Free entry in a Cournot market with overlapping ownership*

Xavier Vives† Orestis Vravosinos‡

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Abstract

We examine the effects of overlapping ownership in a Cournot oligopoly with free entry. If firms develop overlapping ownership only after entering, then an increase in the degree of overlapping ownership spurs entry but causes price to increase and total surplus to fall. Also, entry is never insufficient by more than one firm as in the case without overlapping ownership. If potential entrants have overlapping ownership before entering, then an increase in their degree of overlapping ownership can limit or spur entry. Although entry is excessive under non-decreasing returns to scale, if returns to scale are decreasing enough, then entry is insufficient under high levels of overlapping ownership. Under common assumptions, we find that pre-entry (resp. post-entry) overlapping ownership magnifies (resp. alleviates) the negative impact of an increase of entry costs on entry.

Keywords: common ownership, cross ownership, minority shareholdings, oligopoly, entry, competition policy
JEL classification codes: D43, E11, L11, L13, L21, L41

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†IESE Business School; e-mail: xvives@iese.edu.

‡New York University; e-mail: orestis.vravosinos@nyu.edu.

1 Introduction

Overlapping ownership, be it in the form of common or cross ownership, has generated concern for its potential anti-competitive impact (Elhauge, 2016; Posner et al., 2017), especially due to the rising shares of large investment funds in multiple competitors in several industries; for example airlines (Azar et al., 2018), banks and supermarkets (Schmalz, 2018). Azar and Vives (2019, 2021a) and Backus et al. (2021a) document the dramatic rise in common ownership in the S&P 500 firms in the last decades.

At the same time, firm entry patterns have been argued to pose a significant impact on the aggregate economy. Using a panel of U.S. states over the period 1982–2014, Gourio et al. (2016) find that (positive) shocks to the number of new firms have sizable and lasting (positive) effects on a state's real GDP, productivity, and population.[1] Gutiérrez and Philippon (2021) document a decline in entry of firms in the U.S. economy and estimate the elasticity of entry with respect to Tobin's Q to have dropped to zero since the late 1990s, up to which point it was positive and significant. Apart from a generalized decline in entry, Decker et al. (2016) document a particular decline in high-growth young firms in the U.S. since 2000, when such firms could have had a major contribution to job creation. Gutiérrez et al. (2021) argue that increases in entry costs have had a considerable impact on the U.S. economy over the past 20 years, leading to higher concentration, as well as lower entry, investment and labor income.

The literature above documents the decline in firm entry rates (accompanied by a milder decrease in firm exit rates) and a concurrent increase in common ownership over close to 40 years in the U.S economy. There are several explanations for the decreased entry dynamism, an increase in entry costs (for technological or regulatory reasons) being a prominent one. It is possible also that apart from softening competition in pricing, overlapping ownership also contributes to diminishing entry dynamism. Some recent empirical work points in this direction. Newham et al. (2019) find that in the U.S. pharmaceutical industry higher common ownership between the brand firm and potential generic entrants leads to fewer generic entrants. Relatedly, Xie and Gerakos (2020) analyze patent infringement lawsuits filed by brand-name drug manufacturers against generic manufacturers to find that common institutional ownership of the brand and generic

[1] They attribute the population increase to the fact that states with higher economic activity attract residents from other states.

firms increases the likelihood that the two litigants enter into a settlement whereby the brand firm often pays the generic to delay entry. Ruiz-Pérez (2019) estimates a structural model of market entry and price competition under common ownership in the U.S. airline industry to find that the higher the common ownership between the incumbents and a potential entrant, the lower the likelihood of entry.

In this paper we provide a framework to study the effects of overlapping ownership in a Cournot oligopoly with free entry. We examine two possibilities of free entry equilibria with symmetric firms. A new industry that is to mostly be populated by start-ups can be modelled as an industry where overlapping ownership will develop post-entry. On the other hand, an industry or product market which established firms with existing ownership ties consider whether to enter can be thought of as exhibiting pre-entry overlapping ownership.

Post-entry overlapping ownership If firms develop overlapping ownership only after entering (for example if they are new firms), then they do not internalize the negative externality their entry has on other firms, as in the standard Cournot model with free entry. Thus, modulo the integer constraint on the number of firms, firms enter until the individual gross profit is equal to the entry cost, so that in equilibrium the net profit is zero. Nevertheless, when deciding whether to enter, they take into account how overlapping ownership will affect product market outcomes.

We find that even though an increase in the degree of overlapping ownership causes firms to suppress production thus allowing more firms to enter, the anti-competitive effect of overlapping ownership prevails causing price to increase and total surplus to fall. Also, the result on the tendency for excessive entry under business-stealing competition (*i.e.*, when individual quantity decreases with the number of firms in the Cournot game) generalizes. Entry is never insufficient by more than one firm (compared to the level of entry chosen by a planner that can regulate entry but not the extent of overlapping ownership) as in the standard Cournot model with free entry (see Mankiw and Whinston, 1986; Amir et al., 2014). Last, apart from spurring entry, under common assumptions overlapping ownership also alleviates the negative impact of an increase in the entry cost on entry. Thus, post-entry overlapping ownership could dampen the negative macroeconomic implications described in Gutiérrez et al. (2021).

Pre-entry overlapping ownership We then study the case where potential entrants are already established firms and have overlapping ownership before entering.[2] This way we address the main concern about how overlapping ownership can suppress entry: by inducing firms to internalize the negative externality that their entry would pose on other firms. Modulo the integer constraint, entry is lower than what it would be without the internalization of the entry externality, so that in equilibrium the net profit is positive. If we compare this with the post-entry overlapping ownership case, where net profit is zero, we see that an investor considering whether to become a common owner in the industry will prefer to become an owner before rather than after entry.

We are interested in four main questions. Will overlapping ownership indeed suppress entry and what forces are at play? Does entry still tend to be excessive as in the case without overlapping ownership? What level of overlapping ownership is socially optimal? How does overlapping ownership mediate the (negative) effect of entry costs on entry?

First, we distinguish the three channels through which an increase in the level of pre-entry overlapping ownership affects entry; these channels are not specific to our assumption of Cournot competition. On the one hand, it tends to limit entry by increasing the degree of internalization of the negative externality of entry on other firms' profits. On the other hand, overlapping ownership tends to increase equilibrium profits in the product market competition stage, which tends to increase entry. There also is a channel with an ambiguous effect on entry: overlapping ownership changes the magnitude of the entry externality. The effect of overlapping ownership on entry will depend on the size of the different channels and the direction of the ambiguous channel's effect. We find that an increase in the degree of overlapping ownership can limit or, counter-intuitively, spur entry. For low levels of overlapping ownership, the rise in own profit due to increases in overlapping ownership can dominate. However, for high levels, competition in the product market is already soft enough, so that further increases in overlapping ownership suppress entry. Common ownership among U.S.-listed firms is indeed already high enough, so that if private firms are treated as a competitive fringe, then further increases in common ownership are likely to limit entry by public firms into product markets where other public firms already operate. In the extreme case of complete indexation of the industry, where firms jointly maximize aggregate industry profits, only one firm enters so that fixed entry

[2]For example, one can think of pharmaceuticals considering whether to incur in R&D costs to enter a new drug market.

costs are minimized unless there are substantial decreasing returns to scale (DRS).

