidder in a Bayesian equilibrium chooses a demand function that maximizes her conditional profit (while taking as given the other traders’ strategies).\(^{20}\) Our attention will be restricted to anonymous linear Bayesian equilibria in which strategies are linear and identical among traders of the same type (for short, equilibria).

**Definition.** An *equilibrium* is a linear Bayesian equilibrium such that the demand functions for traders of type \( i, i = 1, 2, \) are identical and equal to

\[
X_i(s_i, p) = b_i + a_i s_i - c_i p,
\]

where \( b_i, a_i, \) and \( c_i \) are constants.

The equilibrium is characterized in Subsection 3.1, together with some particular cases, and the equilibrium comparative static properties are examined in Subsection 3.2.

### 3.1 Equilibrium characterization

Consider a trader of type \( i \). If rivals’ strategies are linear and identical among traders of the same type and if the market clears, that is, if \((n_i - 1)X_i(s_i, p) + x_i + n_jX_j(s_j, p) = Q\), for \( j = 1, 2 \) and \( j \neq i \), then this trader faces the inverse residual supply \( p = I_i + d_i x_i \), where

\[
I_i = ((n_i - 1) (b_i + a_i s_i) + n_j (b_j + a_j s_j) - Q) / ((n_i - 1) c_i + n_j c_j) \text{ and } \quad (1)
\]

\[
d_i = 1 / ((n_i - 1) c_i + n_j c_j). \quad (2)
\]

The expression for inverse residual supply disentangles the capacity of a bidder to influence the market price \( (d_i) \) from learning from the price \( (I_i) \). Thus, the slope of inverse residual supply \( (d_i) \) is an index of the trader’s market power or price impact.\(^{21}\) Indeed, by putting one more unit in the market, a trader of type \( i \) will move the price by \( d_i \). A competitive trader would face a flat inverse residual supply \( (d_i = 0) \). The slope \( d_i \) increases, and the inverse residual supply becomes less elastic, the steeper are the demand functions submitted by the other traders (i.e., the lower are \( c_i \) and \( c_j \)).

As a consequence, this trader’s information set \( (s_i, p) \) is informationally equivalent to \( (s_i, I_i) \). Note that only the intercept of the inverse residual supply for a trader of type \( i \) is informative about the signal of traders of type \( j \). The bidder, therefore, chooses \( x_i \) to maximize

\[
E [\pi_i | s_i, p] = (E [\theta_i | s_i, I_i] - I_i - d_i x_i) x_i - \lambda_i x_i^2 / 2.
\]

\(^{20}\) As in Kyle (1989), demands may be considered in the class of upper-hemicontinuous, convex-valued correspondences mapping prices \( p \) into non-empty subsets of the interval \([-\infty, \infty]\). If there is no market clearing price, the market shuts down, and if there are multiple clearing prices, the auctioneer chooses the one that maximizes volume traded.

\(^{21}\) We assume that \((n_i - 1) c_i + n_j c_j \neq 0\).
The first-order condition (FOC) is given by $X_i(s_i, p) = \left( \mathbb{E}[\theta_i|s_i, p] - p \right) / (d_i + \lambda_i)$. The second-order condition (SOC) that guarantees a maximum is $2d_i + \lambda_i > 0$, which implies that $d_i + \lambda_i > 0$.

Using the expression for $I_i$ and assuming that $a_j \neq 0$, we can show that $(s_i, p)$ is informationally equivalent to $(s_1, s_2)$. Therefore, since $\mathbb{E}[\theta_i|s_i, p] = \mathbb{E}[\theta_i|s_i, I_i]$, it follows that

$$\mathbb{E}[\theta_i|s_i, p] = \mathbb{E}[\theta_i|s_1, s_2].$$

According to Gaussian distribution theory,

$$\mathbb{E}[\theta_i|s_i, s_j] = \bar{\theta}_i + \Xi_i (s_i - \bar{\theta}_i) + \Psi_i (s_j - \bar{\theta}_j),$$

where

$$\Xi_i = \frac{1 - \rho^2 + \sigma^2_{\xi_j}}{(1 + \sigma^2_{\xi_j})} \text{ and } \Psi_i = \frac{\rho \sigma^2_{\xi_j}}{(1 + \sigma^2_{\xi_j})}.$$

We remark that Equation (5) has the following implications.

1. The private signal $s_i$ is useful for predicting $\theta_i$ ($\Xi_i \neq 0$) whenever $1 - \rho^2 + \sigma^2_{\xi_j} \neq 0$, that is, when either the liquidation values are not perfectly correlated ($\rho \neq 1$) or type-$j$ traders are imperfectly informed about $\theta_j$ ($\sigma^2_{\xi_j} \neq 0$).

2. The private signal $s_j$ is useful for predicting $\theta_i$ ($\Psi_i \neq 0$) whenever $\rho \sigma^2_{\xi_j} \neq 0$, that is, when the private liquidation values are correlated ($\rho \neq 0$) and type-$i$ traders are imperfectly informed about $\theta_i$ ($\sigma^2_{\xi_i} > 0$). Note that the weight given to $s_j$ in estimating $\theta_i$, $\Psi_i$, increases with the correlation coefficient of valuations ($\rho$).

Using the expression for demand given in (3) and the updating formulae, we can derive a linear equilibrium by identifying coefficients. Our first proposition summarizes the result. It shows the relationship between $a_i$ and $c_i$ in equilibrium and also indicates that these coefficients are positive (see Lemma A1 and A2 in Appendix A for more details).

**Proposition 1.** Let $\rho < 1$. If equilibrium exists, then it is unique and the demand function of a trader of type $i$ ($i = 1, 2$) is given by $X_i(s_i, p) = \left( \mathbb{E}[\theta_i|s_i, p] - p \right) / (d_i + \lambda_i)$. In addition, we have that signal and price responsiveness ($a_i$ and $c_i$) move together, $a_i = \Delta_i c_i > 0$, where $\Delta_i = 1 / \left( 1 + (1 + \rho)^{-1} \sigma^2_{\xi_i} \right)$, with

$$c_i = (1 - \Lambda_i) / (d_i + \lambda_i),$$

(6)
where $\Lambda_i \equiv \Psi_i \left( \frac{n_i c_i}{n_j c_j} + 1 \right) / \Delta_j$, and the ratio $c_1/c_2$ is the unique positive solution of a cubic polynomial.

**Remark 1.** Since $a_i > 0$ and $c_i > 0$, for $i = 1, 2$, it follows that in equilibrium, the higher the value of the trader’s observed private signal (or the lower the price), the higher the quantity she will demand. When $\sigma_{\varepsilon_i}^2 > 0$ we have $a_i < c_i$, since $\Delta_i < 1$ in this case; when $\sigma_{\varepsilon_i}^2 = 0$, we have $\Delta_i = 1$ and $a_i = c_i$. Observe that we can write the demand as $X_i(s_i, p) = b_i + c_i (\Delta_i s_i - p)$.

Given that $p$ is a linear function of $s_1$ and $s_2$, for $i = 1, 2$, Equation (4) holds. The equilibrium price is therefore privately revealing, in other words, the private signal and the price enable a type $i$ trader to learn about $\theta_i$ as much as if she had access to all the information available in the market, $(s_1, s_2)$.\textsuperscript{22}

The term $\Lambda_i$ defined in Proposition 1 is the coefficient of the price in $E[\theta_i|s_i, p]$ and, therefore, it represents the information-sensitivity weight of the price. From the market clearing condition, and taking into account that $a_i = \Delta_i c_i$, we obtain that $\frac{\partial p}{\partial s_i} = \Delta_j / \left( \frac{n_i c_i}{n_j c_j} + 1 \right)$ and, therefore, the coefficient of the price in $E[\theta_i|s_i, p]$, which equals $E[\theta_i|s_i, s_j]$, will be given by $\frac{\partial E[\theta_i|s_i, s_j]}{\partial s_i} \left( \frac{\partial p}{\partial s_j} \right)^{-1}$, which yields the desired expression for $\Lambda_i$. Moreover, note that Expression (6) indicates that the larger $\Lambda_i$, the lower the responsiveness of the demand to the price ($c_i$). From the perspective of a bidder in group $i$, a high price conveys the news that $s_j$ is high and, therefore, that the value $\theta_i$ will tend to be high because of the positive correlation between $\theta_i$ and $s_j$. If the price is more informative about $\theta_i$, then a bidder in group $i$ is more cautious and, consequently, the reduction in the quantity demanded by this bidder due to an increase in $p$ is smaller.

Next, we study how the slope of the optimal demand for a trader of group $i$ ($c_i$) varies due to a change in the slope in the optimal demand for bidders of type $j$ ($c_j$), with the slopes of demands for other bidders of group $i$ remaining fixed. Combining Expression (6) and the expression of $\Lambda_i$, it follows that

$$c_i = \left( 1 - \frac{\Psi_i d_i^{-1}}{\Delta_j n_j c_j} \right) \left( d_i + \lambda_i + \frac{\Psi_i}{\Delta_j n_j c_j} \right). \quad (7)$$

Note that the optimal slope of a type $i$ trader depends on its price impact ($d_i$) and the slope of the demand functions of bidders of the rival group ($c_j$). When $\Psi_i = 0$ prices are uninformative (that is, when either the valuations are uncorrelated, $\rho = 0$; the private signal $s_i$ is perfectly informative, $\sigma_{\varepsilon_i}^2 = 0$; or the signal $s_j$ is useless, $\sigma_{\varepsilon_j}^2 = \infty$), Expression (7) becomes $c_i =$

\textsuperscript{22}This would not be the case if there were more than two groups or if the traders in each group were to receive idiosyncratic signals. In this case, an information externality would appear, inducing additional inefficiencies in the market. The situation would be similar to the case of a noisy equilibrium (e.g., Vives 2017).
Given that $d_i$ is decreasing in $c_j$, as shown in Expression (2), we observe a basic strategic complementarity in the slopes of the demands submitted by the traders. According to this strategic effect, if the type $j$ rivals of a type $i$ trader bid a demand function with a lower $c_j$, then the slope of the inverse residual supply $d_i$ for this trader increases and so he also has an incentive to bid a demand function with a lower $c_i$. However, if $\Psi_i > 0$, then there is also an inference effect from the information conveyed by the price (i.e., a type $i$ trader will recover $s_j$ from the price). A lower $c_j$ increases the terms $\Psi_i \frac{d_i^{-1}}{n_j c_j}$ and $\Psi_i \frac{d_i^{-1}}{n_j c_j}$ in Expression (7), which will also tend to depress $c_i$. This leads to the conclusion that the basic strategic complementarity in slopes due to the strategic effect is reinforced by the inference effect.

Our next proposition summarizes when an equilibrium exists. If an equilibrium does exist, then Proposition 1 implies that it is unique.

**Proposition 2.** Equilibrium exists iff $c_i > 0$, $i = 1, 2$.

- **Uninformative prices.** When valuations are uncorrelated ($\rho = 0$), private signals are perfectly informative or useless ($\sigma^2_{\xi_i} = 0$, or $\sigma^2_{\xi_i} = \infty$, $i = 1, 2$), equilibrium exists iff $n_1 + n_2 \geq 3$.

- **Informative prices.** When the private signal $s_j$ is useful for predicting $\theta_i$ ($\rho \sigma^2_{\xi_i} > 0$ and $\sigma^2_{\xi_j} \geq 0$), $j \neq i$, and valuations are not perfectly correlated ($\rho < 1$), we find a necessary and sufficient condition for $c_i > 0$, $i = 1, 2$ (see Proposition 2A in Appendix A). It follows that equilibrium exists if: (i) both groups of bidders are large enough ($n_1$ and $n_2$ are large enough); (ii) given the number of bidders in group $i$ ($n_i$), the number of bidders in the other group ($n_j$) is large enough, and the correlation coefficient between valuations ($\rho$) low enough, for $i, j = 1, 2, j \neq i$, (iii) $\sigma^2_{\xi_j} = 0$ and $n_j \geq 2$.

**Remark 2.** Equilibrium does not exist for $\rho$ close to 1 and low $n_i$. This is so because in those cases the market power of traders explodes and the demand schedules would become vertical (with $c_i \to 0$, $i = 1, 2$). As $\rho$ increases, the informational role of the price is more important and traders submit steeper demand schedules (see Proposition 3 below). Neither does an equilibrium exist when $\rho = 1$. If the price reveals a sufficient statistic for the common valuation, then no trader has an incentive to place any weight on her signal. But if traders put no weight on signals, then the price contains no information about the common valuation. This conundrum is related to the Grossman-Stiglitz (1980) paradox. In fact, $\rho < 1$ and $n_1 + n_2 \geq 3$ are necessary conditions for the existence of equilibrium with incomplete information (in line with Kyle 1989; Vives 2011).  

---

23 In this case, the equilibrium coincides with the full-information equilibrium (denoted by superscript $f$). In the full (shared) information setup, traders can access $(s_1, s_2)$ and, consequently, the price does not provide any additional information.

24 The result applies to the linear equilibrium.

25 Du and Zhu (2017b) consider ex post nonlinear equilibria in a bilateral divisible double auction and show
Let us illustrate the existence of equilibrium result in the particular case of symmetric groups, i.e., $n_i = n$, $\lambda_i = \lambda$, and $\sigma^2_{\varepsilon_i} = \sigma^2_{\tilde{\varepsilon}}$, $i = 1, 2$. We find that equilibrium exists iff $n > 1 + \rho \tilde{\sigma}^2_{\varepsilon}/((1 - \rho) (1 + \rho + \tilde{\sigma}^2_{\varepsilon}))$, where recall that $\tilde{\sigma}^2_{\varepsilon} = \sigma^2_{\varepsilon}/\sigma^2_{\tilde{\varepsilon}}$. Therefore, the equilibrium’s existence is guaranteed provided either that $n$ is high enough or that $\rho$ or $\tilde{\sigma}^2_{\varepsilon}$ is low enough. In the model of Vives (2011) bidders receive different private signals and the condition that guarantees existence of an equilibrium is $n > 1 + n\rho \tilde{\sigma}^2_{\varepsilon}/((1 - \rho) (1 + (2n - 1)\rho + \tilde{\sigma}^2_{\varepsilon}))$. Direct computation yields that the condition derived in the model of Vives is more stringent than the condition derived in our setup. The reason is that, in Vives (2011), the degree of asymmetry in information (and induced market power) is greater because each of the $2n$ traders receives a private signal.

### 3.2 Comparative statics

We start by considering how the model’s underlying parameters affect the equilibrium and, in particular, price impact (Proposition 3). We then explore how the equilibrium is affected when there are two distinct groups of traders, that is, a strong group and a weak group (Corollary 1). Our theme is to explore the interaction between strategic and inference effects when a payoff or an information parameter changes.

**Proposition 3.** Let $\rho \sigma^2_{\varepsilon_i}\sigma^2_{\varepsilon_j} > 0$. Then, for $i = 1, 2, i \neq j$, the following statements hold.

(i) An increase in the expected valuation of group $i$ or in the quantity offered ($\bar{\theta}_i$ or $Q_i$), or a decrease in the expected valuation of group $j$ ($\bar{\theta}_j$), raises the demand intercept ($b_i$).

(ii) An increase in transaction costs ($\lambda_i$ or $\lambda_j$), a decrease in the precision of private signals (i.e., an increase in $\sigma^2_{\varepsilon_i}$ or $\sigma^2_{\varepsilon_j}$), or an increase in the correlation coefficient between valuations ($\rho$), makes demand less responsive to private signals and prices (lower $a_i$ and $c_i$) and increases price impact ($d_i$).

(iii) If the number of bidders ($n_i$ or $n_j$) increases, then $d_i$ decreases.

**Remark 3.** Note that when $\rho = 0$, $d_i$ is independent of $\sigma^2_{\varepsilon_i}$, $i = 1, 2$; and when $\sigma^2_{\varepsilon_i} = 0$ or $\sigma^2_{\varepsilon_i} = \infty$, for $i = 1, 2$, $d_i$ is independent of $\rho$. Those cases correspond to a setting where prices do not convey information; $d^L_i$ and $d^U_i$, $j \neq i$, are independent of $\sigma^2_{\varepsilon_i}$; and comparative statics of $d^L_i$ and $d^U_j$ on $\lambda_i$ and $n_i$ hold as in the previous proposition. Indeed, if $\rho = 0$, then: (a) both that with more than three symmetric traders there are no nonlinear equilibria in the class of smooth demands downward sloping in price and upward sloping in signals.

---

26 See our working paper Manzano and Vives (2019) for the cases of a monopsony competing with a fringe and of an informed group facing an uninformed one as in Grossman and Stiglitz (1980).

27 Akgün (2004) considers a linear equilibrium in a certainty common value model and shows (in our notation) that an increase in $\lambda_i$ reduces $c_i$ and $c_j$. 

12
c_i and d_i (as well as c_j and a_j, j ≠ i) are independent of \( \sigma^2_{\xi_i} \); (b) \( a_i \) decreases with \( \sigma^2_{\xi_i} \); and (c) \( b_i \) is independent of both \( Q \) and \( \bar{\theta}_j \). If \( \sigma^2_{\xi_i} = 0 \) for \( i = 1, 2 \), then \( b_i = 0 \), and \( c_i, c_j, a_i, a_j, d_i, \) and \( d_j, i = 1, 2, j ≠ i \), are independent of \( \rho \). That is, for the information parameters to matter for price impact, it is necessary that prices convey information. Proposition 3(ii) implies that, if \( \rho \sigma^2_{\xi_1} \sigma^2_{\xi_2} > 0 \), then \( d_i^T < d_i \), \( i = 1, 2 \). Thus, asymmetric information increases the price impact of traders in both groups beyond the full-information level.

Remark 4. When groups are symmetric, then results (ii) and (iii) hold when \( \lambda_i = \lambda_j, \sigma^2_{\xi_i} = \sigma^2_{\xi_j}, \) and \( n_i = n_j \) move together. This is the result obtained in Vives (2011). The proposition disentangles the impact of, say, \( \sigma^2_{\xi_i} \) on equilibrium parameters, keeping \( \sigma^2_{\xi_j} \) constant.

The only equilibrium coefficient affected by the quantity offered in the auction \( Q \) and by the prior mean of the valuations \( (\bar{\theta}_i \) and \( \bar{\theta}_j) \) is \( b_i \). Proposition 3(i) indicates that if \( Q \) increases, then all the bidders will increase their demand (higher \( b_i \) and \( b_j \)). This is because an increase in \( Q \) reduces the intercepts of the inverse residual supply functions -see Expression (1)-, i.e., more residual supply at any given price, which causes an increase in the demands for the risky asset (higher \( b_i \) and \( b_j \)). Moreover, if the prior mean of the valuation of group \( j \) \( (\bar{\theta}_j) \) increases, then the bidders in this group will demand a greater quantity of the risky asset (higher \( b_j \)). Then, the intercept \( I_i \) of the inverse residual supply for the group \( i \) bidder rises (less residual supply at any given price). That reaction leads the traders in group \( i \) to reduce their demand for the risky asset (and lower \( b_i \)). Thus, here we can see a strategic substitutability in demand intercepts.

All these comparative static results provide testable predictions. Furthermore, if we have estimates of transaction costs, precision of the signals, and correlation of valuations, we can predict how changes in these parameters lead to changes in the slopes of submitted demands and the price impact of the two groups.