Second, we find that whether entry is excessive or insufficient will depend on (i) the level of overlapping ownership, (ii) the magnitude of the entry externality, (iii) whether returns to scale are increasing (IRS) or decreasing and to what degree, and (iv) whether competition is business-stealing or business-enhancing (*i.e.*, whether individual quantity decreases or increases, respectively, with the number of firms in the Cournot game), and to what degree. Under business-stealing competition, increases in the level of overlapping ownership or the magnitude of the entry externality, tend to make entry insufficient with the two forces being complements in inducing insufficient entry. Also, DRS tend to make entry insufficient, since the planner takes advantage of variable-cost savings due to entry to a greater extent than firms do. On the other hand, increases in the magnitude of the business-stealing effect and IRS tend to make entry excessive. Under business-enhancing competition, entry is always insufficient. This can be seen as a generalization of the result of Amir et al. (2014), who prove that entry is insufficient under business-enhancing competition without overlapping ownership.

In the standard case of business-stealing competition, we find that with non-DRS entry is indeed excessive under general conditions. However, if returns to scale are decreasing enough, then entry is insufficient under high levels of overlapping ownership. On the one hand, the planner combats DRS by having many firms enter. On the other hand, while with positive overlapping ownership a firm also prefers lower average costs both for itself and for the other firms, the internalization of the price effect that its entry will have suppresses entry relative to the planner's solution.

Third, a total welfare-maximizing planner that can only regulate overlapping ownership and then allow firms to freely enter may optimally choose (a) no overlapping ownership, (b) intermediate levels of it, or even (c) complete indexation of the industry. For instance, with constant returns to scale (CRS), the average variable cost of production is not affected by entry. The planner has to balance the effect of overlapping ownership—in the Cournot game and in the entry stage—on the total quantity and price, and on the total entry costs. In numerical simulations we find that for low entry cost, the planner chooses no overlapping ownership, while for higher entry cost intermediate levels of overlapping ownership or complete indexation can be optimal. With IRS, complete indexation of the industry (leading to a monopoly) can be optimal under both a total surplus and a

consumer surplus standard.

Also, we show that under common assumptions, overlapping ownership exacerbates the negative impact of an increase in the entry cost on entry. Therefore, pre-entry overlapping ownership could magnify the negative macroeconomic implications documented in Gutiérrez et al. (2021). This result is opposite to the one under post-entry overlapping ownership. However, especially with regard to the macroeconomic impact, the case of large firms with existing ownership ties deciding whether to enter new markets seems most relevant given the widespread holdings of the large diversified funds.

Last, we examine the effects of overlapping ownership in the case where apart from the (pre-entry) commonly-owned firms, there are also maverick firms (price-taking and without ownership ties), which may enter the market. The presence of maverick firms essentially changes the demand faced by the commonly-owned firms by depressing it and making it more elastic. This suppresses entry by commonly-owned firms and makes it less sensitive to the level of overlapping ownership. Our results on the effects of overlapping ownership on the price, entry by commonly-owned firms, as well as our comparison of equilibrium and socially optimal levels of entry extend to this case with the demand appropriately adjusted.

After this introduction, section 2 discusses related literature and section 3 presents the model and studies the existence, uniqueness, stability and comparative statics of the pricing stage equilibrium. Section 4 studies the model with post-entry overlapping ownership, while section 5 examines pre-entry overlapping ownership. Last, section 6 concludes. Proofs are gathered in Appendix A, and supplementary results and proofs thereof in Appendices B and C.

2 Related literature

Research attention to the possible anti-competitive effects of overlapping ownership dates back to at least Rubinstein and Yaari (1983) and Rotemberg (1984). Recently, interest on the topic has revived given the rising shares of large diversified funds, because as Banal-Estañol et al. (2020) show, the profit loads firms place on competing firms increase if the holdings of more diversified investors increase relative to those of less diversified investors. Multiple empirical studies have been conducted and there is a debate on whether

and how common ownership affects corporate conduct and softens competition.[3]

Theoretical work has considered models where the effects of overlapping ownership are not only through product market competition: when (i) there are diversification benefits because investors are risk-averse (Shy and Stenbacka, 2020) or (ii) firms choose cost-reducing or quality-enhancing R&D investment possibly with R&D spillovers (Bayona and López, 2018; López and Vives, 2019), product positioning (Li and Zhang, 2021) or qualities (Brito et al., 2020), (iii) firms invest in a preemption race (Zormpas and Ruble, 2021), (iv) firms may choose to transfer their innovation technology to a rival firm (Papadopoulos et al., 2019). Last, other studies have examined the effects of overlapping ownership in a general equilibrium setting (Azar and Vives, 2019, 2021a,b) or under alternative models of corporate control (Vravosinos, 2021).

All of the models above treat the number of firms in the industry as exogenous. However, Li et al. (2015) show that in a Cournot duopoly the incumbent firm can strategically develop cross ownership to deter the other firm from entering. Sato and Matsumura (SM; 2020) provide a model of free entry under pre-entry common ownership. In their model, incumbents under a symmetric common ownership structure choose whether to enter a new market. Using a circular-market model with CRS they show that entry always decreases with the level of common ownership. In that model, welfare only depends on the number of firms, the cost of transportation and the entry cost.[4] It is independent of all equilibrium objects except entry. Therefore, the welfare effects of common ownership are directly implied by its effects on entry, namely whether it leads to or exacerbates excessive or insufficient entry. They show that entry always decreases with common ownership. Thus, given that in their setting for low levels of common ownership entry is excessive while for high it is insufficient, welfare has an inverted-U shaped relationship with the degree of common ownership, which implies a strictly positive optimal degree of common ownership.

[3]While He and Huang (2017), Azar et al. (2018), Park and Seo (2019), Boller and Morton (2020), Banal-Estañol et al. (2020) and Anton et al. (2021, 2022) find evidence in favor of this hypothesis, others have found little to no effect (e.g. Koch et al., 2021; Lewellen and Lowry, 2021; Backus et al., 2021c). Backus et al. (2021b) outline the limitations of the empirical approaches used so far and argue that these limitations make it difficult to draw clear conclusions.

[4]Consumers have a unit demand and need to pay transportation costs proportional to their distance from the firm that they choose to buy from. Thus, it is assumed that their willingness to pay is high enough so that they always buy, regardless of how many firms have entered (where the number of firms affects the distance of consumers from them). The planner's problem is then equivalent to minimizing the total transportation and entry costs; the former decrease with the number of firms, while the latter increase with it.

Our model differs from theirs in several ways. First, we consider quantity instead of price competition. Further, we derive some results respecting the integer constraint on the number of firms, which is ignored in SM. Last, in our setting total surplus depends on equilibrium objects not only through the number of firms. For example, this means that higher overlapping ownership can induce a social planner that regulates entry (but not overlapping ownership) to allow fewer firms to enter, since overlapping ownership decreases the effectiveness of entry in reducing the price.

The differences in the generality, scope and implications of the analysis are even more pronounced. We derive results under general demand and cost functions and consider examples of parametric assumptions for ease of interpretation. Importantly, we delineate three channels through which pre-entry overlapping ownership affects entry. While all three channels are also present in the model of SM, the authors do not discuss the following channel: that overlapping ownership changes the magnitude of the externality a firm's entry poses on other firms. Still, one can check that in their model the magnitude of the entry externality monotonically increases with the extent of overlapping ownership, while we show that in our model the direction of this effect is ambiguous. Further, they find that entry always decreases with (pre-entry) overlapping ownership, while in our case both possibilities are present. We also describe four forces that determine whether entry will be excessive or insufficient under pre-entry overlapping ownership. In addition, in our model equilibrium total surplus can behave in multiple different ways as the extent of overlapping ownership changes—contrary to the inverted-U relationship found in SM. Last, we study how overlapping ownership mediates the effect of the entry cost on entry, which is not examined in SM.