Part (ii) of Proposition 3 shows how the response to private information and price varies with several parameters. If the transaction costs for a bidder of type \( j \) \( (\lambda_j) \) increase, then the bidder sets lower \( a_j \) and \( c_j \). Moreover, any increase in a group’s transaction costs also affects the behavior of traders in the other group. If \( \lambda_j \) increases, then the decrease in \( c_j \) results in an increase of the slope of the inverse residual supply for group \( i \) (higher \( d_i \)) as well as the terms related to the inference of \( \theta_i \) from the price \( (\frac{\Psi_i}{\Delta_j n_i c_j} \) and \( \frac{\Psi_i}{\Delta_j n_j c_j} \) in Expression (7). As both the strategic and the inference effects work in the same direction, an increase in \( \lambda_j \) leads group-\( i \) traders to reduce their demand sensitivity to the private signal and price (lower \( a_i \) and \( c_i \)). We can therefore see how an increase in the transaction costs for group-\( j \) traders (say, a deterioration of their collateral in liquidity auctions that raises \( \lambda_j \)) leads not only to steeper demand schedules for bidders in group \( j \), but also, as a reaction, to steeper demands for group-\( i \)
traders.

We analyze how the response to private information and price varies with a change in the precision of private signals. If the private signal of type-i bidders is less precise (higher $\sigma_{\varepsilon_i}^2$), then their demand is less sensitive to private information. A private signal of reduced precision also gives the type-i bidder more incentive to consider prices when predicting $\theta_i$ (higher $\Lambda_i$),\textsuperscript{28} which leads, in turn, to this bidder having a demand function with lower $c_i$. The same can be said for a bidder of type $j$ because of strategic complementarity in the slopes of demand functions (the decrease in $c_i$ due to a rise in $\sigma_{\varepsilon_i}^2$ leads to lower $c_j$ and $a_j$, in turn).\textsuperscript{29} This result (in the supply competition model interpretation) may help explain why, in the Texas balancing market, small firms use steeper supply functions than predicted by theory (Hortaçsu and Puller 2008). Indeed, smaller firms may receive lower-quality signals owing to economies of scale in information gathering.

We also find that the more highly the valuations are correlated (higher $\rho$), the less trader responsiveness to private signals is (lower $a_i$, $i = 1, 2$) and the steeper demand schedules are (lower $c_i$, $i = 1, 2$). We can explain these results by recalling that, when the valuations are correlated ($\rho > 0$), a type-i trader learns about $\theta_i$ from prices. In fact, the information-sensitivity weight on the price ($\Lambda_i$, $i = 1, 2$) is higher when $\rho$ is larger,\textsuperscript{30} in which case demand is less sensitive to private information. The rationale for the relationship between the correlation coefficient ($\rho$) and the slopes of demand functions is as follows. An increment in the price of the risky asset makes an agent more optimistic about her valuation, which leads to less of a reduction in demand quantity than in the case of uncorrelated valuations.\textsuperscript{31}

Cassola et al. (2013) show how distressed bidders after the August 2007 shock suffered a large decline in the valuation of their collateral in the interbank market, which in terms of our model shows up in an increased $\lambda_i$. Those banks also had an increase in the valuation for liquidity (which in our model shows up as an increased $\bar{b}_i$).\textsuperscript{32} In Cassola et al. (2013) it is assumed that the private valuations of the traders are independent, since the common component is known. This means that there are no information effects. However, if the common value component is

\textsuperscript{28}It is easy to check that $\Psi_i/\Delta_j$, $i \neq j$, $i = 1, 2$, is increasing in $\sigma_{\varepsilon_i}^2$.

\textsuperscript{29}The decrease in $c_i$ due to an increase in $\sigma_{\varepsilon_i}^2$ leads to an increase in $d_j$ and in the terms associated with the inference effect, although these last terms become less significative because of the reduction in the precision of group $i$.

\textsuperscript{30}It is easy to check that $\Psi_i/\Delta_j$, $i \neq j$, $i = 1, 2$, is increasing in $\rho$.

\textsuperscript{31}A high price conveys the good news that the private signal received by other group’s traders is high. When valuations are positively correlated, a bidder infers from the high private signal of the other group that her own valuation is high.

\textsuperscript{32}The marginal valuation of a bidder of type $i$ is $\theta_i - \lambda_i x_i$. This is akin to the marginal valuation in Figure 4 in Cassola et al. (2013). There, a decreased collateralized borrowing capacity of a bidder ($K$) will make the slope of the marginal valuation steeper.
not known, as is plausible to believe, and if in a crisis the signals of the groups become noisier, in particular for those of the group hit by the shock, and the correlation of valuations increases, then all these effects reinforce the steepening of the demand schedules (as found in Cassola et al. (2013)).

Figure 1 illustrates the case of initially identical groups (with the grey demand in the figure) that become differentiated after a shock for group 1 induces a higher $\lambda_1$, higher noise in the signal ($\sigma_{\epsilon_1}^2$), and higher correlation $\rho$ as well as increased willingness to pay for liquidity for both groups ($\bar{\theta}_1$ and $\bar{\theta}_2$). This corresponds to the case of a group of banks being hit a the crisis and the quality of their information deteriorating (or their perceived uncertainty increasing) as well as the correlation of valuations raising. We see that the shock to group 1 induces this group to steepen its demand schedule substantially while the demand schedule of group 2 also steepens in response (but much less). The assumed increased willingness to pay for liquidity for both groups ($\bar{\theta}_1$ and $\bar{\theta}_2$) only affects the intercepts of the demand functions. Note that the expected marginal valuation of liquidity for the safe group need not change a lot (consistent with the findings of Cassola et al. (2013)).

Figure 1: Comparative statics on demand functions $n_i = 5; s_i = \bar{\theta}_i; \sigma^2_{\theta} = 5; Q = 4$. The decreasing grey line corresponds to the optimal demand curve for a bidder in the symmetric case, while the black lines represent the optimal demand curves for each group when the expected valuations, the correlation coefficient among valuations, and the transaction costs and the noise in the signal for group 1 have increased.

Our comparative static results highlight the interaction between the strategic and inference effects resulting from a parameter change. We have seen how a steepening of demand schedule
by one group leads to the steepening of demand schedule by another group because of a strategic
effect, which is reinforced by an inference effect. The result is strategic complementarity in the
slopes of demands. The presence of private information and learning from prices compounds
the strategic effect that would be present with full information and makes the impact of the
change of a parameter larger.

Proposition 3(iii) formalizes the anticipated result that an increase in the number of auction
participants (higher \( n_i \) or \( n_j \)) reduces the price impact of traders in both groups.\(^{33}\)

**Corollary 1 (Strong and weak groups).** Suppose that group 1 is less informed, has higher
transaction costs, and is more numerous than group 2 (i.e., \( \sigma_{e_1}^2 \geq \sigma_{e_2}^2 \), \( \lambda_1 \geq \lambda_2 \), and \( n_1 \geq n_2 \)), and suppose that at least one of these inequalities is strict. Then, in equilibrium, the stronger
group (here, group 2) reacts more both to private information and to prices (\( a_1 < a_2 \), \( c_1 < c_2 \))
and has more price impact (\( d_1 < d_2 \)) than does the weaker group.

Corollary 1 shows that if a group of traders is less informed, has higher transaction costs,
and is more numerous, then it reacts less both to private signals and to prices. Observe in
particular that group-1 traders, having less precise private information, rely more on the price
for information (higher \( \Lambda_1 \)); as a result, their overall price response (\( c_1 = (1 - \Lambda_1) / (d_1 + \lambda_1) \))
is smaller. Similarly, group-1 traders, for whom \( n_1 \) is larger, put more information-sensitivity
weight on the price (which depends more strongly on \( s_1 \)). As we will see below, some results
will depend on the comparison between the "total transaction costs" \( d_1 + \lambda_1 \) and \( d_2 + \lambda_2 \) of the
two groups. While with full information we have that \( d_1 + \lambda_1 > d_2 + \lambda_2 \) whenever \( \lambda_1 > \lambda_2 \), in
our model we may have \( d_1 + \lambda_1 < d_2 + \lambda_2 \) with \( \lambda_1 > \lambda_2 \). This, in fact, will happen whenever \( \rho \)
is large, since then price impact induced by private information is large.\(^{34}\)

4 Welfare analysis

We identify factors that affect, in equilibrium, quantities, expected price and revenue in the
auction in Subsection 4.1 (see Appendix B for an analysis of bid shading); the equilibrium

\(^{33}\)Rostek and Weretka (2015) address the question of whether encouraging trader participation enhances
market competitiveness and liquidity also in a linear-Gaussian uniform-price double auction with a finite number
of traders whose valuations are potentially asymmetrically correlated. They assume that each trader’s value is,
on average, correlated with other traders’ values in the same way and find that, in general, the price impact is not
monotone in market size. This is so because the arrival of an additional trader may change the informativeness
of the market price so that the market power of all traders increases and the gains to trade are lower. In our
model, since the equilibrium price is privately revealing, the informativeness of the market price does not change
as the number of bidders increases.

\(^{34}\)See our working paper Manzano and Vives (2019) for the details and proof.
and efficient allocations in Subsection 4.2 to be used as a benchmark; deadweight losses in Subsection 4.3; and market integration in Subsection 4.4.

4.1 Quantities, prices and revenue

Let \( t_i = \mathbb{E} [ \theta_i | s_1, s_2], \) \( i = 1, 2, \) be the predicted values with full information \((s_1, s_2)\), and \( t = (t_1, t_2). \) After some algebra, it follows that equilibrium quantities as functions of \( t \) are given by:

\[
x_i (t) = \frac{n_j (t_i - t_j)}{n_i (d_j + \lambda_j) + n_j (d_i + \lambda_i)} + \frac{d_j + \lambda_j}{n_i (d_j + \lambda_j) + n_j (d_i + \lambda_i)} Q, \ i = 1, 2, j \neq i. \quad (8)
\]

Observe that, according to this expression, the equilibrium quantities can be decomposed into two terms: a valuation trading term (the first), which depends on the relative valuations of the groups, and a clearing trading term (the second), which is related to the absorption of \( Q \) by the traders and which vanishes when \( Q = 0. \) With regard to the valuation term, it vanishes when both groups have the same valuation, \( t_1 = t_2, \) and it is positive (resp., negative) for the group with the higher (resp., lower) value of \( t_i. \) Higher total transaction costs \((d_i + \lambda_i)\) lower the response to valuation differences \( t_i - t_j. \) As for the clearing trading term, when \( Q > 0 \) it is lower (resp., higher) for the group with higher (resp., lower) \( d_i + \lambda_i \) and total clearing demands add up to \( Q. \)

Let \( \tilde{t} = (n_1 t_1 + n_2 t_2) / (n_1 + n_2). \) Using the optimal demand of bidders, it follows that \( p(t) = t_i - (d_i + \lambda_i)x_i (t), \ i = 1, 2. \) Therefore,

\[
p(t) = \tilde{t} - ((d_1 + \lambda_1)n_1 x_1 (t) + (d_2 + \lambda_2)n_2 x_2 (t)) / (n_1 + n_2).
\]

From the above expressions, we can derive the expected price:

\[
\mathbb{E} [p(t)] = \left( \frac{n_1}{d_1 + \lambda_1} \bar{\theta}_1 + \frac{n_2}{d_2 + \lambda_2} \bar{\theta}_2 - Q \right) \bigg/ \left( \frac{n_1}{d_1 + \lambda_1} + \frac{n_2}{d_2 + \lambda_2} \right). \quad (9)
\]

It is worth noting that, in the double auction case \((Q = 0), \) \( \mathbb{E} [p] \) is a convex combination of \( \bar{\theta}_1 \) and \( \bar{\theta}_2. \) In this case, \( \mathbb{E} [x_2 (t)] < 0 < \mathbb{E} [x_1 (t)] \) iff \( \bar{\theta}_1 > \bar{\theta}_2, \) with group 1 consisting of buyers and group 2 of sellers. Also, for symmetric groups (except possibly with respect to the means) we have \( \mathbb{E} [p] = (\bar{\theta}_1 + \bar{\theta}_2) / 2. \)

Proposition 4. Let \( \rho \sigma_{s_1} \sigma_{s_2} > 0. \) In equilibrium, the following statements hold.

(i) If the two groups have the same expected valuations \((\bar{\theta}_1 = \bar{\theta}_2), \) then the expected price is increasing in the number of bidders \((n_i, i = 1, 2), \) but is decreasing in transaction costs \((\lambda_i, i = 1, 2), \) the variances of error terms in private signals \((\sigma^2_{s_i}, i = 1, 2), \) and the correlation between valuations \((\rho). \) Otherwise, if \(|\bar{\theta}_1 - \bar{\theta}_2| \) is large enough, then these results need not hold.
The seller’s expected revenue $\mathbb{E}[p]Q$:
- increases with the expected valuations $(\bar{\theta}_i, i = 1, 2)$;
- is maximum when the quantity offered is
  \[ Q = \left( \frac{n_i}{d_i + \lambda_i} \bar{\theta}_1 + \frac{n_2}{d_2 + \lambda_2} \bar{\theta}_2 \right) / 2, \]
  which increases with $n_i$ and $\bar{\theta}_i$, and decreases with $\rho, \lambda_i$ and $\sigma^2_{\varepsilon_i}, i = 1, 2$.

**Corollary 2.** The expected revenue is between (a) the larger expected revenue of the auction in which both groups are ex ante identical with a large number of bidders (each group with $\max\{n_1, n_2\}$), high expected valuation ($\max\{\bar{\theta}_1, \bar{\theta}_2\}$), low transaction costs ($\min\{\lambda_1, \lambda_2\}$) and precise signals ($\min\{\sigma^2_{\varepsilon_1}, \sigma^2_{\varepsilon_2}\}$) and (b) the smaller expected revenue of the auction in which both groups are ex ante identical but with the opposite characteristics (i.e., $\min\{n_1, n_2\}$, $\min\{\bar{\theta}_1, \bar{\theta}_2\}$, $\max\{\lambda_1, \lambda_2\}$, and $\max\{\sigma^2_{\varepsilon_1}, \sigma^2_{\varepsilon_2}\}$).

**Remark 5.** If valuations are uncorrelated ($\rho = 0$), then $\mathbb{E}[p]$ is independent of the precision of private signals ($\sigma^2_{\varepsilon_i}, i = 1, 2$), and if private signals are perfectly informative ($\sigma^2_{\varepsilon_i} = 0, i = 1, 2$), then $\mathbb{E}[p]$ is independent of $\rho$. The reason is that in both cases, $d_i$ is independent of $\sigma^2_{\varepsilon_i}$ and $\rho$. If $\sigma^2_{\varepsilon_1} = \infty$, then $\mathbb{E}[p]$ is independent of $\sigma^2_{\varepsilon_2}$.

When $\bar{\theta}_1 = \bar{\theta}_2$, we confirm that the price increases when the number of bidders increases or when the asset becomes more attractive for the traders because of a reduction in their transaction costs or an increase in the precision of their private signals. To understand the negative relationship between the expected price and $\rho$, note that Proposition 3 indicates that an increase in $\rho$ makes bidders behave more cautiously.

Proposition 4 also shows that, in general, the relationship between the expected price and $\lambda_i, \sigma^2_{\varepsilon_i}$, and $\rho$, $i = 1, 2$ is potentially ambiguous. A reduction in a group’s transaction costs, an increase in the precision of its private signal, or an increase in the number of bidders of a group can decrease the revenue provided that $|\bar{\theta}_i - \bar{\theta}_j|$ is large enough, which never happens in the symmetric model, where $\mathbb{E}[p] = \bar{\theta} - (d + \lambda) Q / 2n$ (see Vives 2010, Prop. 2). With enough asymmetry in the expected valuations, the bidders of one group turn into sellers instead of buyers of the asset. Then, decreasing their transaction costs, increasing the precision of their private signal or increasing their number can lead to an increase in supply and, hence, a lower price is expected.\textsuperscript{35}

We would like to understand how ex ante differences among bidders affect the seller’s expected revenue. Suppose that group 2 is our strong group; it has lower transaction costs ($\lambda_2 < \lambda_1$), is less numerous ($n_2 < n_1$), and is better informed ($\sigma^2_{\varepsilon_2}$ is large enough, which never happens in the symmetric model, where $\mathbb{E}[p] = \bar{\theta} - (d + \lambda) Q / 2n$ (see Vives 2010, Prop. 2). With enough asymmetry in the expected valuations, the bidders of one group turn into sellers instead of buyers of the asset. Then, decreasing their transaction costs, increasing the precision of their private signal or increasing their number can lead to an increase in supply and, hence, a lower price is expected.\textsuperscript{35}

\textsuperscript{35} Note that this would not happen in a Treasury auction, but it could happen in the wholesale electricity market (with supply competition), where integrated firms can also submit demand bids and where a firm may become a net buyer.
higher) than in the case where $\bar{\theta}_1 = \bar{\theta}_2$. If $\bar{\theta}_t \approx \bar{\theta}_2$, then Proposition 4(i) suggests that group 2's relatively small size ($n_2 < n_1$) reduces the seller's expected revenue, although both its relatively low transaction costs ($\lambda_2 < \lambda_1$) and its relatively precise signals ($\sigma^2_{\tilde{\xi}_2} < \sigma^2_{\tilde{\xi}_1}$) have the opposite effect. So in general, the ex ante differences between the two groups have an ambiguous effect on the seller's expected revenue. Nonetheless, part (ii) of Proposition 4 directly follows from part (i).

4.2 Characterizing the equilibrium and efficient allocations

Recall that $t = (t_1, t_2)$ denotes the vector of predicted values with full information $(s_1, s_2)$. The strategies in the equilibrium induce outcomes as functions of $t$ and are given in Equation (8). One can easily show that the equilibrium outcome solves the following distorted benefit maximization program:

$$\max_{x_1, x_2} \mathbb{E} \left[ n_1 (\theta_1 x_1 - (d_1 + \lambda_1) x_1^2/2) + n_2 (\theta_2 x_2 - (d_2 + \lambda_2) x_2^2/2) \right]$$

$$\text{s.t. } n_1 x_1 + n_2 x_2 = Q,$$

where $d_1$ and $d_2$ are the equilibrium parameters. The efficient allocation would be obtained if we set $d_1 = d_2 = 0$, which corresponds to a price-taking equilibrium (denoted by superscript $o$). The equilibrium strategy of a type-$i$ bidder ($i = 1, 2$) will be of the form $X^o_i(s_i, p) = b^o_i + a^o_i s_i - c^o_i p$, $i = 1, 2$, and is derived by maximizing the following program:

$$\max_{x_i} (\mathbb{E} [\theta_i | s_i, p] - p) x_i - \lambda_i x_i^2/2,$$

while taking prices as given. The FOC of this optimization problem yields

$$\mathbb{E} [\theta_i | s_i, p] - p - \lambda_i x_i = 0.$$

After identifying coefficients and solving the corresponding system of equations, we find that there exists a unique equilibrium in this setup.

**Proposition 5.** Let $Q = (n_1 + n_2)q$ and let $\mu_i = n_i/(n_1 + n_2)$ for $i = 1, 2$. Then, there exists a unique price-taking equilibrium, and the equilibrium coefficients of the demand functions are

---

[36] See Lemma A3 in Appendix A.

[37] The efficient allocation maximizes the expected total surplus. Here, the revenue collected is just a transfer from the bidders to the auctioneer and washes out. If the social objective is just the surplus of the bidders or the revenue of the auctioneer, the objective function should be modified accordingly.
given by
\[
\begin{align*}
    b_i^o &= \frac{\sigma_i^2 (\mu_j (\overline{b}_i - \rho \overline{b}_j) + \rho \lambda_j q)}{\mu_i \rho \sigma_i^2 \lambda_j + \mu_j (1 - \rho^2 + \sigma_i^2) \lambda_i},
    q_i^o &= \frac{\mu_j (1 - \rho^2)}{\mu_i \rho \sigma_i^2 \lambda_j + \mu_j (1 - \rho^2 + \sigma_i^2) \lambda_i},
    \end{align*}
\]

Remark 6. It can be shown that, with strategic agents, if \( n_1 \) and \( n_2 \) both approach infinity and \( n_i/(n_1 + n_2) \) converges to \( \mu_i \) (\( 0 < \mu_i < 1 \)), for \( i = 1, 2 \), then the demand equilibrium coefficients converge to the equilibrium coefficients of a continuum economy setup, which coincide with the equilibrium coefficients of the price-taking equilibrium.\(^{38}\) In the continuum economy, there is a continuum of bidders along the interval \([0, 1]\); a fraction \( \mu_i \) (\( 0 < \mu_i < 1 \)) of these bidders are traders of type \( i \), \( i = 1, 2 \); and \( q \) represents the aggregate (average) quantity supplied in the market.

Under symmetry or when both groups value the asset equally, it can be shown that market power lowers prices from the competitive benchmark (recall that we are in bidding/demand competition) whenever \( Q > 0 \). However, in our setup the competitive price could be smaller than in the strategic case. This may happen, for example, when the quantity auctioned is low and the group that values the asset more is the weak group. In such a case, traders of the weak group are buyers, whereas agents of the strong group are sellers. Then, we expect that the strong group will obtain a higher price in the strategic framework.

Our next corollary provides some comparative statics results.

**Corollary 3.** Let \( \rho \sigma_i^2 \sigma_j^2 > 0 \). Then, the only equilibrium coefficients affected by the quantity offered and the expected valuations \((Q, \overline{b}_i, \text{ and } \overline{b}_j)\) are the intercepts of the demand functions (with \( b_i^o \) increasing in \( \overline{b}_i \) and \( Q \), and decreasing in \( \overline{b}_j \)), for \( i, j = 1, 2 \) and \( i \neq j \). Furthermore, when there is a decrease in the proportion of bidders of group \( j \) (\( \mu_j \)) or an increase in transaction costs (\( \lambda_i \) and \( \lambda_j \)), the correlation coefficient between valuations (\( \rho \)), the variance of the error term in the private signal of group \( i \) (\( \sigma_i^2 \)), or the proportion of bidders of group \( i \) (\( \mu_i \)), the demands of group \( i \) become less sensitive to private signals and prices (lower \( a_i \) and \( c_i \)); however, group \( i \)'s demands are not affected by the precision of the private signal of group \( j \) (\( \sigma_j^2 \)).

Observe that, under competitive behavior, we can derive an additional comparative statics result: the relationship between the equilibrium coefficients and the proportion of individuals in group 1. In particular, increasing the proportion \( \mu_1 \) of type-1 traders leads, for those traders, to an increased information-sensitivity weight of the price (higher \( \Psi_1 (n_1 c_1^o + n_2 c_2^o) (n_2 a_2^o)^{-1} \)) and

\(^{38}\)See our working paper Manzano and Vives (2019) for a proof of the statement.
so, a lower overall response to the price \( c_i^0 = \lambda_i^{-1} (1 - \Psi_1 (n_1 c_i^0 + n_2 c_j^0) (n_2 a_2^0)^{-1}) \); the opposite holds for type-2 traders.

Thus, as outlined at the beginning of this subsection, the auction outcome can be obtained as the solution to a maximization problem with a more concave objective function than the expected total surplus, which suggests that inefficiency may be eliminated by quadratic subsidies \((\kappa_i x_i^2/2, i = 1, 2)\) that compensate for the distortions. The per capita subsidy rate \((\kappa_i)\) to a trader of type \(i\) must be such that it compensates for the distortion \(d_i(\kappa_i)\) while accounting for the subsidy. Since the aim is to induce competitive behavior, the trader should be led to respond with \(c_i^0\) to the price. This means that the exact amount of \(\kappa_i\) must be \(d_i(c_i^0, c_j^0)\), since that would be the distortion arising when traders use the competitive linear strategies. The following proposition shows that, if subsidies are selected properly, then bidders behave competitively and so the equilibrium allocation is efficient.

**Proposition 6.** Let \(i = 1, 2\) and \(i \neq j\). Then the efficient allocation is induced by the quadratic subsidies \(\kappa_i x_i^2/2\), where \(\kappa_i = d_i(c_i^0, c_j^0) = 1/ ((n_i - 1) c_i^0 + n_j c_j^0)\). If \(\rho \sigma_{\xi_i}^2 \sigma_{\xi_j}^2 > 0\), then the per capita subsidy rates \((\kappa_i, i = 1, 2)\) increase with transaction costs \((\lambda_i, i = 1, 2)\), the variances of error terms in private signals \((\sigma_{\xi_i}^2, i = 1, 2)\), and the correlation coefficient between valuations \((\rho)\), but decrease with the number of bidders \((n_i, i = 1, 2)\). We have that \(\kappa_1 < \kappa_2\) iff \(c_1^0 < c_2^0\).

The combination of propositions 5 and 6 yields closed-form expressions for the optimal subsidy rates:

\[
\kappa_i = \frac{1}{n_j (1 - \rho)} \left( \frac{(n_i - 1) (1 + \tilde{\sigma}_{\xi_i}^2 + \rho)}{n_i \lambda_j \rho \tilde{\sigma}_{\xi_i}^2 + n_j \lambda_i (1 - \rho^2 + \tilde{\sigma}_{\xi_j}^2)} + \frac{n_i \left(1 + \tilde{\sigma}_{\xi_j}^2 + \rho\right)}{n_j \lambda_j (1 - \rho^2 + \tilde{\sigma}_{\xi_j}^2) + n_j \lambda_i \rho \tilde{\sigma}_{\xi_j}^2} \right)^{-1},
\]

\(i = 1, 2, i \neq j\). If \(\rho = 0\) (or, with full information, if \(\sigma_{\xi_i}^2 = 0, i = 1, 2\)), then \(\kappa_i^f = 1/ ((n_i - 1) \lambda_i^{-1} + n_j \lambda_j^{-1})\), \(i = 1, 2\). Proposition 6 implies that the optimal subsidy rates with incomplete information and learning from prices are higher than with full information: \(\kappa_i > \kappa_i^f\) if (a) \(\rho > 0\) and (b) at least one of \(\tilde{\sigma}_{\xi_i}^2\) or \(\tilde{\sigma}_{\xi_j}^2\) is strictly positive. If \(\sigma_{\xi_i}^2 = \infty\), then \(\kappa_i, i = 1, 2,\) is independent of \(\sigma_{\xi_i}^2\).

**Remark 7.** With symmetric groups we obtain symmetric optimal subsidies which vary as described in Proposition 6 with changes in \(\lambda_i = \lambda_j\), \(\sigma_{\xi_i}^2 = \sigma_{\xi_j}^2\), and \(n_i = n_j\).

The optimal subsidy rates are decreasing in the number of traders because, when there are many agents, competitive behavior is already being approached in the market without subsidies. Moreover, the fact that \(\kappa_i = 1/ ((n_i - 1) c_i^0 + n_j c_j^0)\) implies that (i) the remainder of the comparative statics results stated in Proposition 6 simply follows from Corollary 3, and (ii) \(\text{sgn} \{\kappa_1 - \kappa_2\} = \text{sgn} \{c_1^0 - c_2^0\}\). Hence, \(\kappa_1 < \kappa_2\) iff \(c_1^0 < c_2^0\), i.e., the bidders who require a higher
per capita subsidy rate are the ones whose demands are more sensitive to price. Corollary 3 allows us to conclude that, if there is a group with more precise private information, with lower transaction costs, and that is less numerous, then it is the group meriting a higher per capita subsidy rate. The reason is that the stronger group’s strategic behavior is more pronounced and so it must receive more compensation in order to become competitive.

The expected (total) optimal subsidy for group \( i \) is \( \kappa_i \mathbb{E} \left[ (x_{i1}^o(t))^2 \right] / 2 \), where \( (x_{i1}^o(t), x_{i2}^o(t)) \) corresponds to the price-taking equilibrium allocation, with

\[
x_{i1}^o(t) = \frac{n_j (t_i - t_j)}{n_i \lambda_j + n_j \lambda_i} + \frac{\lambda_j}{n_i \lambda_j + n_j \lambda_i} Q, \quad i = 1, 2, \quad j \neq i, \quad i = 1, 2, \quad j \neq i.
\] (10)

If \( \bar{\theta}_2 \geq \bar{\theta}_1 \), then \( \mathbb{E} \left[ (x_{i2}^o(t))^2 \right] > \mathbb{E} \left[ (x_{i1}^o(t))^2 \right] \), from which it follows that the bidders from the stronger group (group 2) should receive the higher expected subsidy. However, if \( \bar{\theta}_1 > \bar{\theta}_2 \), then there are parameter configurations under which bidders from the weaker group (group 1) should receive the higher expected subsidy, even though \( \kappa_1 < \kappa_2 \). These conclusions would have to be revised if redistributive considerations come into play.\(^{39}\)

Our result has policy implications. It implies, for example, that a central bank seeking an efficient distribution of liquidity among banks should relax collateral requirements (i.e., provide a larger subsidy) to the strong group. This prescription sounds apparently counterintuitive because the efficiency motive may conflict with the central bank’s function as lender of last resort, which often involves shoring up weak banks (e.g., the European Central Bank relaxing the collateral requirements for Greek banks to avoid a meltdown of that country’s banking system).

Another example is that of a wholesale electricity market characterized by a small (oligopolistic) group and a fringe; in this case, a regulator looking to improve productive efficiency should set a higher subsidy rate for the oligopolistic group. This could be accomplished by offering differential subsidies to renewable energy technologies, for instance, that lower the marginal cost of production.

It is worth noting that primary dealers in the US Treasury are required to bid at least the pro-rate share of those dealers present in the auction ("demonstrate substantial presence") and in exchange enjoy privileges such as exclusive intermediation of OMO, and in the crisis period access to the QE auction mechanism as well as to the Primary Dealer Credit Facility. This may be interpreted as a subsidy that lowers the effective transaction cost of the dealers since they have the obligation to bid a minimum amount.

\(^{39}\)Athey et al. (2013) find with regard to US Forest Service timber auctions that restricting entry increases small business participation but substantially reduces efficiency and revenue. In contrast, subsidizing small bidders directly increases revenue and the profits of small bidders without much cost in efficiency. See also Loertscher and Marx (2017) and Pai and Vohra (2012).
4.3 Deadweight loss

The expected deadweight loss, $\mathbb{E}[DWL]$, at an anonymous allocation $(x_1(t), x_2(t))$ is the difference between expected total surplus at the efficient allocation, $\mathbb{E}[TS^o]$, and at the baseline allocation, denoted by $\mathbb{E}[TS]$. Lemma A4 in Appendix A shows that

$$\mathbb{E}[DWL] = \frac{1}{2} \lambda_1 n_1 \mathbb{E}[(x_1(t) - x_1^o(t))^2] + \frac{1}{2} \lambda_2 n_2 \mathbb{E}[(x_2(t) - x_2^o(t))^2].$$

(11)

Using (8) and (10), it follows that

$$\mathbb{E}[DWL] = \phi((n_2 d_1 + n_1 d_2)^2 \mathbb{E}[(t_1 - t_2)^2] + 2(n_2 d_1 + n_1 d_2)(\lambda_1 d_2 - \lambda_2 d_1)(\bar{\theta}_2 - \bar{\theta}_1) Q + (\lambda_1 d_2 - \lambda_2 d_1)^2 Q^2),$$

where $\phi = n_1 n_2 / (2(n_2 \lambda_1 + n_1 \lambda_2)(n_1 (d_2 + \lambda_2) + n_2 (d_1 + \lambda_1))^2)$, and

$$\mathbb{E}[(t_1 - t_2)^2] = (\bar{\theta}_1 - \bar{\theta}_2)^2 + (1 - \rho)^2 \sigma_\theta^2 + \frac{2(1 + \rho)}{(1 + \sigma_{\xi_1}^2)(1 + \sigma_{\xi_2}^2)}.$$  

(12)

In a double auction ($Q = 0$) or when $\lambda_2 d_1 = \lambda_1 d_2$, Equations (8) and (10) indicate that the clearing trading terms corresponding to the equilibrium with imperfect competition and the ones corresponding to the competitive equilibrium are equal and, therefore, the expected deadweight loss only derives from differences in the valuation trading terms. Strategic behavior restricts trade and generates inefficiency.\(^\text{40}\)

Consider now the other extreme case where the differences between the equilibrium and efficient quantities mainly arise in the clearing trading terms (i.e., $Q$ large enough and $\lambda_2 d_1 \neq \lambda_1 d_2$). If group 1 has higher transaction costs ($\lambda_1 > \lambda_2$), is more numerous ($n_1 > n_2$), and is less informed ($\sigma_{\xi_1}^2 > \sigma_{\xi_2}^2$) than group 2, then $d_1 / d_2 < \lambda_1 / \lambda_2$ (and, therefore, $\lambda_1 d_2 - \lambda_2 d_1 > 0$). In this case, the weaker group (group 1) gets more than the efficient allocation $x_1(t) > x_1^o(t)$ (and the stronger one, group 2, less, $x_2(t) < x_2^o(t)$). Suppose that the weaker group is also the one that ex-ante values the asset less, then $(\lambda_1 d_2 - \lambda_2 d_1)(\bar{\theta}_2 - \bar{\theta}_1) > 0$. Thus, the differences between the valuation trading terms and the clearing trading terms go in the same direction and inefficiency increases (the second term in the expression of $\mathbb{E}[DWL]$ is positive). This is the case with primary dealers in a Treasury auction (strong group) who may value the bonds more than other direct bidders because they have more clout in reselling them. In Hortaçsu et al. (2018) it is found that the willingness of primary dealers to pay is no lower than that of other

\(^{40}\)Note that if we interpret the traders as producers competing to supply a fixed demand $Q$, then the condition $d_1 / d_2 = \lambda_1 / \lambda_2$ means that the ratio of the production of the two types of firms is aligned with the ratio of the slopes of their respective marginal costs. This condition guarantees productive efficiency provided that $\bar{\theta}_1 = \bar{\theta}_2$ and $\rho = 1$ and, since demand is fixed, this coincides with overall efficiency.
direct bidders (as well as of indirect bidders). However, group strength and preference strength need not be always aligned. This is the case, for example, when a weaker group of banks is hit by a crisis, then its valuation for liquidity may increase more than for a strong group of banks.

We can derive the following results.

**Proposition 7.**

1. Price impact \((d_1, d_2)\) and \(\mathbb{E}[DWL]\) may be negatively associated.
2. \(\mathbb{E}[DWL]\) increases with payoff asymmetry (i.e., \(|\overline{\theta}_2 - \overline{\theta}_1|\)) and with \(Q\) whenever group strength and preference strength are aligned (i.e., when the stronger group 2, with \(\lambda_1 > \lambda_2\), \(n_1 > n_2\), and \(\sigma_{\xi_1}^2 > \sigma_{\xi_2}^2\), values the asset no less than does the weaker group, \(\overline{\theta}_1 \leq \overline{\theta}_2\)).
3. When groups are symmetric, the expected deadweight loss is independent of \(Q\), and price impact \(d\) and \(\mathbb{E}[DWL]t\) are positively associated, given predicted values \(t\), for changes in information parameters. This need not be the case with asymmetric groups (e.g., for large \(Q\), \(d_i/d_j > \lambda_i/\lambda_j\) implies that \(\mathbb{E}[DWL]t\) increases in \(d_i\) and decreases in \(d_j\)).

Several comments are in order. First, in the empirical literature, price impact is typically a measure of deadweight loss, because there is an implicit assumption that price impact and \(\mathbb{E}[DWL]\) are positively associated. Proposition 7(i) shows that this may not always be the case. To see this result, suppose that \(Q = 0\). Then,

\[
\mathbb{E}[DWL] = \phi (n_2d_1 + n_1d_2)^2 \mathbb{E}[(t_1 - t_2)^2].
\]

So, \(\mathbb{E}[DWL]\) is equal to the product of two factors, \(\phi (n_2d_1 + n_1d_2)^2\) which increases in \(d_1\) and \(d_2\), and \(\mathbb{E}[(t_1 - t_2)^2]\). On the one hand, as \(d_1\) and \(d_2\) increase in \(\rho\) and in \(\sigma_{\xi_i}^2\), we have that the first factor increases in \(\rho\) and \(\sigma_{\xi_i}^2\). On the other hand, Equation (12) shows that the second factor, the difference in predicted values \(\mathbb{E}[(t_1 - t_2)^2]\), decreases when values are more correlated, \(\rho\) is higher, or signals are noisier (higher \(\sigma_{\xi_i}^2\)).\(^{41}\) Hence, it follows that \(\mathbb{E}[DWL]\) may increase or decrease in \(\rho\) and \(\sigma_{\xi_i}^2\). In particular, under full information (i.e., \(\sigma_{\xi_1}^2 = \sigma_{\xi_2}^2 = 0\)), both \(d_1\) and \(d_2\) are independent of \(\rho\); in this case \(\mathbb{E}[DWL]\) decreases with \(\rho\).\(^{42}\) By continuity, when both \(\sigma_{\xi_1}^2\) and \(\sigma_{\xi_2}^2\) are small enough, price impacts slightly increase when \(\rho\) increases, while \(\mathbb{E}[DWL]\) decreases. Consequently, in this case, price impacts and \(\mathbb{E}[DWL]\) are negatively associated.

\(^{41}\)\(\mathbb{E}[(t_1 - t_2)^2]\) vanishes when \(\rho\) approaches 1 or when there is no uncertainty (\(\sigma_{\theta}^2 = 0\)) provided \(\overline{\theta}_1 = \overline{\theta}_2\).

\(^{42}\)It is worth noting that both \(\mathbb{E}[TS]\) and \(\mathbb{E}[TS]\) increase in \(\mathbb{E}[(t_1 - t_2)^2]\), which is associated to gains from trade by the two groups. However, the weight of \(\mathbb{E}[(t_1 - t_2)^2]\) in \(\mathbb{E}[TS]\) is larger than in \(\mathbb{E}[TS]\) and, therefore, \(\mathbb{E}[DWL]\) increases in \(\mathbb{E}[(t_1 - t_2)^2]\). Similarly, if \(\rho = 0\), then \(d_1\) and \(d_2\) are independent of \(\sigma_{\xi_1}^2\) and \(\sigma_{\xi_2}^2\), from which it follows that \(\mathbb{E}[DWL]\) decreases with \(\sigma_{\xi_1}^2\) and \(\sigma_{\xi_2}^2\). If \(\sigma_{\xi_i}^2 = \infty\), then \(d_i\) is independent of \(\sigma_{\xi_j}^2\), \(i = 1, 2, j \neq i\), and \(\mathbb{E}[DWL]\) decreases with \(\sigma_{\xi_j}^2\).
Second, the impact of a small amount of asymmetry may be large. Suppose, for example, that the initial situation is symmetric for the groups and that $\sigma^2_2$ is low. Then, $\mathbb{E}[DWL]$ is close to zero since we have that $d_1/d_2 = \lambda_1/\lambda_2$. However, if $\lambda_2$ is lowered, then we can check that $d_1/d_2$ decreases and, therefore, $d_1/d_2 < \lambda_1/\lambda_2$, in which case $\mathbb{E}[DWL]$ may be quite large if $Q$ is large since $\mathbb{E}[DWL]$ is increasing in $(\lambda_1 d_2 - \lambda_2 d_1)^2 Q^2$. This is consistent with the results in Hortaçsu et al. (2018), who document a significant amount of efficiency losses due to heterogeneity at long maturities in US Treasury auctions.