Our work can be seen as an extension of the literature on free entry in Cournot markets. Mankiw and Whinston (1986) show that in a symmetric Cournot market with free entry and non-IRS where in the pricing stage (i) the total quantity increases with the number of firms, and (ii) the business-stealing effect is present, entry is never insufficient by more than one firm. Amir et al. (2014) extend these results to the case of limited IRS, showing that still under business-stealing competition entry is never insufficient by more than one firm. Our results on the post-entry overlapping ownership case closely mirror these findings. In the case of pre-entry overlapping ownership, we extend the result of Amir et al. (2014) to the case of competition under overlapping ownership, showing that under

business-enhancing competition, entry is always insufficient. However, we show that under business-stealing competition, pre-entry overlapping ownership can lead to insufficient entry (by more than one firm) when returns to scale are decreasing enough.

The setting of symmetric firms with a symmetric overlapping ownership structure that we consider preserves the properties of the Cournot game being symmetric, which allows for extensions of existing results (e.g. see Vives, 1999) to the case of competition under overlapping ownership. Namely, we extend the results of Amir and Lambson (2000), who use lattice-theoretic methods to study equilibrium existence and comparative statics with respect to the (exogenous) number of firms in a symmetric Cournot market, and of Amir et al. (2014), who build on the latter to study free entry. Also, we add to the literature on the stability of a Cournot equilibrium extending the results of al Nowaihi and Levine (1985).

3 The Cournot-Edgeworth λ-oligopoly model with free entry

There is a (large enough) finite set $\mathcal{F} := \{1,2,\ldots,N\}$ of N symmetric firms that can potentially enter a market. The game has two stages, the entry stage and the pricing stage. In the first stage, each firm chooses whether to enter by paying a fixed cost $f > 0$.[5] Assume n firms have entered. In the pricing stage, entrants compete à la Cournot. Namely, each firm i chooses its production quantity, $q_i \in \mathbb{R}_+$, simultaneously with the other firms. We denote by $s_i := q_i/Q$ firm i's share of the total quantity $Q := \sum_{i=1}^{n} q_i$. We also write $\boldsymbol{q}$ and $\boldsymbol{q}_{-i}$ to denote the production profile of all firms, and all firms expect i, respectively; also, $Q_{-i} := \sum_{j\neq i} q_j$.

3.1 The pricing stage

Firm i's $\in N$ production cost is given by the function $C : \mathbb{R}_+ \to \mathbb{R}_+$ with $C(q_i) \geq 0$ and $C'(q_i) > 0$ for every q_i. Denote by $E_C(q) := C'(q)q/C(q)$ the elasticity of the cost function. When we say that firms have constant elasticity costs, we mean that $C(q_i) = cq_i^{\kappa}/\kappa$ for some $c,\kappa > 0$. Under constantly elastic costs, $E_C(q) = \kappa$ for every q. (i) For $\kappa = 1$ we have CRS, (ii) for $\kappa \in (0,1)$ we have IRS, (iii) for $\kappa > 1$ DRS (for $\kappa = 2$ costs are quadratic).

[5] We study pure strategy equilibria. If firms decide whether to enter sequentially, this is indeed without loss of generality. However, if they choose simultaneously whether to enter, then although the pure equilibrium is still an equilibrium, there can also be equilibria where firms mix in their entry decisions (e.g. see Cabral, 2004).

$AC(q) \coloneqq C(q)/q$ is the average cost.

The inverse demand function $P : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $P'(Q) < 0$ for every $Q \in \left[0,\overline{Q}\right)$, where $\overline{Q} \in (0,+\infty]$ is such that $P(Q) > 0$ if and only if $Q \in \left[0,\overline{Q}\right)$.[6] We assume that there exists $\overline{q} > 0$ such that $P(q) < AC(q)$ for every $q > \overline{q}$, and that P and C are three-times differentiable.[7] For $Q < \overline{Q}$ we denote by $\eta(Q) \coloneqq -P(Q)/(QP'(Q))$ the elasticity of demand, by $E_{P'}(Q) \coloneqq -P''(Q)Q/P'(Q)$ the elasticity of the slope of inverse demand, and by $E_{P''}(Q) \coloneqq P'''(Q)Q/P''(Q)$ the elasticity of the curvature of inverse demand. An inverse demand function with constant elasticity of slope (CESL), $E_{P'}(Q) \equiv E$ for every Q allows for log-concave and log-convex demand encompassing linear and constant elasticity specifications. When we refer to linear demand, we mean $P(Q) = \max\{a - bQ, 0\}$. Every result applies to generic cost function and inverse demand function unless otherwise stated. We assume that the optimal (gross) monopoly profit is higher than the entry cost, that is, $\max_{Q\geq 0}\{P(Q)\,Q - C(Q)\} > f$.

Suppose n firms have entered. A quantity profile $\boldsymbol{q}^*$ is an equilibrium of the pricing stage if for each firm $i \in \{1,\ldots,n\}$

$$q_i^* \in \arg\max_{q_i\geq 0}\left\{\pi_i\left(q_i,\boldsymbol{q}_{-i}^*\right) + \lambda\sum_{j\neq i}\pi_j\left(q_i,\boldsymbol{q}_{-i}^*\right)\right\},$$

where $\pi_i(\boldsymbol{q}) \coloneqq P(Q_{-i} + q_i)\,q_i - C(q_i)$ and $\lambda \in [0,1]$ is the (exogenous) Edgeworth (1881) coefficient of effective sympathy among firms.[8] This coefficient can for example arise from a symmetric overlapping ownership structure (be it common or cross ownership) as in López and Vives (2019) or Azar and Vives (2021a). In comparative statics exercises, an increase in λ can for example represent an expansion of an investment fund's holdings across all firms in the industry.[9]

Given a quantity profile $\boldsymbol{q}$ where the number of firms that have entered is $n \equiv \dim(\mathbf{q})$,

[6] More precisely, $P(Q)$ and its derivatives may be undefined for $Q = 0$ (e.g. with $\lim_{Q\downarrow 0} P(Q) = +\infty$ and $\lim_{Q\downarrow 0} P'(Q) = -\infty$).

[7] This assumption is only used for some of the results; different orders of differentiability are used for different results. P is required to be differentiable for $Q < \overline{Q}$.

[8] Section A.1 in Appendix A presents models that give rise to this objective function. Often we implicitly assume $\lambda < 1$ either to divide by $1 - \lambda$ or because $\lambda = 1$ can indeed be ill-behaved.

[9] In a market with more than two firms, a trade that only involves two firms (e.g. a firm buying shares of another firm) will generally not shift the coefficient of effective sympathy uniformly for all firm pairs.

total surplus is given by

$$\mathrm{TS}(\boldsymbol{q}) := \int_0^Q P(X)dX - \sum_{i=1}^n C(q_i) - nf,$$

while the Herfindahl–Hirschman index (HHI) and modified HHI (MHHI) are given by

$$\mathrm{HHI}(\boldsymbol{q}) := \sum_{i=1}^n s_i^2, \qquad \mathrm{MHHI}(\boldsymbol{q}) \equiv (1-\lambda)\,\mathrm{HHI}(\boldsymbol{q}) + \lambda.$$

We denote the MHHI at a symmetric equilibrium by $H_n := (1+\lambda(n-1))/n$.

3.2 Equilibrium in the pricing stage

In this section we study equilibrium in the pricing stage.