It is worth highlighting that Proposition 7 shows that the total quantity traded does not affect the deadweight loss in symmetric environments, but in asymmetric environments this does matter. This means that how the asset is allocated across the two asymmetric groups can generate inefficiency.

### 4.4 Market integration

Our analysis can also shed light on the effects of integrating separated markets. Suppose that groups 1 and 2 operate in separate markets (auctions), that is, in market $i$ all the buyers ($n_i$) are of type $i$ and supply is $n_i Q / (n_1 + n_2)$. In this framework, given that all the individuals are identical in market $i$, the market clearing condition implies that the equilibrium quantities are given by $Q / (n_1 + n_2)$. Hence, the expected total surplus in market $i$, denoted by $\mathbb{E}[TS]_{Market\ i}$, satisfies

$$\mathbb{E}[TS]_{Market\ i} = \frac{n_i \bar{\theta}_i}{n_1 + n_2} Q - \frac{\lambda_i n_i}{(n_1 + n_2)^2} \frac{Q^2}{2}$$

and, consequently, the sum of expected total surplus in this setting

$$\mathbb{E}[TS]_{Market\ 1} + \mathbb{E}[TS]_{Market\ 2} = \frac{n_1 \bar{\theta}_1 + n_2 \bar{\theta}_2}{n_1 + n_2} Q - \frac{n_1 \lambda_1 + n_2 \lambda_2}{(n_1 + n_2)^2} \frac{Q^2}{2}.$$ 

Note that the previous expression is equal to the expected total surplus at the equally distributed allocation in the integrated market.\footnote{Wittwer (2017) compares ‘connected’ with ‘disconnected’ financial markets in which agents trade two perfectly divisible assets. In a connected market traders can make their demand for one security contingent on the price of the other. By contrast, interlinking demands across assets is not possible when each asset is traded in a separate disconnected market. This paper shows under which conditions both market structures generate the same allocation.}

As the allocation of the perfect competitive equilibrium maximizes $\mathbb{E}[TS]$ in this setup, then market integration increases expected total surplus, $\mathbb{E}[TS]$, if bidders behave as price-takers (strictly except if $\bar{\theta}_1 = \bar{\theta}_2$, $\sigma^2_{\varepsilon_1} = \sigma^2_{\varepsilon_2} = \infty$, and $\lambda_1 = \lambda_2$). In the latter case payoffs are symmetric among bidders of the two groups and there is no information on values. Therefore, there are no gains from trade among the groups.
Another framework in which it is readily seen that market integration increases $\mathbb{E}[TS]$ is when $Q$ is small enough and $|\bar{\theta}_1 - \bar{\theta}_2|$ is large enough. When $Q = 0$, we have that $\mathbb{E}[TS]_{\text{Market 1}} + \mathbb{E}[TS]_{\text{Market 2}} = 0$ and $\mathbb{E}[TS]$ at the integrated market will be positive provided that $E \left[ (t_1 - t_2)^2 \right] > 0$, a sufficient condition for which is that $\bar{\theta}_1 \neq \bar{\theta}_2$. By continuity, this result holds when $Q$ is small enough and $|\bar{\theta}_1 - \bar{\theta}_2|$ is large enough. To understand this result, note that in the integrated market, the bidders of the group who values the asset less become sellers, while in separated markets both groups are buyers. Thus, in the integrated market, the group that values the asset more keeps a higher quantity of the asset than in separated markets. Consequently, market integration increases $E[TS]$.

In general, the expression of expected deadweight loss given in (11) allows us to analyze the effect of integrating separated markets on the expected total surplus. This expression implies that if the optimal allocation is expected to be closer to the equilibrium allocation than the equally distributed allocation, then the expected deadweight loss at the equilibrium allocation will be lower than at the equally distributed allocation. This leads us to conclude that $\mathbb{E}[TS]$ is higher when the market is integrated. This is the case of symmetric bidders ($\bar{\theta}_1 = \bar{\theta}_2$, $\sigma^2_{\epsilon_1} = \sigma^2_{\epsilon_2}$, $\lambda_1 = \lambda_2 = \lambda$, and $n_1 = n_2 = n$). The optimal allocation, the equilibrium allocation and the equally distributed allocation are respectively given for $i \neq j$, $i = 1, 2$ by:

$$
  x^o_i (t) = \frac{t_i - t_j}{2\lambda} + \frac{Q}{2n}, \\
  x_i (t) = \frac{t_i - t_j}{2(d + \lambda)} + \frac{Q}{2n}, \quad \text{and} \\
  x^E_i = \frac{Q}{2n}.
$$

Notice that $x^o_i (t) > x_i (t) > x^E_i$ and $x^o_i (t) < x_j (t) < x^E_j$, whenever $t_i > t_j$, resulting in a positive effect of market integration on the expected total surplus.

On the basis of the above, market integration may only decrease $\mathbb{E}[TS]$ if $Q$ is large enough or $|\bar{\theta}_1 - \bar{\theta}_2|$ low enough, bidders behave strategically and are asymmetric (apart from the potential asymmetry in expected valuations). An illustrative example is the following. Suppose that $\bar{\theta}_1 = \bar{\theta}_2$, $\sigma^2_{\epsilon_1} = \sigma^2_{\epsilon_2} = \infty$, $\lambda_1 = \lambda_2 = \lambda$, and $n_1 > n_2$. In this case, for $i \neq j$, $i = 1, 2$,

$$
  x^o_i (t) = x^E_i = \frac{Q}{n_1 + n_2}, \quad i = 1, 2, \\
  x_i (t) = \frac{(d_j - d_i) Q n_j}{(n_i (d_j + \lambda) + n_j (d_i + \lambda)) (n_i + n_j)} + \frac{Q}{n_1 + n_2}.
$$

Hence, $x_1 (t) > x^E_1 = x^o_1 (t)$ and $x_2 (t) < x^E_2 = x^o_2 (t)$, i.e., the optimal allocation coincides with the equally distributed allocation and differs from the equilibrium allocation. In this case, we conclude that integrating separated markets reduces the expected total surplus. With
asymmetric precision of information \( (\sigma_{\xi_1}^2 \neq \sigma_{\xi_2}^2) \) and informative prices \( (\rho > 0) \) but otherwise symmetric groups, integration may be also welfare decreasing for large \( Q \). Note that this would not happen with uninformative prices, \( \rho = 0 \).

In summary, under symmetry or under perfect competition, market integration increases the expected total surplus. Market integration is also good in terms of \( \mathbb{E}[TS] \) when bidders of one group become sellers because they value the asset much less than the other group and \( Q \) is low enough. To find that market integration decreases the expected total surplus, we have to restrict our attention to a setup with strategic behavior, asymmetric groups and with bidders of both groups expected to be buyers when markets are unified. In such a case, gains from trade of integration may be overwhelmed by the inefficiency generated by group asymmetries and price impact.\(^{44}\)

5 Oligopsony with competitive fringe

We have claimed in Remark 6 that the equilibrium under imperfect competition converges to a price-taking equilibrium in the limit as the number of traders of both groups becomes large. We examine here what happens when only one group (group 1) is large. Let \( q \) denote the fixed per capita supply, that is, \( Q = (n_1 + n_2)q \).

**Proposition 8.** Let \( \rho \sigma_{\xi_1}^2 > 0 \). Suppose that \( n_1 \to \infty \) and \( n_2 < \infty \). Then:

(i) An equilibrium exists iff \( n_2 > \bar{n}_2 (\rho, \sigma_{\xi_1}^2, \sigma_{\xi_2}^2) \), where \( \bar{n}_2 \) is increasing in \( \rho \) and \( \sigma_{\xi_1}^2 \) and where \( \bar{n}_2 \) is decreasing in \( \sigma_{\xi_2}^2 \) whenever \( (2\rho - 1) \sigma_{\xi_1}^2 < 1 - \rho^2 \).\(^{45}\)

(ii) An agent in the large group absorbs the inelastic per capita supply in the limit \( \lim b_1 = q, \lim a_1 = \lim c_1 = 0 \) and retains some price impact \( \lim d_1 > 0 \), while an agent in the small group commands a higher degree of market power \( \lim d_2 > \lim d_1 \).\(^{46}\)

\(^{44}\)The results derived in this section are in line with Malamud and Rostek (2017). In a model with independent private information, these authors show that if traders are symmetric, then an integrated market maximizes welfare. By contrast, if traders have different risk preferences, then fragmented markets can allocate risk more efficiently, thus realizing gains from trade that cannot be reproduced in an integrated market. Babus and Kondor (2017) examine the effect of trade decentralization, comparing a centralized market as described in Vives (2011) and a decentralized market in which dealers can engage in bilateral transactions with other dealers. The paper shows that the effect of trade decentralization on welfare and liquidity is in general ambiguous.

\(^{45}\)In the particular case where \( n_2 = 1 \), the existence condition boils down to \( (2\rho - 1) \sigma_{\xi_1}^2 < 1 - \rho^2 \).

\(^{46}\)The limit expected quantity of a bidder of group 2 is given by

\[
\lim_{n_1 \to \infty} \mathbb{E}[x_2(t)] = \left( \bar{v}_2 - \bar{v}_1 + \left( \lim_{n_1 \to \infty} d_1 + \lambda_1 \right) q \right) / \left( \lim_{n_1 \to \infty} d_2 + \lambda_2 \right).
\]
(iii) In the limit, the price depends only on the valuations and price impact of agents in the competitive fringe: \( \lim_{n_1 \to \infty} p = \mathbb{E} [\theta_1 | s_1, s_2] - \left( \lim_{n_1 \to \infty} d_1 + \lambda_1 \right) q. \)

Equation (50) in Appendix A shows that, when \( n_2 = \tilde{n}_2 (\rho, \tilde{\sigma}_{\varepsilon_1}^2, \tilde{\sigma}_{\varepsilon_2}^2) \), the demand functions for bidders in group 2 would be completely inelastic \( \left( \lim_{n_1 \to \infty} c_2 = 0 \right) \). This explains why the inequality \( n_2 > \tilde{n}_2 (\rho, \tilde{\sigma}_{\varepsilon_1}^2, \tilde{\sigma}_{\varepsilon_2}^2) \) is required for the existence of equilibrium.

Neither group 1 nor group 2 has flat (i.e. price-taking) aggregate demand in the limit, and each group has some price impact. We see that an agent in the large group just absorbs the inelastic per capita supply, behaving like a "Cournot quantity setter", and keeping some power. Moreover, in this case we have that \( c_2 = 0 \). The reason why the large group retains price impact in the limit is because there is learning from the price (incomplete information and correlation of values, \( \rho \sigma_{\varepsilon_1}^2 > 0 \)). In this case, the aggregate demand of group 1 does not become flat, \( \lim_{n_1 \to \infty} n_1 c_1 < \infty \).

As \( n_1 \to \infty \), the weight of the price in \( \mathbb{E} [\theta_1 | s_1, p], \Lambda_1 \), tends to 1 (at the rate of \( 1/n_1 \)) and, since \( c_1 = (1 - \Lambda_1)/(d_1 + \lambda_1) \), we have that price responsiveness of group 1 (\( c_1 \)) converges to zero. Moreover, in this case we have that \( c_1 \to 0 \) at the rate of \( 1/n_1 \), leading to \( \lim_{n_1 \to \infty} d_i > 0, i = 1, 2 \). However, if \( \rho \sigma_{\varepsilon_2}^2 = 0 \), \( \lim_{n_1 \to \infty} n_1 c_1 = \infty \) and there is no price impact in the limit: \( \lim_{n_1 \to \infty} d_i = 0, i = 1, 2 \). Here group 1 does not learn from the price (\( \Lambda_1 = 0 \)) and, consequently, \( c_1 \) does not tend to 0 as \( n_1 \to \infty \). In this case, \( \lim_{n_1 \to \infty} c_1 = 1/\lambda_1 \), whereas \( \lim_{n_1 \to \infty} c_2 = 1/\lambda_2 \) when \( \rho = 0 \) and

\[ \lim_{n_1 \to \infty} c_2 = \left( 1 - \rho \tilde{\sigma}_{\varepsilon_2}^2 / (1 + \tilde{\sigma}_{\varepsilon_2}^2 - \rho^2) \right) / \lambda_2 \] when \( \tilde{\sigma}_{\varepsilon_1}^2 = 0 \).

If the small group is fully informed (\( \sigma_{\varepsilon_2}^2 = 0 \)) and the large group is entirely uninformed (\( \sigma_{\varepsilon_1}^2 \to \infty \)), then: \( \tilde{n}_2 = 2\rho \); an equilibrium always exists for \( n_2 > 2 \); and the equilibrium coefficients for group 2 are \( \lim_{n_1 \to \infty} b_2 = 0 \), and \( \lim_{n_1 \to \infty} a_2 = \lim_{n_1 \to \infty} c_2 = (n_2 - 2\rho) / ((n_2 - \rho) \lambda_2) \). In this case, the groups’ relative price impact is given by \( \lim_{n_1 \to \infty} (d_2/d_1) = 1 + \rho / (n_2 - \rho) \). As \( \rho \) increases, the relative price impact of group 2 also increases. This accords with the general theme that increased correlation in the presence of asymmetric information raises price impact.

### 6 Concluding remarks

The comparative static results obtained provide testable predictions. For example, an increase in transaction cost or noise in the signals in any group, or an increase in correlation of values

\(^{47}\)Baisa and Burkett (2018) also obtain that in a uniform-price auction with independent private values a single large bidder (with multi-unit demand) retains market power when he competes against many small bidders, each with single-unit demands. In our case, with correlated values, the fringe also retains market power when learning from the price.
across groups, should increase the price impact of traders in both groups. Furthermore, co-
movements in those parameters magnify the impact. The group that is stronger (because it has
more precise private information, faces lower transaction costs, and is more oligopsonistic) has
more price impact and must therefore receive a higher subsidy to behave competitively. This
result is consistent with the evidence of US Treasury auctions (Hortaçsu et al. 2018), where
primary dealers (strong group) exercise market power and earn significant surplus, on top of
having privileges in exchange for bidding minimum amounts in the auctions. The expected
deadweight loss increases with the quantity auctioned and with the degree of payoff (expected
valuation) asymmetry, provided the stronger group values the asset no less than does the weaker
group. A small amount of asymmetry may generate large deadweight losses. The link between
heterogeneity and efficiency losses is corroborated empirically for Treasury auctions by Hortaçsu
et al. (2018).

Our findings have policy implications. Consider a regulator who wants to reduce inefficiency
in an industry with two groups of firms (e.g., a small oligopolistic group and a competitive
fringe). This regulator must bear in mind that any intervention directed toward one group will
also affect the other’s behavior. In addition, for the regulator to induce competitive behavior it
should set a higher subsidy rate for the group that has better information, is more oligopson-
istic, and has lower transaction costs. The framework developed here can be adapted to study
competition policy, analyzing the effects of mergers and industry capacity redistribution.

Several extensions could be considered. A first one is to see how the results would be
modified in a discriminatory auction. A second one is to allow for traders in each group to
receive different signals. The latter is not a minor departure since, in general, the equilibrium
would be no longer privately revealing.

Appendix A

Proposition 1 follows from Lemmata A1 and A2.

Lemma A1. Let \( \rho < 1 \). In equilibrium, the demand function for a trader of type \( i \), \( i = 1, 2 \),
is given by \( X_i(s_i, p) = (\mathbb{E} [\theta_i | s_i, p] - p) / (d_i + \lambda_i) \), with \( d_i + \lambda_i > 0 \). The equilibrium coefficients

\footnote{Ausubel et al. (2014) find that, in symmetric auctions with decreasing linear marginal utility, the seller’s revenue is greater in a discriminatory auction than in a uniform-price auction. Pycia and Woodward (2017) demonstrate that a discriminatory pay-as-bid auction is revenue-equivalent to the uniform-price auction provided that supply and reserve prices are set optimally.}
satisfy the following system of equations:

\[
\begin{align*}
\frac{b_i}{d_i + \lambda_i} &= \left( (1 - \Xi_i) \bar{\theta}_i - \Psi_i \bar{\psi}_j - \frac{\Psi_i (n_i b_i + n_j b_j - Q)}{n_j a_j} \right), \\
\frac{a_i}{d_i + \lambda_i} &= \left( \Xi_i - \frac{n_i a_i}{n_j a_j} \Psi_i \right), \\
\frac{c_i}{d_i + \lambda_i} &= \left( 1 - \frac{\Psi_i (n_i c_i + n_j c_j)}{n_j a_j} \right),
\end{align*}
\]

where \( i, j = 1, 2, j \neq i \). Moreover, in equilibrium, \( a_i > 0, i = 1, 2 \).

**Proof:** Consider a trader of type \( i \). Recall that at the beginning of Subsection 3.1 we obtain

\[
X_i(s, p) = (\mathbb{E} [\theta_i | s, p] - p) / (d_i + \lambda_i) \quad \text{and} \quad \mathbb{E} [\theta_i | s, p] = \mathbb{E} [\theta_i | s, s_j].
\]

Since we are looking for strategies of the form \( X_i(s, p) = b_i + a_i s_i - c_i p \), from the market clearing condition we have that

\[
s_j = (n_i c_i + n_j c_j) p + Q - n_i (b_i + a_i s_i - n_j b_j) / (n_j a_j).
\]

Thus, from (5), it follows that

\[
\mathbb{E} [\theta_i | s, p] = (1 - \Xi_i) \bar{\theta}_i - \Psi_i \bar{\psi}_j + \Psi_i \left( \frac{Q - n_i b_i - n_j b_j}{n_j a_j} \right) + \left( \Xi_i - \frac{n_i a_i}{n_j a_j} \Psi_i \right) s_i + \Psi_i \left( \frac{n_i c_i + n_j c_j}{n_j a_j} \right) p.
\]

Substituting the foregoing expression in (3), and then identifying coefficients, we obtain the expressions for the demand coefficients given in (13)-(15).