3.2.1 Existence, uniqueness and stability of pricing stage equilibrium

Having described the environment we first derive conditions for equilibrium existence and uniqueness in the pricing stage using lattice-theoretic methods as in Amir and Lambson (AL; 2000). Let $\Delta(Q,Q_{-i}) := 1 - \lambda - C''(Q-Q_{-i})/P'(Q)$ be defined on the lattice $L := \left\{(Q,Q_{-i}) \in \mathbb{R}^2_{++} : \overline{Q} > Q \geq Q_{-i}\right\}$. $\Delta > 0$ allows for decreasing, constant and mildly increasing returns to scale, while $\Delta < 0$ allows for more significant IRS.

Proposition 1. The following statements hold:

(i) Assume $\Delta(Q,Q_{-i}) > 0$ on L. Then, in the pricing stage

  (a) there exists a symmetric equilibrium and no asymmetric equilibria,

  (b) if also $E_{P'}(Q) < (1+\lambda+\Delta(Q,Q_{-i})/n)/H_n$ on L, then there exists a unique and symmetric equilibrium,

(ii) Assume that $\Delta(Q,Q_{-i}) < 0$ and $E_{P'}(Q) < \frac{1+\lambda+\Delta(Q,Q_{-i})}{1-(1-\lambda)(1-s_i)}$ on L for any i. Then,

  (a) for every $m \in \{1,2\ldots,n\}$ there exists a unique quantity q_m such that any quantity profile where each of m firms produces quantity q_m and the remaining $n-m$ firms produce 0 is an equilibrium of the pricing stage,

  (b) no other equilibria exist in the pricing stage.

Remark: The second order of differentiability of $P(Q)$ is inessential. However, it simplifies the arguments and interpretation, and emphasizes the tension between the assumption $\Delta < 0$ and the one on $E_{P'}(Q)$. The latter guarantees that π_i is strictly concave in q_i whenever $P(Q) > 0$. IRS are needed for $\Delta < 0$ but at the same time tend to violate profit concavity.[10]

Corollary 1.1 studies existence and uniqueness of the pricing stage equilibrium under linear demand and linear-quadratic costs. The linear-quadratic cost function is of the form $C(q) = c_1 q + c_2 q^2/2$, where $c_1 \geq 0$, for (i) $q \in [0, +\infty)$ if $c_2 \geq 0$, (ii) $q \in [0, -c_1/c_2]$ if $c_2 < 0$.[11]

Corollary 1.1. Assume demand is linear, $P(Q) = \max\{a - bQ, 0\}$, and cost is linear-quadratic with $a > c_1 > 0$ and $c_2 > -2bc_1/a$. Then,

(i) if $c_2 > -b(1-\lambda)$, then $\Delta > 0$ on L and a unique, interior and symmetric equilibrium exists,

(ii) if $c_2 < -b(1-\lambda)$, then $\Delta < 0$ on L and a unique (in the class of symmetric equilibria), interior, symmetric equilibrium exists.

In light of Proposition 1 we maintain from now on the following assumption unless otherwise stated in a specific result. The assumption should be understood to hold at the relevant values of (n,λ) for each result.[12]

Maintained Assumption. The conditions in part (i-a,b) or part (ii) of Proposition 1 hold.

[10] In the $\Delta > 0$ case, for $\lambda = 0$ we recover the condition $C'' - P' > 0$, under which AL show that a symmetric equilibrium exists and there are no asymmetric equilibria (Theorem 2.1). In the $\Delta < 0$ case, the assumption on $E_{P'}$ guarantees that the firm's objective is quasiconcave in its own quantity, under which condition AL show the same result. For $\lambda = 1$, DRS are necessary for uniqueness of the (symmetric) equilibrium. For example, with CRS, there are infinitely many equilibria (the symmetric one included), all with the same fixed total quantity arbitrarily distributed across firms, since each firm maximizes aggregate industry profits. Analogously, with $C'' < 0$ it is an equilibrium for firms to concentrate all production in one firm to take advantage of the IRS, as indicated in part (ii-a) of the proposition.

[11] Cost is indeed increasing over $q \leq -c/d$ when $d < 0$. The value of C for higher q will not matter in applications, as parameter values will be such that firms do not produce more than $-c_1/c_2$.

[12] For example, for global comparative statics of the Cournot game as λ changes, the assumption is assumed to hold for fixed n and every $\lambda \in [0,1]$. For existence of a free entry equilibrium in the post-entry overlapping ownership case for a fixed λ, it is sufficient that the assumption hold for every $n \in \mathbb{R}_{++}$ and that fixed λ.

Remark 1: When we relax the conditions in part (i-a,b) of Proposition 1, the second-order condition (SOC) of the firm's problem, that is $E_{P'}(Q) < (1 + \lambda + \Delta(Q,Q_{-i}))/H_n$, will be assumed to hold strictly in any symmetric pricing stage equilibrium.

Remark 2: When in a result we assume $\Delta > 0$ (resp. $\Delta < 0$) it is thus understood that the additional assumption of part (i) (resp. part (ii)) of Proposition 1 also holds.

For some of the results we will discuss what happens in the case where the maintained assumption may fail in this manner: $\Delta > 0$ on L but the assumption that $E'_P < (1 + \lambda + \Delta/n)/H_n$ on L is dropped. In this case the Cournot game equilibrium set may consist of multiple symmetric equilibria. Propositions under this relaxed version of the maintained assumption will be marked with an apostrophe (').

The maintained assumption guarantees that firms will play a symmetric equilibrium in the pricing stage subgame of any SPE. Given that monopoly profit is positive, that equilibrium will be interior. When $\Delta < 0$, we have seen that the pricing subgame also has asymmetric equilibria; however, these cannot be part of an SPE of the complete game, since the entering firms that do not produce would prefer to avoid the entry cost by not entering.

Proposition 2 examines local asymptotic stability of the pricing stage equilibrium in the sense of the myopic continuous adjustment process, as described in al Nowaihi and Levine (1985).

Proposition 2. If $\Delta > 0$, then the pricing stage equilibrium is locally stable.

Proposition 2' studies stability with the maintained assumption relaxed.

Proposition 2'. Assume $\Delta > 0$ but drop the assumption that $E_{P'}(Q) < (1 + \lambda + \Delta(Q,Q_{-i})/n)/H_n$ on L, so that multiple symmetric equilibria may exist. Then, a pricing stage equilibrium is locally stable if and only if $E_{P'}(Q) < (1 + \lambda + \Delta(Q,Q_{-i})/n)/H_n$ in that equilibrium.

Remark: For $\lambda = 0$ we recover the sufficient local (in)stability conditions implied by Theorems 3, 4 and 5 of al Nowaihi and Levine (1985).[13]

[13] al Nowaihi and Levine (1985) deal with a possibly asymmetric equilibrium; they provide analogous conditions where expressions such as Δ vary across firms.