Finally, we show the positiveness of the coefficients \( a_i, i = 1, 2 \). From (14), we get

\[
a_i = \Xi_i / (d_i + \lambda_i + n_i \Psi_i / (n_j a_j)), \quad i, j = 1, 2, j \neq i.
\]

Combining the previous expressions, we have

\[
a_i = \frac{n_j (\Xi_i \Xi_j - \Psi_i \Psi_j)}{n_i \Psi_i (d_j + \lambda_j) + \Xi_j n_j (d_i + \lambda_i)}, \quad i, j = 1, 2, j \neq i.
\]

Direct computation yields \( \Xi_i \Xi_j - \Psi_i \Psi_j = (1 - \rho^2) / ((1 + \sigma_{z_1}^2) (1 + \sigma_{z_2}^2) - \rho^2) > 0 \), whenever \( \rho < 1 \). Moreover, using the positiveness of \( d_i + \lambda_i, \Xi_i, \) and \( \Psi_i, i = 1, 2 \), we conclude that, in equilibrium, the coefficients \( a_i, i = 1, 2 \), are strictly positive. □

**Lemma A2.** In equilibrium,

\[
\begin{align*}
\frac{b_i}{
\Delta_i \frac{c_i}{d_i + \lambda_i}} &= \frac{\Xi_i n_i \Xi_j a_i}{n_i n_j \Xi_i \Xi_j - \Psi_i \Psi_j} Q + a_i \left( \frac{\Xi_i \bar{\theta}_i - \Psi_i \bar{\psi}_j}{\Xi_i \Xi_j - \Psi_i \Psi_j} - \bar{\theta}_i \right), \\
\frac{a_i}{\Delta_i} &= \Delta_i c_i, \\
\frac{c_1}{\lambda_1} &= \left( 1 - \Xi_1 \Delta_1^{-1} - \frac{n_1}{n_2} \left( 1 - \Xi_1 \Delta_1^{-1} \right) \frac{z}{z - \frac{1}{(n_1 - 1) \frac{z}{n_2}} \right), \quad \text{and} \\
\frac{c_2}{\lambda_2} &= \left( 1 - \Xi_2 \Delta_2^{-1} - \frac{n_2}{n_1} \left( 1 - \Xi_2 \Delta_2^{-1} \right) \frac{1}{z - \frac{1}{n_1 z + n_2 - 1}} \right)
\end{align*}
\]

where \( \Delta_i = 1 / \left( 1 + (1 + \rho^{-1}) \sigma_{z_i}^2 \right) \), \( i, j = 1, 2, j \neq i \). Moreover, \( z \equiv c_1 / c_2 \) is the unique positive
Proposition 1. is equivalent to (18). Using (18) in (15), we get the expression for previous formula and substituting the resulting expression in (23), we obtain a formula which

\[n \quad \text{Hence,} \quad n_i c_i + n_j c_j = n_i a_i \frac{n_j a_j - \Psi_i (n_i c_i + n_j c_j)}{n_j a_i \Xi_i - n_i a_j \Psi_i} + n_j a_j \frac{n_i a_i - \Psi_j (n_i c_i + n_j c_j)}{n_i a_i \Xi_i - n_j a_j \Psi_j}. \]

Isolating \( n_i b_i + n_j b_j \) in the previous formula and substituting the resulting expression in (22), Expression (17) is obtained.

Concerning the expression for \( a_i \), substituting (21) in (15), it follows that

\[ c_i = a_i \left( 1 - \frac{n_i c_i + n_j c_j}{n_j a_j} \right) \left( \Xi_i - \frac{n_i a_i}{n_j a_j} \Psi_i \right), i, j = 1, 2, j \neq i. \]

Hence, \( n_i c_i + n_j c_j \) in the previous formula and substituting the resulting expression in (23), we obtain a formula which is equivalent to (18). Using (18) in (15), we get the expression for \( c_i \) given in the statement of Proposition 1.

In relation to \( c_1 \) and \( c_2 \), using the expression for \( d_i \) and (18), (21) implies that

\[ \lambda_i = \left( \frac{\Xi_i}{\Delta_i} - \frac{n_i \Psi_i c_i}{n_j \Delta_j c_j} \right) c_i^{-1} - ((n_i - 1) c_i + n_j c_j)^{-1}, i, j = 1, 2, j \neq i. \]

or, since

\[ \Psi_i \Delta_i^{-1} = 1 - \Xi_i \Delta_i^{-1}, \]
\[ \lambda_i = \left( \Xi_i \Delta_i^{-1} - \frac{n_i}{n_j} (1 - \Xi_i \Delta_i^{-1}) \frac{w_i}{c_j} \right) c_i^{-1} - ((n_i - 1) c_i + n_j c_j)^{-1}, \quad i, j = 1, 2, j \neq i, \] which imply (19) and (20) since \( z = c_1 / c_2 \). Moreover, dividing the previous two equalities, it follows that

\[
\frac{\lambda_1}{\lambda_2} = \frac{\Xi_1 \Delta_1^{-1} - \frac{n_1}{n_2} (1 - \Xi_1 \Delta_1^{-1}) z - z ((n_1 - 1) z + n_2)^{-1}}{\Xi_2 \Delta_2^{-1} z - \frac{n_2}{n_1} (1 - \Xi_2 \Delta_2^{-1}) z (n_1 z + n_2 - 1)^{-1}}.
\tag{25}
\]

After some algebra, (25) is equivalent to \( G(z) = 0 \), where \( G(\zeta) \) is the polynomial given in the statement of this lemma. Notice that \( G(0) < 0 \) and \( \lim_{\zeta \to -\infty} G(\zeta) = \infty \). Consequently, there exists a positive root of \( G(\zeta) \). Furthermore, we have that \( g_2/n_1 > g_1/n_2 \). Combining this inequality with the fact that \( g_3 > 0 \) and \( g_0 < 0 \) allows us to conclude that at least there is only one sign change of the coefficients of \( G(\zeta) \). To show that, we distinguish 3 cases:

- (1) \( 0 < \frac{g_2}{n_1} < \frac{g_1}{n_2} \). This implies that \( 0 \geq g_2 \) and \( 0 > g_1 \). As \( g_3 > 0 \) and \( g_0 < 0 \), we conclude that there is only one sign change of the coefficients of \( G(\zeta) \).
- (2) \( \frac{g_2}{n_1} > 0 \geq \frac{g_1}{n_2} \). This implies that \( g_2 > 0 \geq g_1 \). As \( g_3 > 0 \) and \( g_0 < 0 \), we conclude that there is only one sign change of the coefficients of \( G(\zeta) \).
- (3) \( \frac{g_2}{n_1} > \frac{g_1}{n_2} > 0 \). This implies that \( g_2 > 0 \) and \( g_1 > 0 \). As \( g_3 > 0 \) and \( g_0 < 0 \), we conclude that there is only one sign change of the coefficients of \( G(\zeta) \).

This property implies that there exists only one sign change in the coefficients of \( G(\zeta) \). Applying the Descartes’ rule, we conclude that there exists a unique positive root of \( G(\zeta) \).

**Proposition 2A.** Let \( \rho < 1 \).

- (a) There exists an equilibrium if and only if \( c_i > 0, i = 1, 2, \) where

\[ c_1 = \frac{H_N(z)}{(n_1 - 1) z + n_2} \quad \text{and} \quad c_2 = \frac{H_D(z)}{(n_1 z + n_2 - 1) n_1 z \lambda_2}, \tag{26} \]

where \( z = c_1 / c_2 \) and the expressions of \( H_N(\zeta) \) and \( H_D(\zeta) \) are given by

\[
H_N(\zeta) = n_2^2 \Xi_1 \Delta_1^{-1} + n_2 (\Xi_1 \Delta_1^{-1} (2n_1 - 1) - (n_1 + 1)) \zeta - (n_1 - 1) (1 - \Xi_1 \Delta_1^{-1}) n_1 \zeta^2 \quad \text{and}
\]
\[
H_D(\zeta) = -n_2 (n_2 - 1) (1 - \Xi_2 \Delta_2^{-1}) + n_1 (\Xi_2 \Delta_2^{-1} (2n_2 - 1) - (n_2 + 1)) \zeta + n_1^2 \Xi_2 \Delta_2^{-1} \zeta^2.
\]

- (b) Uninformative prices. When \( \rho = 0, \sigma_{\epsilon_i}^2 = 0, \) or \( \sigma_{\epsilon_i}^2 = \infty, i = 1, 2, \) equilibrium exists iff \( n_1 + n_2 \geq 3 \).

- (c) Informative prices.

  - (c.1) Let \( \rho \sigma_{\epsilon_i}^2 \sigma_{\epsilon_j}^2 > 0 \). Then, \( c_i > 0 \) (i.e, 1, 2) if and only if \( \overline{z}_N > \overline{z}_D \), where \( \overline{z}_N \) and \( \overline{z}_D \) denote the highest root of \( H_N(\zeta) \) and \( H_D(\zeta) \), respectively.

  - (c.2) Let \( \rho \sigma_{\epsilon_i}^2 > 0 \) and \( \sigma_{\epsilon_i}^2 = 0, i \neq j \). Then, \( c_i > 0 \) (i.e, 1, 2) if \( n_j \geq 2 \), or if \( n_j = 1, n_i \) large enough and \( \rho \) low enough.

**Remark 8.** For an equilibrium to exist we must have \( c_i > 0 \) (i.e, 1, 2) and these inequalities hold if and only if \( \overline{z}_D < z < \overline{z}_N \). If \( n_1 = 1 \) and \( n_2 = 1 \), then \( \overline{z}_N = 1 / (2 \Delta_1 \Xi_1^{-1} - 1) \) and
\[ z_D = 2\Delta_2 \Xi^{-1} - 1. \] Since \( \Delta_i \Xi^{-1} \geq 1, i = 1, 2, \) and \( \Delta_1 \Xi^{-1} = \Delta_2 \Xi^{-1} = 1 \) do not hold, we can use direct computation to obtain \( z_N < z_D. \) Applying Proposition 2A, we conclude that no equilibrium exists in this case. Therefore, \( n_1 + n_2 \geq 3 \) is a necessary condition for the existence of an equilibrium.

**Remark 9.** In c.1, we obtain that \( \lim_{\lambda_1 \to 0} z = z_N \) and \( \lim_{\lambda_2 \to 0} z = z_D. \)

**Remark 10.** In c.2, when \( \sigma_{\varepsilon_i}^2 = 0, z_D = 1/n_1 \) if \( n_2 = 1, \) whereas \( z_D = 0 \) if \( n_2 \geq 2. \)

**Proof:** (a) (Necessity). From Proposition 1 we know that \( a_i > 0, i = 1, 2, \) whenever \( \rho < 1. \) Combining this property with expressions given in (18), we have that, in equilibrium, the coefficients \( c_i, i = 1, 2, \) are strictly positive. Moreover, (19) and (20) can be rewritten as the expressions given in (26).

(Sufficiency). Suppose that the candidates equilibrium coefficients \( c_1 \) and \( c_2 \) are positive and satisfy (26). Then, the ratio \( z = c_1/c_2 > 0 \) and satisfies (25). Then, we conclude that an equilibrium exists and it is unique since we know that (25) has a unique positive solution. Finally, substituting this value of \( z \) in the expressions stated in Lemma A2, we obtain the equilibrium coefficients of the demand functions.

(b) When \( \rho = 0 \) or \( \sigma_{\varepsilon_i}^2 = 0, i = 1, 2, \) the demand functions are given by

\[ X_i(s_i, p) = (\mathbb{E}[\theta_i | s_i] - p) / (d_i + \lambda_i), i = 1, 2, \]

while when \( \sigma_{\varepsilon_i}^2 = \infty, i = 1, 2, \) the demand functions hold

\[ X_i(s_i, p) = (\bar{\theta}_i - p) / (d_i + \lambda_i), i = 1, 2. \]

Moreover recall that the SOCs imply \( d_i + \lambda_i > 0. \) Moreover, in all these cases we can express the coefficients of the demand functions in terms of \( d_i, i = 1, 2. \) In particular, \( c_i = 1 / (d_i + \lambda_i) > 0, i = 1, 2. \) Given our expression for \( d_i, \) we characterize \( d_i, i = 1, 2, \) as the positive solutions of the following system of equations:

\[ d_i = \left( \frac{n_i - 1}{d_i + \lambda_i} + \frac{n_j}{d_j + \lambda_j} \right)^{-1} \text{ for } i, j = 1, 2 \text{ and } j \neq i. \]

After some algebra, we conclude that this system has positive solutions if and only if \( n_1 + n_2 \geq 3. \)

(c.1) (Necessity). Let \( z_N \) and \( z_D \) denote the highest root of \( H_N(\zeta) \) and \( H_D(\zeta), \) respectively. Notice that the positiveness of \( c_i, i = 1, 2, \) is equivalent to \( z_N > z > z_D. \) Therefore, \( z_N > z_D. \)

(Sufficiency). Suppose that \( z_N > z_D. \) Recall that Lemma A2 shows that there exists a unique positive value of \( z \) that solves (25), which can be rewritten as

\[ \frac{\lambda_1}{\lambda_2} = \frac{n_1 (n_2 - 1 + n_1 z) H_N(z)}{(n_2 + (n_1 - 1) z) n_2 H_D(z)}. \] (27)
This implies that $\bar{z}_N > z > \bar{z}_D$. Notice that these inequalities guarantee the positiveness of $c_i$, $i = 1, 2$.

**Case 2** Suppose that $\rho \sigma_{\xi_1}^2 > 0$ and $\sigma_{\xi_2}^2 = 0$. In this case $\Xi_2 \Delta_2^{-1} = 1$ and, hence, $H_D(\zeta) = \zeta n_1 (n_2 + \zeta n_1 - 2)$. On the one hand, if $n_2 = 1$, then $\bar{z}_D = 1/n_1$. As in c.1) the condition that guarantees the existence of equilibrium is $\bar{z}_N > \bar{z}_D$, which is equivalent to $n_1 (2 \Xi_1 \Delta_1^{-1} - 1) > \Xi_1 \Delta_1^{-1}$, i.e., $\Xi_1 \Delta_1^{-1} > 1/2$ and $n_1 > \Xi_1 \Delta_1^{-1}/(2 \Xi_1 \Delta_1^{-1} - 1)$ or, using the expressions of $\Xi_1$ and $\Delta_1$, $1 - \rho^2 + (1 - 2 \rho) \hat{\sigma}_{\xi_1}^2 > 0$ and $n_1 > 1 + \hat{\sigma}_{\xi_1}^2 \rho/(1 - \rho^2 + (1 - 2 \rho) \hat{\sigma}_{\xi_1}^2)$, which implies when $\rho$ is low enough and $n_1$ is large enough.

On the other hand, if $n_2 \geq 2$, $H_D(\zeta) > 0$ for all $\zeta > 0$ and, therefore, we have that $c_2 > 0$ is satisfied. The positiveness of $c_1$ requires that $\bar{z}_N > z$. But, this inequality holds since $z$ solves Equation (27). To sum up, when $\sigma_{\xi_2}^2 = 0$, an equilibrium exists if $n_2 = 1$, $n_1$ large enough and $\rho$ low enough, or if $n_2 \geq 2$.

Now, suppose that $\rho \sigma_{\xi_2}^2 > 0$ and $\sigma_{\xi_1}^2 = 0$. In this case $\Xi_1 \Delta_1^{-1} = 1$ and, hence, $H_N(\zeta) = n_2^2 + n_2 (n_1 - 2) \zeta$. On the one hand, if $n_1 = 1$, then $\bar{z}_N = n_2$. As in c.1) the condition that guarantees the existence of equilibrium is $\bar{z}_N > \bar{z}_D$, which is equivalent to $n_2 (2 \Xi_2 \Delta_2^{-1} - 1) > \Xi_2 \Delta_2^{-1}$, i.e., $\Xi_2 \Delta_2^{-1} > 1/2$ and $n_2 > \Xi_2 \Delta_2^{-1}/(2 \Xi_2 \Delta_2^{-1} - 1)$ or, using the expressions of $\Xi_2$ and $\Delta_2$, $1 - \rho^2 + (1 - 2 \rho) \hat{\sigma}_{\xi_2}^2 > 0$ and $n_2 > 1 + \hat{\sigma}_{\xi_2}^2 \rho/(1 - \rho^2 + (1 - 2 \rho) \hat{\sigma}_{\xi_2}^2)$, which implies when $\rho$ is low enough and $n_2$ is large enough.

On the other hand, if $n_1 \geq 2$, $H_N(\zeta) > 0$ for all $\zeta > 0$ and, therefore, we have that $c_1 > 0$ is satisfied. The positiveness of $c_2$ requires that $z > \bar{z}_D$. But, this inequality holds since the equilibrium value, $z$, solves Equation (27). To sum up, when $\sigma_{\xi_1}^2 = 0$, an equilibrium exists if $n_1 = 1$, $n_2$ large enough and $\rho$ low enough, or if $n_1 \geq 2$.

**Lemma 2A.** The condition $\bar{z}_N > \bar{z}_D$ given in the statement of Proposition 2A is satisfied in the following cases:

(i) if $\rho < 1$ and $n_1, n_2$ are large enough;

(ii) given $n_i, n_j$ is large enough and $\rho$ low enough, for $i, j = 1, 2$ and $j \neq i$.

**Proof:** We distinguish two cases: $n_1 > 1$ and $n_1 = 1$.

**Case 1:** $n_1 > 1$. In this case

$$
\bar{z}_N = \frac{n_2 \left( (n_1 - 1) (2 \Xi_1 \Delta_1^{-1} - 1) - (2 - \Xi_1 \Delta_1^{-1}) + \sqrt{(2 - \Xi_1 \Delta_1^{-1})^2 + (n_1 - 1) (n_1 + 3 - 6 \Xi_1 \Delta_1^{-1})} \right)}{2n_1(n_1 - 1) (1 - \Xi_1 \Delta_1^{-1})}
$$

and

$$
\bar{z}_D = \frac{n_2 + 1 - \Xi_2 \Delta_2^{-1} (2n_2 - 1) + \sqrt{(2 - \Xi_2 \Delta_2^{-1})^2 + (n_2 - 1) (n_2 + 3 - 6 \Xi_2 \Delta_2^{-1})}}{2 \Xi_2 \Delta_2^{-1} n_1}.
$$

(28) (29)
Proposition 2A indicates that an equilibrium exists if and only if \( \Xi_N > \Xi_D \), or equivalently, \( n_1 \Xi_N/n_2 > n_1 \Xi_D/n_2 \). Using the expressions of \( \Xi_N \) and \( \Xi_D \), we have that \( n_1 \Xi_N/n_2 \) is increasing in \( n_1 \) and \( n_1 \Xi_D/n_2 \) is decreasing in \( n_2 \). Taking limits, it follows that

\[
\lim_{n_1 \to \infty} n_1 \Xi_N/n_2 = \Xi_1 \Delta_1^{-1}/ (1 - \Xi_1 \Delta_1^{-1}) \quad \text{and} \quad \lim_{n_2 \to \infty} n_1 \Xi_D/n_2 = (1 - \Xi_2 \Delta_2^{-1})/(\Xi_2 \Delta_2^{-1}).
\]

Moreover, using the expressions of \( \Xi_i \) and \( \Delta_i \), \( i = 1, 2 \), we have that

\[
\frac{\Xi_1 \Delta_1^{-1} - 1 - \Xi_2 \Delta_2^{-1}}{1 - \Xi_1 \Delta_1^{-1}} = \frac{(1 - \rho^2) \left(1 + \rho + \bar{\sigma}_{\varepsilon_1}^2\right) \left(1 + \bar{\sigma}_{\varepsilon_1}^2\right)}{\rho \bar{\sigma}_{\varepsilon_1}^2 \left(1 + \rho + \bar{\sigma}_{\varepsilon_2}^2\right) \left(1 - \rho^2 + \bar{\sigma}_{\varepsilon_1}^2\right)} > 0.
\]

Hence, we get that, as \( \rho < 1 \), \( \lim_{n_1 \to \infty} n_1 \Xi_N/n_2 > \lim_{n_2 \to \infty} n_1 \Xi_D/n_2 \). This implies that whenever \( \rho < 1 \) and \( n_1 \) and \( n_2 \) large enough, the existence of the equilibrium is guaranteed.