Under $\Delta > 0$, when we drop the condition $E_{P'} < (1+\lambda+\Delta/n)/H_n$ on L guaranteeing uniqueness, multiple symmetric equilibria may exist, some of which stable and some unstable. These two sets of equilbria are differentiated by a local version of the dropped condition. An equilibrium is stable if and only if the dropped condition holds *in that equilibrium.*

We denote by $\boldsymbol{q}_n$ the symmetric Cournot equilibrium when n firms are in the market (which is unique under out maintained assumption), and with some abuse of notation by q_n the quantity each firm produces in that profile, where the subscript n now does *not* refer to the identity of the n-th firm; we also write $Q_n := nq_n$, $\mathrm{TS}_n := \mathrm{TS}(\boldsymbol{q}_n)$.[14] To simplify notation, for any $n > 0$ we also denote by $\Pi(n,\lambda) := P(Q_n)\, q_n - C(q_n)$ the individual (gross) profit in the symmetric equilibrium of the Cournot game with n firms and Edgeworth coefficient λ; define $\Pi(0,\lambda) := 0$ for any λ. When we ignore the integer constraint on n, we allow all equilibrium objects, such as $\Pi(n,\lambda)$, to be defined for $n \in \mathbb{R}_{++}$. We refer to $\Pi(n,\lambda) - f$ as net profit. The equilibrium pricing formula is

$$\frac{P(Q_n) - C'(q_n)}{P(Q_n)} = \frac{H_n}{\eta(Q_n)}. \tag{1}$$

3.2.2 Comparative statics of pricing stage equilibrium

Proposition 3 describes some comparative statics for the pricing stage (*i.e.*, under a fixed number of firms). We treat n as a continuous variable and differentiate with respect to it.

Proposition 3. The following statements hold:

(i) total and individual quantity, and total surplus are decreasing in λ,

(ii) if $E_{P'}(Q) < 2$ (resp. $E_{P'}(Q) > (1+\lambda)/\lambda$) for every $Q < \overline{Q}$, then individual quantity is decreasing (resp. increasing) in n over $n \geq 2$,[15]

(iii) if $\Delta > 0$, then total quantity is increasing in n,

(iv) if $\Delta < 0$, then total quantity is decreasing in n.

[14] All of these objects also depend on λ, which we omit in the subscript to simplify notation.

[15] Of course, the condition $E_{P'}(Q) > (1+\lambda)/\lambda$ is very strong, especially given the assumption $E_{P'}(Q) < (1+\lambda+\Delta/n)/H_n$ on L. Also, it pushes against profit concavity in own quantity, which can make even make the monopolist's problem ill-behaved. For example, with CESL demand, when $E > 2$, $\lim_{Q\downarrow 0}(P(Q)Q - C(Q)) = +\infty$.

Further, individual quantity may be non-monotone in n. For example, for $\lambda = 0$, CESL demand and CRS with $a = b = 1$, $c = 2$ and $E = 1.8$, $q_1 < q_2$ and q_n is decreasing in n for $n \geq 2$.

Remark 1: If $\Delta > 0$ and also for every $Q < \overline{Q}$, $E_{P'}(Q) < 2$, $E_{P'}(Q)\left[E_{P'}(Q) + E_{P''}(Q)\right] \geq -2$ and for every $q < \overline{q}$, $C''(q)$,$C'''(q) \geq 0$,[16] then $(\partial Q_n)^2/(\partial\lambda\partial n) < 0$,

Remark 2: $\partial^2 q_n/(\partial\lambda\partial n)$ can be negative or positive (and change sign as λ and/or n changes). For example, for CRS and CESL demand

$$\operatorname{sgn}\left\{\frac{\partial^2 q_n}{\partial\lambda\partial n}\right\} = \operatorname{sgn}\left\{n + \lambda(n-1) - 3 - \left(H_n(n-1) - 1\right)E\right\}.$$

Competition is business-stealing (*i.e.*, q_n is decreasing in n) under standard assumptions. As in AL, for $\Delta > 0$ the Cournot market is quasi-competitive (*i.e.*, Q_n is increasing in n) while for $\Delta < 0$ it is quasi-anticompetitive (*i.e.*, Q_n is decreasing in n). Also, increases in overlapping ownership cause the price to increase and total surplus to fall.[17] Under the assumptions of Remark 1, this effect of overlapping ownership is strongest in industries with large number of firms, which would otherwise be the most competitive ones.

Proposition 3' studies comparative statics of the pricing stage with the maintained assumption relaxed.

Proposition 3'. Assume $\Delta > 0$ but drop the assumption that $E_{P'}(Q) < (1 + \lambda + \Delta(Q,Q_{-i})/n)/H_n$ on L, so that multiple symmetric equilibria may exist. Then, in extremal equilibria:

(i) total and individual quantity, and total surplus are non-increasing in λ,

(ii) total quantity is non-decreasing in n.

Under $\Delta > 0$, when we drop the condition $E_{P'} < (1+\lambda+\Delta/n)/H_n$ on L guaranteeing uniqueness, the results of Proposition 3 still hold weakly for extremal equilibria.[18] They also hold strictly but only locally around stable equilibria.[19] As for example observed in AL, a discrete change (e.g. in the integer number n of firms) may even lead to a change in the number of equilibria rendering it hard to make meaningful comparisons between non-extremal equilibra.

[16] If $P''(Q) = 0$, cancel P'' in $E_{P'}(Q)$ with the one in $E_{P''}(Q)$. Under CESL demand, $E_{P'}(Q)\left[E_{P'}(Q) + E_{P''}(Q)\right] \geq -2$ holds if and only if $E \leq 2$.

[17] The fact that total surplus is decreasing in λ relies on the fact that firms are symmetric.

[18] By extremal equilibria we mean the equilibrium with minimum quantity among all equilibria and the equilibrium with maximum quantity among all equilibria.

[19] Namely, parts (i), (ii) and (iii) of Proposition 3 hold locally in any stable equilibrium (with n treated as a continuous variable in parts (ii) and (iii)).

Last, notice how aggregate industry profits depend on the number of firms. Under increasing returns to scale, monopoly maximizes aggregate industry profits. As we will see in the next section, this combined with the fact that the Cournot market is quasi-anticompetitive for $\Delta < 0$ will imply that monopoly maximizes total surplus when $\Delta < 0$. Similarly, under constant elasticity costs and non-decreasing returns to scale ($E_C(q_n) \leq 1$), aggregate industry profits decrease with the number of firms. However, under decreasing returns to scale the effect of entry on aggregate industry profits depends on two main factors. On the one hand, entry causes price to fall. On the other hand, as more firms enter, production is distributed across more firms, which induces savings in variable costs. For returns to scale decreasing strongly enough the latter effect dominates and aggregate industry profits increase with the number of firms. This is why multiple firms may enter in that case even when $\lambda = 1$. The Online Appendix analyzes in detail how aggregate profits change with the number of firms.

3.3 The entry stage

The way in which firms make their entry decisions is described in sections 4 and 5. Section 4 considers the case where firms develop overlapping ownership only after entering (for example if they are new firms). In section 5 potential entrants are already established firms and have overlapping ownership before entering, which directly affects their entry decisions. In both cases of we assume that when indifferent, firms enter, show that there is a unique equilibrium and denote by $n^*(\lambda)$ the number of firms that enter in equilibrium when the Edgeworth coefficient is λ. Analogously, $\widehat{n}^*(\lambda)$ is the number of firms that enter in equilibrium if we ignore the integer constraint on n.