Consider now a fixed positive integer \( n_1 \), such that \( n_1 > 1 \). Using the fact that \( \Xi_N \) is the positive root of \( H_N(Z) \), it follow that \( n_1 \Xi_N/n_2 > \Xi_1 \Delta_1^{-1}/(2 - \Xi_1 \Delta_1^{-1}) \). Moreover,

\[
\Xi_1 \Delta_1^{-1}/(2 - \Xi_1 \Delta_1^{-1}) > (1 - \Xi_2 \Delta_2^{-1})/(\Xi_2 \Delta_2^{-1}),
\]

whenever \( \rho \) is low enough. Therefore,

\[
n_1 \Xi_N/n_2 > \Xi_1 \Delta_1^{-1}/(2 - \Xi_1 \Delta_1^{-1}) > (1 - \Xi_2 \Delta_2^{-1})/(\Xi_2 \Delta_2^{-1}) = \lim_{n_2 \to \infty} n_1 \Xi_D/n_2.
\]

Hence, we conclude that if \( n_2 \) is large enough, as \( n_1 \Xi_N/n_2 \) is decreasing in \( n_2 \), the previous inequalities imply that \( n_1 \Xi_N/n_2 > n_1 \Xi_D/n_2 \) or, equivalently, \( \Xi_N > \Xi_D \). Applying Proposition 2A, it follows that in this case there exists an equilibrium provided that \( n_2 \) is high enough and \( \rho \) low enough.

Consider now a fixed positive integer \( n_2 \), such that \( n_2 \geq 1 \), and assume again that \( \rho < 1 \). Using the fact that \( \Xi_D \) is the positive root of \( H_D(\zeta) \), it follow that \( n_1 \Xi_D/n_2 \leq (2 - \Xi_2 \Delta_2^{-1})/(\Xi_2 \Delta_2^{-1}) \). In addition, when \( \rho \) is low enough, then we have that

\[
(2 - \Xi_2 \Delta_2^{-1})/(\Xi_2 \Delta_2^{-1}) < \Xi_1 \Delta_1^{-1}/(1 - \Xi_1 \Delta_1^{-1}) = \lim_{n_1 \to \infty} n_1 \Xi_N/n_2.
\]

Thus, we have that \( n_1 \Xi_D/n_2 < \lim_{n_1 \to \infty} n_1 \Xi_N/n_2 \). Using the fact that \( n_1 \Xi_N/n_2 \) increases with \( n_1 \), we have that, when \( n_1 \) is high enough, \( n_1 \Xi_D/n_2 < n_1 \Xi_N/n_2 \) or, equivalently, \( \Xi_D < \Xi_N \), which guarantees the existence of equilibrium. To sum up, we have that given \( n_2 \), there exists an equilibrium provided that \( n_1 \) is high enough and \( \rho \) low enough.

**Case 2:** \( n_1 = 1 \). In this case, we have that \( \Xi_N = n_2 \Xi_1 \Delta_1^{-1}/(2 - \Xi_1 \Delta_1^{-1}) \) and

\[
\Xi_D = \frac{n_2 + 1 - \Xi_2 \Delta_2^{-1}(2n_2 - 1) + \sqrt{(2 - \Xi_2 \Delta_2^{-1})^2 + (n_2 - 1)(n_2 + 3 - 6\Xi_2 \Delta_2^{-1})}}{2\Xi_2 \Delta_2^{-1}}.
\]
Furthermore, whenever $\rho$ is low enough, (30) holds. Therefore, it follows that

$$
\frac{z}{n_2} = \frac{X_1 \Delta_1^{-1}}{(2 - X_1 \Delta_1^{-1})} > \frac{1 - X_2 \Delta_2^{-1}}{(X_2 \Delta_2^{-1})} = \lim_{n_2 \to \infty} \frac{z_D}{n_2}.
$$

Using the fact that $z_D/n_2$ decreases with $n_2$, the previous inequality implies that $z_N/n_2 > z_D/n_2$ whenever $n_2$ is high enough, i.e., $z_N > z_D$, which guarantees the existence of equilibrium. To sum up, we have that when $n_1 = 1$, there exists an equilibrium provided that $n_2$ is high enough and $\rho$ low enough. $\blacksquare$

**Proof of Proposition 2**: This proposition directly follows from Proposition 2A and Lemma 2A. $\blacksquare$

**Remark (symmetric groups)**. Let $n_i = n$, $\lambda_i = \lambda$, and $\sigma_{\xi_i}^2 = \sigma_{\xi_j}^2$, $i = 1, 2$. Here $z = 1$ in equilibrium. From Proposition 2A we know that, if an equilibrium exists, then the value of $z$ is in the interval $(z_D, z_N)$. It follows that $z_N > 1 > z_D$ or, equivalently, that $H_N(1) > 0$ and $H_D(1) > 0$. After performing some algebra, we find that the foregoing inequalities are satisfied iff $n > 1 + \rho \sigma_{\xi}^2 / ((1 - \rho)(1 + \rho + \sigma_{\xi}^2))$, where $\sigma_{\xi}^2 = \sigma_{\xi_1}^2 / \sigma_{\xi_2}^2$.

**Proof of Proposition 3**: Let $\rho \sigma_{\xi_1}^2 / \sigma_{\xi_2}^2$. In what follows we prove the following comparative statics results for $i, j = 1, 2$, $i \neq j$:

(a) $\partial b_i / \partial \bar{\theta}_i > 0$, $\partial a_i / \partial \bar{\theta}_i = 0$, and $\partial c_i / \partial \bar{\theta}_i = 0$,

(b) $\partial b_i / \partial \bar{\theta}_j < 0$, $\partial a_i / \partial \bar{\theta}_j = 0$, and $\partial c_i / \partial \bar{\theta}_j = 0$,

(c) $\partial b_i / \partial Q > 0$, $\partial a_i / \partial Q = 0$, and $\partial c_i / \partial Q = 0$,

(d) $\partial a_i / \partial \lambda_i < 0$ and $\partial c_i / \partial \lambda_i < 0$,

(e) $\partial a_i / \partial \lambda_j < 0$ and $\partial c_i / \partial \lambda_j < 0$,

(f) $\partial a_i / \partial \rho < 0$ and $\partial c_i / \partial \rho < 0$,

(g) $\partial a_i / \partial \sigma_{\xi_i}^2 < 0$ and $\partial c_i / \partial \sigma_{\xi_i}^2 < 0$,

(h) $\partial a_i / \partial \sigma_{\xi_j}^2 < 0$ and $\partial c_i / \partial \sigma_{\xi_j}^2 < 0$,

(i) $\partial d_i / \partial n_i < 0$ and $\partial d_j / \partial n_i < 0$.

From Lemma A1, we know that the equilibrium coefficients that depend on $\bar{\theta}_i$, $\bar{\theta}_j$ and $Q$ are $b_1$ and $b_2$. Using Lemma A2 and after some algebra, the results given in (a), (b) and (c) are obtained. In what follows, without any loss of generality, let $i = 1$. First, we prove that $\partial z / \partial \lambda_1 < 0$. From Lemma A2, we know that $z$ is the unique positive solution that satisfies:

$$
\frac{\lambda_1}{\lambda_2} - \frac{N(z)}{D(z)} = 0, \quad (31)
$$

36
where
\[ N(z) = \Xi_1 \Delta_1^{-1} - n_1 (1 - \Xi_1 \Delta_1^{-1}) \frac{z}{n_2 - z} \left((n_1 - 1) z + n_2\right)^{-1} \] and
\[ D(z) = \Xi_2 \Delta_2^{-1} - n_2 (1 - \Xi_2 \Delta_2^{-1}) \frac{1}{n_1 - z} \left(n_1 z + n_2 - 1\right)^{-1}, \]
with \( \Xi_i \Delta_i^{-1} = \left(1 - \rho^2 + \hat{\sigma}_{\varepsilon_i}^2\right)(1 + \rho + \hat{\sigma}_{\varepsilon_i}^2) \left((1 + \hat{\sigma}_{\varepsilon_i}^2)(1 + \hat{\sigma}_{\varepsilon_i}^2) - \rho^2\right)(1 + \rho)\). Applying the Implicit Function Theorem,
\[
\frac{\partial z}{\partial \lambda_i} = -\frac{\partial (\lambda_1/\lambda_2 - N(z)/D(z))}{\partial \lambda_i} / \partial \lambda_i, \quad i = 1, 2.
\]
As \( \partial (\lambda_1/\lambda_2 - N(z)/D(z)) \)/\( \partial \lambda_1 > 0 \), \( \partial (\lambda_1/\lambda_2 - N(z)/D(z)) \)/\( \partial \lambda_2 < 0 \), and
\[
\frac{\partial (\lambda_1/\lambda_2 - N(z)/D(z))}{\partial z} > 0,
\]
because of \( z \in (\tau_D, \tau_N) \), we conclude that \( \partial z/\partial \lambda_1 < 0 \) and \( \partial z/\partial \lambda_2 > 0 \).

Next, we study the relationship between \( \ell \)'s and \( \lambda_1 \). Differentiating (20), we have
\[
\frac{\partial c_2}{\partial \lambda_1} = \frac{\partial c_2}{\partial z} \frac{\partial z}{\partial \lambda_1} = \frac{1}{\lambda_2} \left( \frac{n_2 (1 - \Xi_2 \Delta_2^{-1})}{n_1 z^2} + \frac{n_1}{(n_1 z + n_2 - 1)^2} \right) \frac{\partial z}{\partial \lambda_1} < 0,
\]
since \( \partial z/\partial \lambda_1 < 0 \). Moreover, as \( c_1 = z c_2 \), it follows that \( \partial c_1/\partial \lambda_1 = (\partial z/\partial \lambda_1) c_2 + z (\partial c_2/\partial \lambda_1) < 0 \), because of the positiveness of \( c_2 \) and \( z \), and the negativeness of \( \partial z/\partial \lambda_1 \) and \( \partial c_2/\partial \lambda_1 \). In relation to \( a_1 \) and \( a_2 \), from (18), direct computation yields \( \partial a_1/\partial \lambda_1 < 0 \) and \( \partial a_2/\partial \lambda_1 < 0 \), since \( \partial c_1/\partial \lambda_1 < 0 \) and \( \partial c_2/\partial \lambda_1 < 0 \).

Now, we study how the correlation coefficient \( \rho \) affects \( a_1 \). Let \( y = a_1/a_2 \). As \( a_1 = \Delta_1 c_1 \) and \( a_2 = \Delta_2 c_2 \), then \( z = \Delta_2 y/\Delta_1 \). Substituting this expression in (25), and after some algebra, we have that
\[
\frac{\lambda_1}{\lambda_2} y = \frac{\tilde{N}(y, \rho)}{\tilde{D}(y, \rho)},
\]
where \( \tilde{N}(y, \rho) = \frac{1 - \rho^2 + \hat{\sigma}_{\varepsilon_1}^2 - \rho n_1 \hat{\sigma}_{\varepsilon_1}^2 \rho_y}{(1 + \hat{\sigma}_{\varepsilon_1}^2)(1 + \hat{\sigma}_{\varepsilon_2}^2) - \rho^2} - \left((n_1 - 1) \frac{1 + \rho + \hat{\sigma}_{\varepsilon_2}^2}{1 + \rho} + n_2 \frac{1 + \rho + \hat{\sigma}_{\varepsilon_2}^2}{1 + \rho} \right) \frac{1}{y} \) and
\[
\tilde{D}(y, \rho) = \frac{1 - \rho^2 + \hat{\sigma}_{\varepsilon_1}^2 - \rho n_1 \hat{\sigma}_{\varepsilon_1}^2 \rho_y}{(1 + \hat{\sigma}_{\varepsilon_1}^2)(1 + \hat{\sigma}_{\varepsilon_2}^2) - \rho^2} - \left(n_1 \frac{1 + \rho + \hat{\sigma}_{\varepsilon_2}^2}{1 + \rho} + (n_2 - 1) \frac{1 + \rho + \hat{\sigma}_{\varepsilon_2}^2}{1 + \rho} \right) \frac{1}{y}. \]
Moreover, \( a_1 = \tilde{N}(y, \rho)/\lambda_1 \) and \( a_2 = \tilde{D}(y, \rho)/\lambda_2 \). Hence,
\[
\frac{\partial a_1}{\partial \rho} = \frac{(\partial \tilde{N}(y, \rho)/\partial y)(\partial y/\partial \rho) + \partial \tilde{N}(y, \rho)/\partial \rho}{\lambda_1}.
\]
Thus, in order to show $\partial a_1/\partial \rho < 0$, it suffices to prove that
\[
\frac{\partial \tilde{N}(y, \rho)}{\partial y} \frac{\partial y}{\partial \rho} + \frac{\partial \tilde{N}(y, \rho)}{\partial \rho} < 0.
\] 
(34)

Direct computation yields $\partial \tilde{N}(y, \rho)/\partial y < 0$. Then, (34) is equivalent to
\[
\frac{\partial y}{\partial \rho} > -\frac{\partial \tilde{N}(y, \rho)/\partial \rho}{\partial \tilde{N}(y, \rho)/\partial y}.
\] 
(35)

Moreover, recall that $y$ in equilibrium is the unique positive value that satisfies (33). Thus, applying the Implicit Function Theorem, it follows that
\[
\frac{\partial y}{\partial \rho} = \frac{\partial \left( \lambda_1 y/\lambda_2 - \tilde{N}(y, \rho)/\tilde{D}(y, \rho) \right) / \partial \rho}{\partial \left( \lambda_1 y/\lambda_2 - \tilde{N}(y, \rho)/\tilde{D}(y, \rho) \right) / \partial y}.
\]

Then, (35) can be rewritten as
\[
-\frac{\partial \left( \lambda_1 y/\lambda_2 - \tilde{N}(y, \rho)/\tilde{D}(y, \rho) \right) / \partial \rho}{\partial \left( \lambda_1 y/\lambda_2 - \tilde{N}(y, \rho)/\tilde{D}(y, \rho) \right) / \partial y} > -\frac{\partial \tilde{N}(y, \rho)/\partial \rho}{\partial \tilde{N}(y, \rho)/\partial y}.
\]

or using the fact that in equilibrium $\partial \left( \lambda_1 y/\lambda_2 - \tilde{N}(y, \rho)/\tilde{D}(y, \rho) \right) / \partial y > 0$, (35) is satisfied iff
\[
-\frac{\partial \left( \lambda_1 y/\lambda_2 - \tilde{N}(y, \rho)/\tilde{D}(y, \rho) \right) / \partial \rho}{\partial \left( \lambda_1 y/\lambda_2 - \tilde{N}(y, \rho)/\tilde{D}(y, \rho) \right) / \partial y} > -\left( \frac{\partial \tilde{N}(y, \rho)/\partial \rho}{\partial \tilde{N}(y, \rho)/\partial y} \right) \frac{\partial \left( \lambda_1 y/\lambda_2 - \tilde{N}(y, \rho)/\tilde{D}(y, \rho) \right) / \partial \rho}{\partial \left( \lambda_1 y/\lambda_2 - \tilde{N}(y, \rho)/\tilde{D}(y, \rho) \right) / \partial y}.
\] 
(36)

Notice that
\[
-\frac{\partial \left( \lambda_1 y/\lambda_2 - \tilde{N}(y, \rho)/\tilde{D}(y, \rho) \right) / \partial \rho}{\partial \left( \lambda_1 y/\lambda_2 - \tilde{N}(y, \rho)/\tilde{D}(y, \rho) \right) / \partial y} = -\left( \frac{\partial \tilde{N}(y, \rho)/\partial \rho}{\partial \tilde{N}(y, \rho)/\partial y} \right) \frac{\partial \left( \lambda_1 y/\lambda_2 - \tilde{N}(y, \rho)/\tilde{D}(y, \rho) \right) / \partial \rho}{\partial \left( \lambda_1 y/\lambda_2 - \tilde{N}(y, \rho)/\tilde{D}(y, \rho) \right) / \partial y}.
\]

or using (31),
\[
-\frac{\partial \left( \lambda_1 y/\lambda_2 - \tilde{N}(y, \rho)/\tilde{D}(y, \rho) \right) / \partial \rho}{\partial \left( \lambda_1 y/\lambda_2 - \tilde{N}(y, \rho)/\tilde{D}(y, \rho) \right) / \partial y} = -\frac{\partial \tilde{N}(y, \rho)/\partial \rho - \lambda_1 y \left( \partial \tilde{D}(y, \rho)/\partial \rho \right)}{\tilde{D}(y, \rho)}.
\]

Analogously,
\[
\frac{\partial \left( \lambda_1 y/\lambda_2 - \tilde{N}(y, \rho)/\tilde{D}(y, \rho) \right) / \partial y}{\partial \left( \lambda_1 y/\lambda_2 - \tilde{N}(y, \rho)/\tilde{D}(y, \rho) \right) / \partial \rho} = \lambda_1 \frac{\partial \tilde{N}(y, \rho)/\partial y - \lambda_1 y \left( \partial \tilde{D}(y, \rho)/\partial y \right)}{\tilde{D}(y, \rho)}/\lambda_2.
\]
Therefore, (36) is equivalent to
\[
\frac{\partial \tilde{N}(y, \rho) / \partial \rho - \lambda_1 y \left( \partial \tilde{D}(y, \rho) / \partial \rho \right)}{\tilde{D}(y, \rho)} > \frac{\partial \tilde{N}(y, \rho) / \partial \rho}{\partial \tilde{N}(y, \rho) / \partial y} \times \left( \frac{\lambda_1}{\lambda_2} - \frac{\partial \tilde{N}(y, \rho) / \partial y - \lambda_1 y \left( \partial \tilde{D}(y, \rho) / \partial y \right)}{\tilde{D}(y, \rho)} / \lambda_2 \right),
\]
or,
\[
-\frac{y \left( \partial \tilde{D}(y, \rho) / \partial \rho \right)}{\tilde{D}(y, \rho)} > \frac{\partial \tilde{N}(y, \rho) / \partial \rho}{\partial \tilde{N}(y, \rho) / \partial y} \left( 1 + \frac{y \left( \partial \tilde{D}(y, \rho) / \partial y \right)}{\tilde{D}(y, \rho)} \right).
\]
(37)

Moreover, recall that \( a_2 = \tilde{D}(y, \rho) / \lambda_2 \). The positiveness of \( a_2 \) tells us that \( \tilde{D}(y, \rho) > 0 \). After some algebra, we have that \( \partial \tilde{D}(y, \rho) / \partial \rho < 0 \), \( \partial \tilde{N}(y, \rho) / \partial \rho < 0 \) and \( \partial \tilde{D}(y, \rho) / \partial y > 0 \). Hence, we conclude that the left-hand side (LHS) of (37) is positive, whereas the right-hand side (RHS) of (37) is negative since \( \partial \tilde{N}(y, \rho) / \partial y < 0 \). Consequently, the fact that (37) is satisfied allows us to conclude that \( \partial a_1 / \partial \rho < 0 \).

Concerning the effect of \( \rho \) on \( c_1 \), recall that \( c_1 = a_1 / \Delta_1 = (1 + \rho + \sigma_i^2) a_1 / (1 + \rho) \). This expression tells us that \( c_1 \) is the product of two decreasing positive functions in \( \rho \). Therefore, \( \partial c_1 / \partial \rho < 0 \).