We will consider the planning problem where a total surplus-maximizing planner can choose the number of firms and then let the firms compete à la Cournot (giving subsidies in case profit Π in the symmetric Cournot equilibrium falls below f). Denote by $n^o(\lambda)$ the number of firms that given λ maximizes total surplus. Clearly, under our usual assumptions if the planner could also choose λ, she would set $\lambda = 0$, since total surplus is decreasing in λ. We will study whether equilibrium entry is excessive or insufficient compared to $n^o(\lambda)$ assuming that TS_n is single-peaked.[20] We also define $\widehat{n}^o(\lambda)$ to be the

[20]Lemma 2 in Appendix B provides sufficient conditions for it to be concave. Note that with the integer constraint on n, there can be two values of it that maximize total surplus even with single-peaked total surplus. However, this is not important for the analysis.

number of firms that given λ maximizes total surplus if we ignore the integer constraint on n:

$$n^o(\lambda) := \arg\max_{n\in\mathbb{N}} \mathrm{TS}_n \quad \text{and} \quad \widehat{n}^o(\lambda) := \arg\max_{n\in\mathbb{R}_+} \mathrm{TS}_n\,.$$

We assume $n^o(\lambda) \geq 1$, $\widehat{n}^o(\lambda) > 0$. The welfare loss is given by

$$\mathrm{WL}(\lambda) := \mathrm{TS}_{\widehat{n}^o(\lambda)} - \mathrm{TS}_{\widehat{n}^*(\lambda)} \geq 0.$$

Last, we will look at comparative statics with respect to λ—including how free entry equilibrium total surplus varies with λ. This way we will deduce the optimal choice of a planner that can only choose the level of overlapping ownership and then allow firms to freely enter. As we will see, in that case a positive level of overlapping ownership or even complete indexation ($\lambda = 1$) may be optimal.

4 Free entry under post-entry overlapping ownership

A main concern is that overlapping ownership may suppress entry by making firms internalize the effect their entry would have on other firms' profits. However, we first study the case where potential entrants have no prior overlapping ownership, but after they enter the market and before they pick quantities in the second stage they develop overlapping ownership, so that they have an Edgeworth coefficient of effective sympathy $\lambda \in [0,1]$. This can be interpreted as a long–run equilibrium whereby start-up firms (or already existing firms but without overlapping ownership) enter the industry and then develop overlapping ownership through time. Appendix A.1 describes explicitly how post-entry overlapping ownership can arise.

The exogeneity of λ is important with post-entry overlapping ownership, since the incentives of firms to allow for ownership ties after entry are not modeled. For instance, if instead the amount of shares that investors buy from the entrepreneurs depended on the extent of entry—since the latter affects profits, then λ would be a function of n. Although the exogeneity of λ is restrictive, if firms become publicly traded after entry (at least in the long–run), they indeed have limited control over their ownership ties, since for instance investment funds are free to buy shares of all firms.

4.1 The entry stage

Each firm only looks at own profit to decide whether to enter as there is no overlapping ownership when it does so.[21] The owners of a firm seek to maximize its profits (and thus its value) when they decide whether to enter the market; in doing so they take into account how common ownership will affect product market outcomes.

We have that $\boldsymbol{q}_n$ is a free entry equilibrium production profile if and only if

$$\Pi\left(n,\lambda\right) \geq f > \Pi\left(n+1,\lambda\right)$$

as in Mankiw and Whinston (1986). When overlapping ownership develops only after firms enter, it does not affect the incentives of firms to enter except for through its effect on individual profits in the Cournot game. We assume that there exists n (large enough) such that $\Pi(n,\lambda) < f$ for any λ.

4.2 Existence and uniqueness of equilibrium

Proposition 4 studies existence and uniqueness of a free entry equilibrium.

Proposition 4. $\Pi(n,\lambda)$ is decreasing in n and a unique free entry equilibrium exists.

Given that $\Pi(n,\lambda)$ is decreasing in n, $\widehat{n}^*(\lambda)$ is pinned down by $\Pi\left(\widehat{n}^*(\lambda),\lambda\right) = f$ and $n^*(\lambda) = \max\left\{n \in \mathbb{N} : \Pi\left(n,\lambda\right) \geq f\right\}$.

Under standard conditions equilibrium individual profit in the pricing stage is decreasing in the number of firms and, therefore, there is a unique free entry equilibrium. In that equilibrium firms enter the market until profits have fallen so much that if an additional firm enters, it will push profits even lower so that they will not manage to cover the entry cost.

Proposition 4' studies existence of a free entry equilibrium with the maintained assumption relaxed so that multiple symmetric equilibria may exist.

Proposition 4'. Assume $\Delta > 0$ but drop the assumption that $E_{P'}(Q) < (1 + \lambda + \Delta(Q,Q_{-i})/n)/H_n$ on L. Then, in extremal equilibria profit is non-increasing in n and a free entry equilibrium where in the pricing stage firms play an extremal equilibrium exists.

[21] Formally, if a firm does not enter, its payoff is 0; if it does, it is $(1 + \lambda(n-1))\left(\Pi\left(n,\lambda\right) - f\right)$. Thus, it is optimal for an n-th firm to enter if and only if $\Pi\left(n,\lambda\right) \geq f$.

If for example there is multiplicity of pricing stage equilibria for every n, this will mean that there will exist at least two free entry equilibria: one where the minimum pricing stage equilibrium is played and one where the maximum pricing stage equilibrium is played.[22]

4.3 Overlapping ownership effects

Proposition 5 studies the effects of overlapping ownership.

Proposition 5. Ignore the integer constraint on n (so that entry is given by $\widehat{n}^*(\lambda)$). Then

(i) the number of firms entering is increasing in λ,

(ii) individual quantity, total quantity, and total surplus are decreasing in λ,

(iii) if $C'' \geq 0$, then the MHHI is increasing in λ.

Remark: there exists a set of thresholds $\mathcal{L} := \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$, $\lambda_1 < \lambda_2 < \cdots < \lambda_k$, such that

(a) for every $\lambda \in \mathcal{L}$, $\Pi\left(n^*(\lambda), \lambda\right) = f$, and $n^*(\lambda) = \widehat{n}^*(\lambda)$,

(b) for λ between two consecutive thresholds $n^*(\lambda)$ remains constant and everything behaves as in the Cournot game with a fixed number of firms.

The results of Proposition 5 are schematically summarized in Figure 1. When we take into account the integer constraint, the number of firms is a step function of in λ, and individual quantity is decreasing with jumps down. Total quantity has a decreasing trend with jumps up (resp. down) for the values of λ at which an extra firm enters under $\Delta > 0$ (resp. $\Delta < 0$). Also, total surplus tends to decrease with λ.[23]

Importantly, even when there is free entry of firms—so that increases in λ lead to the entry of new firms as incumbents suppress their quantities, if the entering firms develop

[22] Observe that extremal equilibria correspond to extremal equilibrium profits. Namely, the minimum (resp. maximum) equilibrium quantity corresponds to the maximum (resp. minimum) equilibrium profit.

[23] To compare total surplus under the integer constraint on n, $\mathrm{TS}_{n^*(\lambda)}$, to its value when we ignore the integer constraint, $\mathrm{TS}_{\widehat{n}^*(\lambda)}$, notice the following. For λ between two consecutive thresholds, $\lambda \in (\lambda_k, \lambda_{k+1})$, it holds that $\widehat{n}^*(\lambda) > n^*(\lambda)$. Thus, given that total surplus is globally concave in n, if there is (weakly) excessive entry under the integer constraint, ignoring the integer constraint exacerbates excess entry. Therefore, between two λ thresholds $\mathrm{TS}_{\widehat{n}^*(\lambda)} < \mathrm{TS}_{n^*(\lambda)}$, and for λ equal to a thresholds $\mathrm{TS}_{n^*(\lambda)}$ has a jump down. But if under the integer constraint entry is insufficient by 1 firm (which is possible), $n^*(\lambda) = n^o(\lambda) - 1$, then the above does not follow.

overlapping ownership after entering (up to the level the incumbents have), consumer and total surplus tend to decrease with λ, as in the symmetric case with a fixed number of firms. Also, if one looks at HHI, it will seem as if competition rises as λ increases, which can even be the case with MHHI, although the latter will increase with λ if we slightly strengthen our assumptions. Last, because of the integer constraint, for appropriate levels of λ a small increase in λ can spur the entry of an extra firm causing the total quantity to rise.

The fact that the price increases with λ is to be expected. Remember that an increase in λ is met with an increase in n so that the zero profit condition $\Pi\left(\widehat{n}^{*}(\lambda),\lambda\right) = f$ is satisfied. When the Cournot market is quasi-anticompetitive ($\Delta < 0$), both the increase in λ and the increase in n cause price to increase. When the Cournot market is quasi-competitive ($\Delta > 0$), the increase in λ tends to increase price, while the increase in n tends to decrease it. The former effect dominates. For example, assume non-DRS and by contradiction that after an increase in λ enough additional firms enter the market to keep the price at its level before the increase in λ (or even make it lower). Then, after the increase in λ (i) each firm has a lower share of the market, (ii) the price has not increased, and (iii) the average (variable) cost of production has not decreased (due to non-DRS and individual quantity having decreased). Thus, individual profit has decreased, violating the zero profit condition. The result still holds under DRS, since under $\Delta > 0$.

$$\frac{\dfrac{\partial \Pi\left(n,\lambda\right)}{\partial \lambda}}{\dfrac{\partial \Pi\left(n,\lambda\right)}{\partial n}} = \frac{\overbrace{\left(1-H_n\right)\dfrac{\partial Q_n}{\partial \lambda}}^{-}}{\underbrace{\left(1-H_n\right)\dfrac{\partial Q_n}{\partial n}}_{+} + \underbrace{H_n\dfrac{Q_n}{n}}_{+}},$$

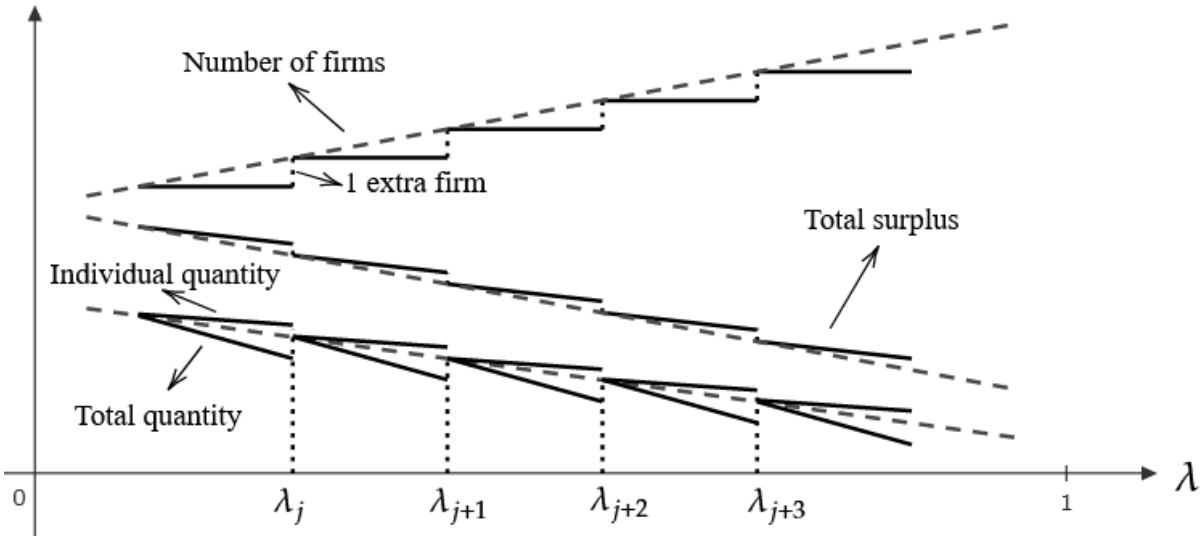
so that

$$\left|\frac{\overbrace{\partial \Pi\left(n,\lambda\right)/\partial \lambda}^{+}}{\underbrace{\partial \Pi\left(n,\lambda\right)/\partial n}_{-}}\right| < \left|\frac{\partial Q_n/\partial \lambda}{\partial Q_n/\partial n}\right| = \left|\frac{\overbrace{dP\left(Q_n\right)/d\lambda}^{+}}{\underbrace{dP\left(Q_n\right)/dn}_{-}}\right|.$$

This means that for individual profit to stay unchanged after an increase in λ, fewer firms need to enter compared to the number of firms that need to enter for the price to remain

unchanged after the increase in λ.

Figure 1: Equilibrium with post-entry overlapping ownership for varying λ



Note: the solid lines represent equilibrium values under the integer constraint; from bottom to top they represent the behavior of total quantity, individual quantity, total surplus, and the number of firms. The dashed lines represent equilibrium values when we ignore the integer constraint; from bottom to top the first line represents the behavior of both the total and the individual quantity, the second of total surplus, and the third of the number of firms. The solid total quantity line is drawn for the case $\Delta > 0$. To draw the solid total surplus line above the dashed ones we assume that the total surplus is concave in n, and that $n^*(\lambda) \geq n^o(\lambda)$ (see footnote 23 and Proposition 7). Only the signs of the slopes of the lines and the directions of the jumps are part of the result; the curvatures of lines, sizes and spacing of the jumps have been chosen for simplicity in depiction (except for the sizes in the jumps in the number of firms).

4.4 Entry cost effect on entry

Proposition 6 studies the effect of the entry cost on entry, as well as how this effect depends on the extent of overlapping ownership. Note that λ affects the slope $d\widehat{n}^*(\lambda)/df$ directly but also through its effect on $n^*(\lambda)$. We are interested in the direct effect so we keep $n^*(\lambda)$ fixed as we vary λ.

Proposition 6. Ignore the integer constraint on n (so that entry is given by $\widehat{n}^*(\lambda)$). Then

(i) entry is decreasing in the entry cost,

(ii) if λ increases and other parameters x (e.g. demand, cost parameters) change infinitesimally so that $\widehat{n}^*(\lambda)$ stays fixed and $\partial^2\Pi(n,\lambda)/(\partial x \partial n) = 0$ (e.g. (f,λ) infinitesimally changes in direction $\mathbf{v} := (-(d\widehat{n}^*(\lambda)/d\lambda)/(d\widehat{n}^*(\lambda)/df),1))$, then $|d\widehat{n}^*(\lambda)/df|$ changes in direction given by $\text{sgn}\left\{ \frac{\partial^2\Pi(n,\lambda)}{\partial\lambda\partial n}\Big|_{n=\widehat{n}^*(\lambda)} \right\}$.

As long as individual profit is decreasing in n, the results of Proposition 6 are not specific to Cournot competition. Part (ii) says that if an increase in λ makes individual profit more (resp. less) strongly decreasing in the number of firms, then an increase in the entry cost needs to be met with a smaller (resp. larger) increase in the number of firms for the zero profit entry condition to continue to hold.

Under Cournot competition, Claim 1 below provides sufficient conditions for the cross derivative of $\Pi(n,\lambda)$ to be negative or positive, which by Proposition 6 implies that overlapping ownership alleviates or exacerbates, respectively, the negative effect of the entry cost on entry.

Claim 1. Assume CRS.

(i) If $\partial E_{P'}(Q)/\partial Q \geq 0$, $E_{P'}(Q_n) \in [0,1]$ and $n \geq 5+E_{P'}(Q_n)$, then $\partial^2\Pi\,(n,\lambda)\,/(\partial\lambda\partial n) < 0$ for every $\lambda \in (0,1)$.