Next, we study how \( a_1 \) and \( c_1 \) vary with a change in \( \sigma_i^2 \), \( i = 1, 2 \). In order to do that first we analyze the effect of \( \sigma_i^2 \) on \( d_1 \) and \( d_2 \). From Proposition 1, we know that \( a_i = \Delta_i c_i > 0 \), \( i = 1, 2 \). Therefore, Expression (2) implies that \( d_i = \left( (n_i - 1) \Delta_i^{-1} a_i + n_j \Delta_j^{-1} a_j \right)^{-1} \), \( i, j = 1, 2, j \neq i \). Substituting the expressions of (16) and the expression for \( \Delta_i \) given in Lemma A2, it follows that
\[
d_i = \left( \frac{(n_i - 1) n_j + n_j n_i}{\Omega_i} \right)^{-1},
\]
where \( \Omega_i = n_j \Upsilon_i (d_i + \lambda_i) + n_i (\Upsilon_i - 1) (d_j + \lambda_j) \) and \( \Omega_j = n_i \Upsilon_j (d_j + \lambda_j) + n_j (\Upsilon_j - 1) (d_i + \lambda_i) \), with \( \Upsilon_i = \Xi_j / (\Xi_j - \Psi_i) = (1 - \rho^2 + \sigma_i^2) / ((1 - \rho^2 + \sigma_i^2)) > 1 \), \( i, j = 1, 2, j \neq i \). Therefore, we derive the following equations that are satisfied in equilibrium: \( F_i (\sigma_{\xi_1}^2, \sigma_{\xi_2}^2, d_1, d_2) = 0 \), \( i = 1, 2 \), where
\[
F_i (\sigma_{\xi_1}^2, \sigma_{\xi_2}^2, d_1, d_2) = \frac{(n_i - 1) n_j d_i}{\Omega_i} + \frac{n_i n_j d_i}{\Omega_j} - 1,
\]
i, \( j = 1, 2, j \neq i \). Let \( DF_{d_1,d_2} (\sigma_{\xi_1}^2, \sigma_{\xi_2}^2, d_1, d_2) \) denote the following matrix:
\[
\begin{bmatrix}
\partial F_1 (\sigma_{\xi_1}^2, \sigma_{\xi_2}^2, d_1, d_2) / \partial d_1 & \partial F_1 (\sigma_{\xi_1}^2, \sigma_{\xi_2}^2, d_1, d_2) / \partial d_2 \\
\partial F_2 (\sigma_{\xi_1}^2, \sigma_{\xi_2}^2, d_1, d_2) / \partial d_1 & \partial F_2 (\sigma_{\xi_1}^2, \sigma_{\xi_2}^2, d_1, d_2) / \partial d_2
\end{bmatrix}.
\]

39
After some tedious algebra, it can be shown that the determinant of \( DF_{d_1,d_2} (\sigma^2_{\xi_1}, \sigma^2_{\xi_2}, d_1, d_2) \) is strictly positive. In particular, it is not null and, therefore, this matrix is invertible. Hence, we can apply the Implicit Function Theorem, we have

\[
\begin{pmatrix}
\frac{\partial d_1}{\partial \sigma^2_{\xi_1}} & \frac{\partial d_1}{\partial \sigma^2_{\xi_2}} \\
\frac{\partial d_2}{\partial \sigma^2_{\xi_1}} & \frac{\partial d_2}{\partial \sigma^2_{\xi_2}}
\end{pmatrix} =
\]

\[- (DF_{d_1,d_2} (\sigma^2_{\xi_1}, \sigma^2_{\xi_2}, d_1, d_2))^{-1} \begin{pmatrix}
\frac{\partial F_1 (\sigma^2_{\xi_1}, \sigma^2_{\xi_2}, d_1, d_2) / \partial \sigma^2_{\xi_1}}{\partial \sigma^2_{\xi_1}} & \frac{\partial F_1 (\sigma^2_{\xi_1}, \sigma^2_{\xi_2}, d_1, d_2) / \partial \sigma^2_{\xi_2}}{\partial \sigma^2_{\xi_2}} \\
\frac{\partial F_2 (\sigma^2_{\xi_1}, \sigma^2_{\xi_2}, d_1, d_2) / \partial \sigma^2_{\xi_1}}{\partial \sigma^2_{\xi_1}} & \frac{\partial F_2 (\sigma^2_{\xi_1}, \sigma^2_{\xi_2}, d_1, d_2) / \partial \sigma^2_{\xi_2}}{\partial \sigma^2_{\xi_2}}
\end{pmatrix} \]

(38)

It is easy to see that all the elements of \( (DF_{d_1,d_2} (\sigma^2_{\xi_1}, \sigma^2_{\xi_2}, d_1, d_2))^{-1} \) are positive. Moreover, \( \partial F_i (\sigma^2_{\xi_1}, \sigma^2_{\xi_2}, d_1, d_2) / \partial \sigma^2_{\xi_i} < 0 \) and \( \partial F_i (\sigma^2_{\xi_1}, \sigma^2_{\xi_2}, d_1, d_2) / \partial \sigma^2_{\xi_j} < 0, i, j = 1, 2, j \neq i \). Hence, (38) implies that \( \partial d_i / \partial \sigma^2_{\xi_i} > 0 \) and \( \partial d_i / \partial \sigma^2_{\xi_j} > 0 \).

Next, we study the comparative statics of \( c_1 \) and \( c_2 \) with respect to \( \sigma^2_{\xi_1} \). Recall that \( c_i = n_j / \Omega_i, \ i, j = 1, 2, j \neq i \). Using the fact that \( \Upsilon_1, d_1, \) and \( d_2 \) are increasing in \( \sigma^2_{\xi_1} \) and that \( \Upsilon_2 \) is independent of \( \sigma^2_{\xi_1} \), we have that \( \Omega_1 \) and \( \Omega_2 \) are increasing in \( \sigma^2_{\xi_1} \), which allows us to conclude that \( c_1 \) and \( c_2 \) are decreasing in \( \sigma^2_{\xi_1} \). Combining these results with the fact that \( \Delta_1 \) is decreasing in \( \sigma^2_{\xi_1} \) and \( \Delta_2 \) is independent of \( \sigma^2_{\xi_1} \), it follows that \( a_1 \) and \( a_2 \) are decreasing in \( \sigma^2_{\xi_1} \), since \( a_1 = \Delta_1 c_1 \) and \( a_2 = \Delta_2 c_2 \).

Finally, concerning h), notice that doing a similar reasoning as before we derive the following equations that are satisfied in equilibrium: \( F_i (n_1, n_2, d_1, d_2) = 0, \ i = 1, 2 \), where

\[
\begin{pmatrix}
\frac{\partial d_1}{\partial n_1} & \frac{\partial d_1}{\partial n_2} \\
\frac{\partial d_2}{\partial n_1} & \frac{\partial d_2}{\partial n_2}
\end{pmatrix} =
\]

\[- (DF_{d_1,d_2} (n_1, n_2, d_1, d_2))^{-1} \begin{pmatrix}
\frac{\partial F_1 (n_1, n_2, d_1, d_2) / \partial n_1}{\partial n_1} & \frac{\partial F_1 (n_1, n_2, d_1, d_2) / \partial n_2}{\partial n_2} \\
\frac{\partial F_2 (n_1, n_2, d_1, d_2) / \partial n_1}{\partial n_1} & \frac{\partial F_2 (n_1, n_2, d_1, d_2) / \partial n_2}{\partial n_2}
\end{pmatrix} \]

Taking into account that all the elements of the previous two matrices are positive, we conclude that \( \partial d_i / \partial n_i < 0 \) and \( \partial d_i / \partial n_j < 0, i, j = 1, 2, j \neq i. \)

**Proof of Corollary 1:** Suppose that \( \sigma^2_{\xi_1} \geq \sigma^2_{\xi_2} \), \( \lambda_1 \geq \lambda_2 \), and \( n_1 \geq n_2 \). Using the expressions of \( \Xi_i \) and \( \Delta_i, i = 1, 2 \), it is easy to see that in this case \( \Xi_2 \Delta_2^{-1} > \Xi_1 \Delta_1^{-1} \). Next, we distinguish two cases:

**Case 1:** \( (n_1 + n_2 - 2) n_1 / ((n_1 + n_2) (n_1 + n_2 - 1)) \geq 1 - \Xi_2 \Delta_2^{-1} \). Evaluating the polynomial \( G(Z) \), stated in the proof of Lemma A2, at \( Z = 1 \), we have that in this case \( G(1) \geq 0 \). This implies that \( z \leq 1 \), and therefore, \( c_1 \leq c_2 \). In addition, using the expressions of \( d_1 \) and \( d_2 \), we
get \( \text{sgn}\{d_1 - d_2\} = \text{sgn}\{c_1 - c_2\} \), which implies \( d_1 \leq d_2 \). Finally, notice that \( \Delta_1 \leq \Delta_2 \) whenever \( \sigma_{i_1}^2 \geq \sigma_{i_2}^2 \). Hence, \( a_1/a_2 = z\Delta_1/\Delta_2 \leq 1 \).

**Case 2:** \((n_1 + n_2 - 2) n_1/((n_1 + n_2)(n_1 + n_2 - 1)) < 1 - \Xi_2 \Delta_2^{-1} \). Notice that

\[
\frac{(n_1 + n_2 - 2) n_2}{(n_1 + n_2)(n_1 + n_2 - 1)} - (1 - \Xi_1 \Delta_1^{-1}) \leq \frac{(n_1 + n_2 - 2) n_1}{(n_1 + n_2)(n_1 + n_2 - 1)} - (1 - \Xi_2 \Delta_2^{-1}),
\]

since \( \Xi_2 \Delta_2^{-1} > \Xi_1 \Delta_1^{-1} \) and \( n_1 \geq n_2 \). Thus, in this case we have that \( H_N(1) < 0 \) and \( H_D(1) < 0 \). Taking into account the shape of these polynomials, the previous two inequalities imply that \( \Xi_D > 1 > \Xi_N \). However, Proposition 2A indicates that in this case there is no equilibrium. ■

**Proof of Proposition 4:** (i) First, suppose that \( \bar{\theta}_1 = \bar{\theta}_2 \). Then \( \mathbb{E}[p] = \bar{\theta}_1 - Q/\left(\frac{n_1}{d_1 + \lambda_1} + \frac{n_2}{d_2 + \lambda_2}\right) \). From Proposition 3 we know that \( d_i \) and \( d_j \) decrease with \( n_i \), and increase with \( \sigma_{i_1}^2 \), \( \lambda_i \), and \( \rho \).

Using these results in the previous expression, we conclude that the expected price is increasing in \( n_i \) and decreasing in \( \lambda_i \), \( \sigma_{i_1}^2 \), and \( \rho \).

Now, suppose that \( \bar{\theta}_1 \neq \bar{\theta}_2 \). The results we have just derived may not hold if \(|\bar{\theta}_1 - \bar{\theta}_2|\) is large enough. For example, let us focus on the relationship between the expected price and \( n_1 \). To study this relationship, we first show that \( n_2 (d_1 + \lambda_1)/(n_1 (d_2 + \lambda_2)) \) decreases with \( n_1 \). Recall that \( d_2 = ((n_2 - 1) n_1/\Omega_2 + n_1 n_2/\Omega_1)^{-1} \). Using the expressions of \( \Omega_i \), \( i = 1, 2 \), we have

\[
1 = \left(\frac{n_2 - 1}{\gamma_2 + (\gamma_2 - 1) \frac{n_2(d_1 + \lambda_1)}{n_1(d_2 + \lambda_2)}} + \frac{n_2}{\tau_1 \frac{n_2(d_1 + \lambda_1)}{n_1(d_2 + \lambda_2)} + \tau_1 - 1}\right)^{-1} + \frac{\lambda_2}{d_2 + \lambda_2}.
\]

The fact that \( d_2 \) decreases with \( n_1 \) implies that \( \lambda_2/(d_2 + \lambda_2) \) increases with \( n_1 \). Then, the previous inequality tells us that \( \frac{n_2 - 1}{\gamma_2 + (\gamma_2 - 1) \frac{n_2(d_1 + \lambda_1)}{n_1(d_2 + \lambda_2)}} + \frac{n_2}{\tau_1 \frac{n_2(d_1 + \lambda_1)}{n_1(d_2 + \lambda_2)} + \tau_1 - 1} \) increases with \( n_1 \). For this to be possible, \( \frac{n_2(d_1 + \lambda_1)}{n_1(d_2 + \lambda_2)} \) needs to be decreasing in \( n_1 \). Given that the expected price satisfies

\[
\mathbb{E}[p] = \left(1 + \frac{n_2 (d_1 + \lambda_1)}{n_1 (d_2 + \lambda_2)}\right)^{-1} \bar{\theta}_1 + \left(1 - \left(1 + \frac{n_2 (d_1 + \lambda_1)}{n_1 (d_2 + \lambda_2)}\right)^{-1}\right) \bar{\theta}_2 - \left(\frac{n_1}{d_1 + \lambda_1} + \frac{n_2}{d_2 + \lambda_2}\right)^{-1} Q,
\]

we have that the relationship between the expected price and \( n_1 \) is ambiguous. For instance, if \( \bar{\theta}_2 \) is low enough, then the fact that \( d_1, d_2, \) and \( n_2 (d_1 + \lambda_1)/(n_1 (d_2 + \lambda_2)) \) are decreasing in \( n_1 \) allows us to conclude that the expected price increases with \( n_1 \). However, if \( \bar{\theta}_2 \) is large and \( \bar{\theta}_1 \) and \( Q \) are low enough, then the expected price decreases with \( n_1 \).

(ii) From the expression for the expected revenue it follows that \( Q \mathbb{E}[p] \) increases with \( \bar{\theta}_i, i = 1, 2 \). In addition, direct computation yields that \( Q = \left(\frac{n_1}{d_1 + \lambda_1} \bar{\theta}_1 + \frac{n_2}{d_2 + \lambda_2} \bar{\theta}_2\right)/2 \). Using Proposition 3, it follows that this quantity increases with \( n_i \) and \( \bar{\theta}_i \) and decreases with \( \rho, \lambda_i \) and \( \sigma_{i_1}^2, i = 1, 2 \). ■
**Proof of Corollary 2:** Using the expression for the expected price, it follows that

\[
\left( \min \{ \bar{v}_1, \bar{v}_2 \} - \frac{Q}{\frac{n_1}{d_1+\lambda_1} + \frac{n_2}{d_2+\lambda_2}} \right) Q \leq \mathbb{E} [p] Q \leq \left( \max \{ \bar{v}_1, \bar{v}_2 \} - \frac{Q}{\frac{n_1}{d_1+\lambda_1} + \frac{n_2}{d_2+\lambda_2}} \right) Q.
\]

Notice both the left-hand side (LHS) and the right-hand side (RHS) of this expression correspond to the expected revenue in an auction where all participants have an expected valuation of \( \min \{ \bar{v}_1, \bar{v}_2 \} \) and of \( \max \{ \bar{v}_1, \bar{v}_2 \} \), respectively. Using Proposition 4(i), we know that both LHS and RHS increase with \( n_i \) but decrease with \( \lambda_i \) and \( \sigma_{e_i}^2 \). Hence, we obtain that \( Q \mathbb{E} [p] \) is lower than the expected revenue of the symmetric auction in which both groups are ex-ante identical, with large size (each group with \( n_1, n_2 \) bidders), with high expected valuation \( \max \{ \bar{v}_1, \bar{v}_2 \} \), low transaction costs \( \min \{ \lambda_1, \lambda_2 \} \), and precise signals \( \min \{ \sigma_{e_1}^2, \sigma_{e_2}^2 \} \), and larger than the expected revenue of the symmetric auction in which both groups are ex-ante identical but with the opposite characteristics (i.e., \( \min \{ n_1, n_2 \} \), \( \min \{ \bar{v}_1, \bar{v}_2 \} \), \( \max \{ \lambda_1, \lambda_2 \} \), and \( \max \{ \sigma_{e_1}^2, \sigma_{e_2}^2 \} \)).

**Lemma A3.** The equilibrium quantities solve the following distorted benefit maximization program:

\[
\max_{x_1, x_2} \mathbb{E} \left[ n_1 (\theta_1 x_1 - (d_1 + \lambda_1) x_1^2 / 2) + n_2 (\theta_2 x_2 - (d_2 + \lambda_2) x_2^2 / 2) \right] \left| \text{s.t. } n_1 x_1 + n_2 x_2 = Q, \right.
\]

taking as given the equilibrium parameters \( d_1 \) and \( d_2 \).

**Proof:** The Lagrangian function of the maximization program is given by

\[
\mathcal{L}(x_1, x_2, \mu) = n_1 (t_1 x_1 - (d_1 + \lambda_1) x_1^2 / 2) + n_2 (t_2 x_2 - (d_2 + \lambda_2) x_2^2 / 2) - \mu (n_1 x_1 + n_2 x_2 - Q),
\]

where \( \mu \) denotes the Lagrange multiplier. Differentiating, we obtain the following FOCs:

\[
\begin{align*}
n_1 (t_1 - (d_1 + \lambda_1) x_1) - \mu n_1 & = 0, \tag{39} \\
n_2 (t_2 - (d_2 + \lambda_2) x_2) - \mu n_2 & = 0, \quad \text{and} \tag{40} \\
n_1 x_1 + n_2 x_2 & = Q. \tag{41}
\end{align*}
\]

From (39) and (40), it follows that \( x_i = (t_i - \mu) / (d_i + \lambda_i), \ i = 1, 2 \). Substituting these expressions in (41) and operating, we have \( \mu = \left( \frac{n_1 t_1}{d_1+\lambda_1} + \frac{n_2 t_2}{d_2+\lambda_2} - Q \right) \left( \frac{n_1}{d_1+\lambda_1} + \frac{n_2}{d_2+\lambda_2} \right)^{-1} \). Then, plugging this expression into (39) and (40), we get the expressions of the equilibrium quantities given in (8). In addition, since the objective function is concave and the constraint is a linear equation, we conclude that the critical point is a global maximum. Hence, the equilibrium quantities are the solutions of the optimization problem stated in Lemma A3. \( \blacksquare \)
Proof of Proposition 5: In the competitive setup, the FOC of the two optimization problems are given by \(E \left[ \theta_i | s_i, p \right] - p - \lambda_i x_i = 0, \ i = 1, 2\). Doing similar computations as in the proof of Lemma A1, we derive the following system of equations:

\[
\begin{align*}
    b_i &= \left( 1 - \Xi_i \right) \bar{\theta}_i - \Psi_i \bar{\theta}_j + \Psi_i \left( \frac{q - \mu_i b_i - \mu_j b_j}{\mu_j a_j} \right) / \lambda_i, \quad (42) \\
    a_i &= \left( \Xi_i - \frac{\mu_i a_i}{\mu_j a_j} \Psi_i \right) / \lambda_i, \quad \text{and} \\
    c_i &= \left( 1 - \Psi_i \left( \frac{\mu_i c_i + \mu_j c_j}{\mu_j a_j} \right) \right) / \lambda_i, \ i, j = 1, 2, j \neq i. \quad (43)
\end{align*}
\]

Note that \(a_i/a_j = \left( \left( \Xi_i - \frac{\mu_i a_i}{\mu_j a_j} \Psi_i \right) / \lambda_i \right) / \left( \left( \Xi_j - \frac{\mu_i a_i}{\mu_j a_j} \Psi_j \right) / \lambda_j \right) \). Hence,

\[
a_i/a_j = \mu_j \left( \Psi_j a_j \mu_j + \Xi_i \lambda_j \mu_i \right) / \left( \mu_i \left( \Psi_i a_j \mu_i + \lambda_i \Xi_j a_j \right) \right).
\]

Then, plugging the previous expression into (43), we get

\[
a_i = \frac{\mu_j (\Xi_i \Xi_j - \Psi_i \Psi_j)}{\mu_j \Xi_j \lambda_i + \mu_i \Psi_i \lambda_j}. \quad (45)
\]

Furthermore, using (42), and after some algebra, we have

\[
\mu_i b_i + \mu_j b_j = \frac{\mu_i}{\lambda_i} \left( 1 - \Xi_i \right) \bar{\theta}_i - \Psi_i \bar{\theta}_j + \frac{\Psi_i}{\lambda_i} q + \frac{\mu_j}{\lambda_j} \left( 1 - \Xi_j \right) \bar{\theta}_j - \Psi_j \bar{\theta}_i + \frac{\Psi_j}{\lambda_j} \frac{q}{\lambda_i} + 1.
\]

Substituting (45) and the last expression into (42), it follows that

\[
b_i = a_i \left( \Xi_i \bar{\theta}_i - \Psi_i \bar{\theta}_j - \bar{\theta}_i \right) + \frac{\lambda_j \Psi_i}{\mu_j \Xi_j \lambda_i + \mu_i \Psi_i \lambda_j} q. \quad (46)
\]

In addition, from (44), and after some algebra, we get

\[
\mu_i c_i + c_j \mu_j = \left( \frac{\mu_i}{\lambda_i} + \frac{\mu_j}{\lambda_j} \right) / \left( \frac{\Psi_i}{\lambda_i} \mu_i + \frac{\Psi_j}{\lambda_j} \mu_j + 1 \right).
\]

Using (45) and the last expression in (44), we have

\[
c_i = \frac{\mu_j (\Xi_j - \Psi_i)}{\mu_j \Xi_j \lambda_i + \mu_i \Psi_i \lambda_j}. \quad (47)
\]

Finally, taking into account the expressions for \(\Xi_i, \Xi_j, \Psi_i, \) and \(\Psi_j\), we obtain the expressions stated in Proposition 5. \(\blacksquare\)

\(^{49}\)To ease the notation the superscript \(o\) is omitted in this proof.
Proof of Proposition 6: Performing similar computations as in the proof of Lemma A1, we obtain that the equilibrium coefficients with subsidies $\kappa_i = d_i(c_1^i, c_2^i)$ satisfy

$$b_i = \frac{(1 - \Xi_i) \bar{\Psi}_i - \Psi_i \bar{\Psi}_j - \Psi_i(n_i b_i + n_i b_j - Q)}{d_i + \lambda_i - d_i(c_1^i, c_2^i)}; a_i = \frac{-\Xi_i - n_i n_j \Psi_i}{d_i + \lambda_i - d_i(c_1^i, c_2^i)} > 0, \text{ and}$$

$$c_i = \frac{1 - \Psi_i(n_i c_i + n_i c_j)}{d_i + \lambda_i - d_i(c_1^i, c_2^i)}, i, j = 1, 2, j \neq i.$$

Comparing this system of equations and the one derived in the proof of Proposition 5, using $Q = (n_1 + n_2)q$ and $\mu_i = n_i/(n_1 + n_2)$ for $i = 1, 2$, we obtain that the equilibrium coefficients of the price-taking equilibrium solves this system. Therefore, we conclude that the quadratic subsidies $\kappa_i x_i^2/2$, with $\kappa_i = d_i(c_1^i, c_2^i), i = 1, 2$, induce an efficient allocation. ■

Lemma A4. The expected deadweight loss at an anonymous allocation $\langle x_1(t), x_2(t) \rangle$ satisfies

$$\mathbb{E}[DWL] = \frac{1}{2} \lambda_1 n_1 \mathbb{E}\left[(x_1(t) - x_1^0(t))^2 \right] + \frac{1}{2} \lambda_2 n_2 \mathbb{E}\left[(x_2(t) - x_2^0(t))^2 \right]. \quad (48)$$

Proof: Notice that $\mathbb{E}[TS] = \mathbb{E}\left[\mathbb{E}[TS|t]\right]$, where

$$\mathbb{E}[TS|t] = \mathbb{E}\left[n_1 \left( \theta_1 x_1(t) - \lambda_1 (x_1(t))^2 / 2 \right) + n_2 \left( \theta_2 x_2(t) - \lambda_2 (x_2(t))^2 / 2 \right) \mid t \right] = n_1 \left( t_1 x_1(t) - \lambda_1 (x_1(t))^2 / 2 \right) + n_2 \left( t_2 x_2(t) - \lambda_2 (x_2(t))^2 / 2 \right).$$

A Taylor series expansion of $\mathbb{E}[TS|t]$ around the price-taking equilibrium $\langle x_1^0(t), x_2^0(t) \rangle$, stopping at the second term due to the fact that $\mathbb{E}[TS|t]$ is quadratic, yields

$$\mathbb{E}[TS|t](x(t)) = \mathbb{E}[TS|t](x^0(t)) + \nabla \mathbb{E}[TS|t](x^0(t))(x(t) - x^0(t)) + \frac{1}{2} (x(t) - x^0(t))^\prime D^2 \mathbb{E}[TS|t](x^0(t))(x(t) - x^0(t)),$$

where $\nabla \mathbb{E}[TS|t](x^0(t))$ and $D^2 \mathbb{E}[TS|t](x^0(t))$ are, respectively, the gradient and the Hessian matrix of $\mathbb{E}[TS|t]$ evaluated at $x^0(t)$. Because of optimality of $x^0(t)$,

$$\nabla \mathbb{E}[TS|t](x^0(t)) = (0, 0).$$

In addition, $D^2 \mathbb{E}[TS|t](x^0(t)) = \begin{pmatrix} -\lambda_1 n_1 & 0 \\ 0 & -\lambda_2 n_2 \end{pmatrix}$. Hence,

$$\mathbb{E}[TS|t](x(t)) - \mathbb{E}[TS|t](x^0(t)) = -\frac{1}{2} \lambda_1 n_1 (x_1(t) - x_1^0(t))^2 - \frac{1}{2} \lambda_2 n_2 (x_2(t) - x_2^0(t))^2$$

and, therefore, (48) is satisfied. ■

Proof of Proposition 7: (i) Suppose that $Q = 0$. Then, $\mathbb{E}[DWL]$ is given by

$$\mathbb{E}[DWL] = \frac{n_1 n_2 (n_2 d_1 + n_1 d_2)^2}{2(n_2 \lambda_1 + n_1 \lambda_2)(n_i (d_j + \lambda_j) + n_j (d_i + \lambda_i))^2} (t_1 - t_2)^2.$$
Hence,
\[
\frac{d\mathbb{E}[DWL]}{d\sigma_{\varepsilon_1}^2} = \frac{\partial \mathbb{E}[DWL]}{\partial d_1} \frac{\partial d_1}{\partial \sigma_{\varepsilon_1}^2} + \frac{\partial \mathbb{E}[DWL]}{\partial d_2} \frac{\partial d_2}{\partial \sigma_{\varepsilon_1}^2} + \frac{\partial \mathbb{E}[DWL]}{\partial \sigma_{\varepsilon_1}^2}.
\]

It is easy to see that in this case \(\frac{\partial \mathbb{E}[DWL]}{\partial d_i} > 0\), \(i = 1, 2\), and \(\frac{\partial \mathbb{E}[DWL]}{\partial \sigma_{\varepsilon_1}^2} < 0\). Combining these results with Proposition 3, we have that the first two terms of \(\frac{d\mathbb{E}[DWL]}{d\sigma_{\varepsilon_1}^2}\) are positive, while the last one is negative.

We know that \(d_1\) and \(d_2\) are independent of \(\sigma_{\varepsilon_1}^2\) when \(\rho = 0\). By continuity, we know that for very low values of \(\rho\) is \(\frac{\partial d_1}{\partial \sigma_{\varepsilon_1}^2}\) and \(\frac{\partial d_2}{\partial \sigma_{\varepsilon_1}^2}\) are positive, but very low. Hence, we conclude that the last term in \(\frac{d\mathbb{E}[DWL]}{d\sigma_{\varepsilon_1}^2}\) dominates and, hence, in this case we have that \(\frac{d\mathbb{E}[DWL]}{d\sigma_{\varepsilon_1}^2} < 0\) although \(\frac{\partial d_1}{\partial \sigma_{\varepsilon_1}^2} > 0\), \(i = 1, 2\). By contrast, if we consider the case in which \(\rho\) is not low and \((\tilde{\theta}_1 - \tilde{\theta}_2)^2\) is high enough, then the first two terms in \(\frac{d\mathbb{E}[DWL]}{d\sigma_{\varepsilon_1}^2}\) dominate, which implies that \(\frac{d\mathbb{E}[DWL]}{d\sigma_{\varepsilon_1}^2} > 0\), \(i = 1, 2\).

(ii) Omitted since it is trivial.

(iii) When groups are symmetric \((n_2 = n_1 = n, \lambda_1 = \lambda_2 = \lambda, \text{ and } \sigma_{\varepsilon_1}^2 = \sigma_{\varepsilon_2}^2)\), \(d_1 = d_2 = d\) and \(\lambda_2 d_1 - \lambda_1 d_2 = 0\). Therefore, the expected deadweight loss consists of only one term, the first one, that is independent of \(Q\). Moreover, we obtain that \(\mathbb{E}[DWL|t] = nd^2(t_2 - t_1)^2/(4(d + \lambda)^2 \lambda)\) and since \(\frac{\partial \mathbb{E}[DWL|t]}{\partial d} > 0\), we conclude that an increase in an information parameter \((\rho\text{ or } \sigma_{\varepsilon}^2)\) raises both \(d\) and \(\mathbb{E}[DWL|t]\).

However, with asymmetric groups the previous results may not hold. In this case, suppose that \(Q\) is large enough. Then, for \(i = 1, 2, j \neq i\),
\[
\text{sgn}\left\{\frac{\partial \mathbb{E}[DWL|t]}{\partial d_i}\right\} = \text{sgn}\left\{\partial \left(\frac{n_1 n_2 (\lambda_1 d_2 - \lambda_2 d_1)^2}{2(\lambda_1 n_2 + \lambda_2 n_1)(n_1 d_j + \lambda_j) + n_2 (d_i + \lambda_i)}\right) Q^2\right\}/\partial d_i\right\} = \text{sgn}\{\lambda_j d_i - \lambda_i d_j\}.
\]

When \(Q\) is large enough we have that if \(d_i/d_j > \lambda_i/\lambda_j\), then \(\frac{\partial \mathbb{E}[DWL|t]}{\partial d_i} > 0\) and \(\frac{\partial \mathbb{E}[DWL|t]}{\partial d_j} < 0\). Thus, with asymmetric groups price impact \((d_1, d_2)\) and the \(\mathbb{E}[DWL|t]\) are not always positively associated, given predicted values \(t\), for changes in information parameters.

Proof of Proposition 8: Using (28) and (29), it follows that \(\lim_{n_1 \to \infty} \bar{z}_N = \lim_{n_1 \to \infty} \bar{z}_D = 0\). Furthermore, after some algebra, we have that the necessary and sufficient condition for the existence of an equilibrium (i.e., \(\lim_{n_1 \to \infty} \bar{z}_N / \bar{z}_D > 1\)) is equivalent to \(n_2 > \tilde{n}_2 (\rho, \sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2)\), where
\[
\tilde{n}_2 (\rho, \sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2) = \frac{\rho ((2 - \rho) \sigma_{\varepsilon_2}^2 + 2 (1 - \rho^2)) \sigma_{\varepsilon_1}^2}{(1 - \rho^2) ((1 + \sigma_{\varepsilon_1}^2) (1 + \sigma_{\varepsilon_2}^2) - \rho^2)}.
\]
Moreover, taking the limit in (25), it follows that \(\lim_{n_1 \to \infty} \bar{z} = 0\) and
\[
\lim_{n_1 \to \infty} n_1 \bar{z} = n_2 \Xi_1 \Delta_1^{-1} / (1 - \Xi_1 \Delta_1^{-1}). \quad (49)
\]
Using the expressions included in the statement of Lemma A2, and after some tedious algebra, we get
\[
\lim_{n_1 \to \infty} b_1 = q, \quad \lim_{n_1 \to \infty} a_1 = 0, \quad \lim_{n_1 \to \infty} c_1 = 0, \quad \lim_{n_1 \to \infty} a_2 = \Delta_2 \lim_{n_1 \to \infty} c_2,
\]
\[
\lim_{n_1 \to \infty} b_2 = \frac{\hat{\sigma}_{\varepsilon_2}^2 \left( (n_2-1)(1-\rho^2) + \frac{(1-\rho^2+\hat{\sigma}_{\varepsilon_1}^2(1-2\rho))}{(1+\hat{\sigma}_{\varepsilon_1}^2)} - \rho^2 \right)}{(1-\rho) \lambda_2 \left( n_2 (1+\rho) - \frac{\rho \hat{\sigma}_{\varepsilon_1}^2 (1+\rho+\hat{\sigma}_{\varepsilon_2}^2)}{(1+\hat{\sigma}_{\varepsilon_1}^2)(1+\hat{\sigma}_{\varepsilon_2}^2)} \right)} + \frac{\rho \hat{\sigma}_{\varepsilon_2}^2 \hat{\sigma}_{\varepsilon_1}^2}{n_2 (1-\rho^2) \left( (1+\hat{\sigma}_{\varepsilon_1}^2)(1+\hat{\sigma}_{\varepsilon_2}^2) - \rho^2 \right)},
\]
\[
\lim_{n_1 \to \infty} c_2 = \frac{\lambda_2 - \hat{\lambda}_2}{1-\rho} \left( \frac{n_2}{1+\rho+\hat{\sigma}_{\varepsilon_2}^2} - \frac{\rho \hat{\sigma}_{\varepsilon_1}^2}{(1+\rho)(1+\hat{\sigma}_{\varepsilon_1}^2)(1+\hat{\sigma}_{\varepsilon_2}^2) - \rho^2} \right).
\]

Next, in relation to the expressions for \(d_1\) and \(d_2\), we have that
\[
\lim_{n_1 \to \infty} d_1 = \lim_{n_1 \to \infty} \left( (n_1 - 1) c_1 + n_2 c_2 \right)^{-1} = \left( \lim_{n_1 \to \infty} \left( \frac{(n_1 - 1)}{n_1} n_1 z + n_2 \right) \lim_{n_1 \to \infty} c_2 \right)^{-1} > 0.
\]

The fact that \(n_1 z\) and \(c_2\) converge to a positive finite number (see (49) and (50)) implies that \(d_1\) does not converge to zero (provided that \(\rho \hat{\sigma}_{\varepsilon_1}^2 > 0\); if \(\rho \hat{\sigma}_{\varepsilon_1}^2 = 0\), then it is easy to see that \(\lim_{n_1 \to \infty} n_1 z = \infty\)). A similar result is obtained with the limit of \(d_2\). In particular,
\[
\lim_{n_1 \to \infty} d_2 = \left( \left( \lim_{n_1 \to \infty} n_1 z + n_2 - 1 \right) \lim_{n_1 \to \infty} c_2 \right)^{-1} > \lim_{n_1 \to \infty} d_1 > 0. \quad \Box
\]

**Appendix B**

**Bid shading and expected discount**

Recall that \(\tilde{t} = (n_1 t_1 + n_2 t_2) / (n_1 + n_2)\). From the demand of bidders it follows that \(p(t) = t_i - (d_i + \lambda_i)x_i(t), i = 1, 2\). For a trader of type \(i\), the expected marginal benefit of buying \(x_i\) units of the asset is \(t_i - \lambda_i x_i\). Hence, the average marginal benefit is given by \(\tilde{t} - (\lambda_1 n_1 x_1 + \lambda_2 n_2 x_2) / (n_1 + n_2)\). The magnitude of (average) *bid shading* is the difference between the average marginal valuation and the auction price, that is, \((d_1 n_1 x_1 + d_2 n_2 x_2) / (n_1 + n_2)\).\(^{50}\)

We can use Equation (8) to write bid shading as
\[
\frac{(n_2 d_2 (d_1 + \lambda_1) + n_1 d_1 (d_2 + \lambda_2)) Q}{(n_1 + n_2) (n_1 (d_2 + \lambda_2) + n_2 (d_1 + \lambda_1))} + \frac{(t_2 - t_1) (d_2 - d_1) n_2 n_1}{(n_1 + n_2) (n_1 (d_2 + \lambda_2) + n_2 (d_1 + \lambda_1))}.
\]

At this juncture, some additional remarks are in order.

---

\(^{50}\)According to Cassola et al. (2013) average bid shading almost tripled after the turmoil in August 2007 in the ECB liquidity auctions.
Bid shading increases with $Q$ and decreases when the group that values more the asset, say group 1, ($t_1 > t_2$) has less price impact ($d_1 < d_2$).

When $d_1 = d_2 = d$ as in the symmetric case, for instance, bid shading consists of only one term (the first one) and it is equal to $dQ / (n_1 + n_2)$.

When $d_1 \neq d_2$, the second term of (51) is negative and bid shading decreases whenever the group that values the asset more highly ($t_i > t_j$) has less price impact ($d_i < d_j$).

If group 1 has higher transaction costs ($\lambda_1 > \lambda_2$), is more numerous ($n_1 > n_2$), and is less informed ($\sigma_{e_1}^2 > \sigma_{e_2}^2$) than group 2, then $c_1 < c_2$, and so $d_1 < d_2$. If $t_1 > t_2$, then the second term of (51) is negative and the two terms have opposite signs. Therefore, if $Q$ is low (e.g., $Q = 0$) or if the difference in predicted values of the asset is high, then negative bid shading obtains.

The expected discount is defined as $\mathbb{E} [\tilde{t}] - \mathbb{E} [p(t)]$. Using (9), some algebra yields the following expression for the expected discount:

$$
\frac{(d_1 + \lambda_1) (d_2 + \lambda_2)}{n_1 (d_2 + \lambda_2) + n_2 (d_1 + \lambda_1)} Q + \frac{n_1 n_2 (d_2 + \lambda_2 - d_1 - \lambda_1) (\tilde{\theta}_2 - \tilde{\theta}_1)}{(n_1 + n_2) (n_1 (d_2 + \lambda_2) + n_2 (d_1 + \lambda_1))}.
$$

Here our related comments are as follows.

- When $d_1 + \lambda_1 = d_2 + \lambda_2 = d + \lambda$ (as in the symmetric case), the expected discount is $(d + \lambda)Q / (n_1 + n_2)$.

- The first term is always positive provided $Q > 0$, whereas the second term is positive whenever $(d_2 + \lambda_2 - d_1 - \lambda_1) (\tilde{\theta}_2 - \tilde{\theta}_1) > 0$. Therefore, the expected discount is lower whenever the group that values the asset more highly ($\tilde{\theta}_2 > \tilde{\theta}_1$) has a lower "total transaction cost" $(d_2 + \lambda_2 < d_1 + \lambda_1)$.

- If group 1 ex ante values the asset more ($\tilde{\theta}_1 > \tilde{\theta}_2$), has higher transaction costs ($\lambda_1 > \lambda_2$), is more numerous ($n_1 > n_2$), and is less informed ($\sigma_{e_1}^2 > \sigma_{e_2}^2$), then $d_1 + \lambda_1 > d_2 + \lambda_2$ whenever (a) the differences between groups are due mostly to transaction costs and (b) $\lambda_1 / \lambda_2$ is high enough. In this case, both terms are positive and so the expected discount is positive. Yet, if both groups have similar transaction costs, then the two terms in (52) have opposite signs. In particular, we expect a negative discount when $Q$ is low.
References