(ii) If $\partial E_{P'}(Q)/\partial Q \leq 0$, $E_{P'}(Q_n) \leq 0$ and $n \leq 6/\,(2 - E_{P'}(Q_n))$, then $\partial^2\Pi\,(n,\lambda)\,/(\partial\lambda\partial n) > 0$ for every $\lambda \in (0,1)$.

Remark: Proposition 15 in Appendix B provides a more detailed result on the sign of $\partial^2\Pi\,(n,\lambda)\,/(\partial\lambda\partial n)$.

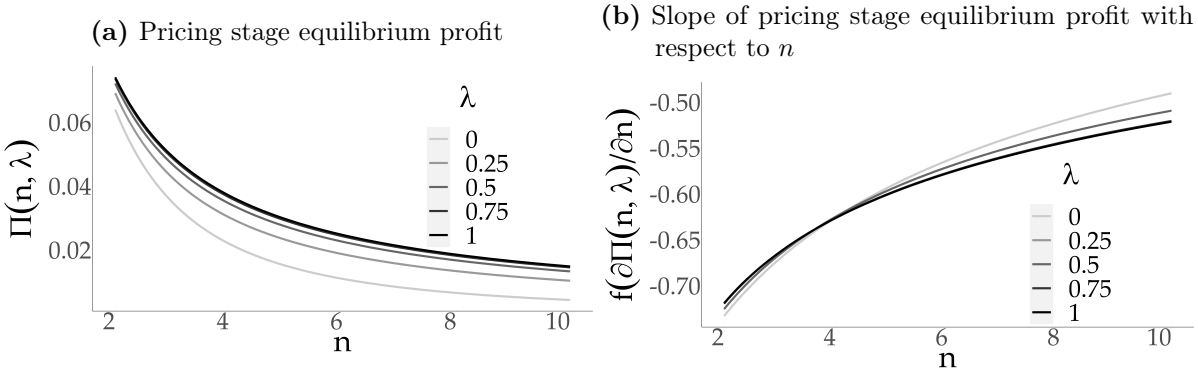
Claim 1 encompasses CESL demand. Therefore, under CESL demand with $E \in [0,1]$ and CRS, in markets with not too low entry ($n \geq 6$ is sufficient), overlapping ownership makes entry less strongly decreasing in the entry cost. This means that as long as it does not induce firms to internalize the entry externality, overlapping ownership could alleviate the negative macroeconomic implications of rising entry costs documented by Gutiérrez et al. (2021) in the U.S. over the past 20 years. The sufficient condition of part (ii) has limited scope, since it requires $n \leq 3$. These can also be seen in Figure 2. If entry is very low, then $\partial\Pi(n,\lambda)/\partial n$ increases (*i.e.* it decreases in absolute value) with λ; otherwise it decreases.

4.5 Equilibrium entry versus the socially optimal level of entry

The derivative of equilibrium total surplus with respect to n is given by

$$\frac{d\,\mathrm{TS}_n}{dn} = P(Q_n)\left(n\frac{\partial q_n}{\partial n} + q_n\right) - C(q_n) - nC'(q_n)\frac{\partial q_n}{\partial n} - f$$

Figure 2: Pricing stage equilibrium profit under CESL demand and CRS



Note: $E = 0.5$, $a = 2$, $b = 1$, $c = 1$. The increasing and sign-preserving transformation in (b) is $f(x) := -(-x)^{1/10}$.

$$= \Pi(n,\lambda) - f + n\left(P(Q_n) - C'(q_n)\right)\frac{\partial q_n}{\partial n},$$

and therefore

$$\left.\frac{d\,\mathrm{TS}_n}{dn}\right|_{n=\widehat{n}^*(\lambda)} = \overbrace{\Pi\left(\widehat{n}^*(\lambda),\lambda\right) - f}^{=0} + n\left(P(Q_n) - C'(q_n)\right)\left.\frac{\partial q_n}{\partial n}\right|_{n=\widehat{n}^*(\lambda)} \propto \left.\frac{\partial q_n}{\partial n}\right|_{n=\widehat{n}^*(\lambda)},$$

so that with TS_n concave in n, under business-stealing (resp. business-enhancing) competition entry is excessive (resp. insufficient). The results of Mankiw and Whinston (1986) and Amir et al. (2014) generalize to the case of post-entry overlapping ownership. Proposition 7 shows that indeed with business-stealing competition and under the integer constraint, entry is never insufficient by more than one firm as in the case without overlapping ownership.

Proposition 7. The following statements hold:

(i) if $\Delta > 0$ and $E_{P'}(Q) < 2$ on L, then $n^*(\lambda) \geq n^o(\lambda) - 1$,

(ii) if $\Delta < 0$, then $n^*(\lambda) \geq n^o(\lambda) = 1$.

How does the welfare loss vary with λ? Under CRS, the welfare loss $\mathrm{WL}(\lambda)$ is locally increasing (resp. decreasing) in λ, $d\,\mathrm{WL}(\lambda)/d\lambda \overset{(<)}{>} 0$, if and only if

$$E_{P'}\left(Q_{\widehat{n}^o(\lambda)}\right) \overset{(>)}{<} 1 + H^{-1}_{\widehat{n}^o(\lambda)} - \frac{\left(\widehat{n}^*(\lambda) - 1\right)\left(1 - H_{\widehat{n}^*(\lambda)}\right)}{H_{\widehat{n}^o(\lambda)}}\underbrace{\frac{\Pi\left(\widehat{n}^o(\lambda),\lambda\right)}{\Pi\left(\widehat{n}^*(\lambda),\lambda\right)}}_{>1},$$

where $\Pi\left(\widehat{n}^o(\lambda),\lambda\right)/\Pi\left(\widehat{n}^*(\lambda),\lambda\right) > 1$ under Prop 4, since $\widehat{n}^*(\lambda) > \widehat{n}^o(\lambda)$.

Even though there is tendency for excessive entry and total surplus decreases with λ, under CRS, the welfare loss will decrease with the level of overlapping ownership if

$$E_{P'}\left(Q_{\widehat{n}^o(\lambda)}\right) \in \left(1 + H^{-1}_{\widehat{n}^o(\lambda)} - \frac{\left(\widehat{n}^*(\lambda)-1\right)\left(1-H_{\widehat{n}^*(\lambda)}\right)}{H_{\widehat{n}^o(\lambda)}}\frac{\Pi\left(\widehat{n}^o(\lambda),\lambda\right)}{\Pi\left(\widehat{n}^*(\lambda),\lambda\right)}, 2\right).$$

Indeed, provided that there are not strong DRS (that would induce variable-cost savings from entry large enough compared to the extra entry costs), under complete indexation ($\lambda = 1$), only one firm enters in equilibrium, which also is the planner's optimum, $n^*(1) = n^o(\lambda) = 1$, so that WL(1) = 0 as can be seen in Figure 3. For example, under CRS, this is because any additional entry will leave the total quantity, price and total variable costs unchanged; it will only increase the total fixed costs. However, for E lower, overlapping ownership will exacerbate the welfare loss. In Figure 3(a), the welfare loss first increases and then decreases with λ.

Proposition 7' compares equilibrium entry to the socially optimal level of entry when the maintained assumption is relaxed. Also, it considers the case of business-enhancing competition. To economize on notation, we are still using q_n, $n^*(\lambda)$ and $n^o(\lambda)$ to denote equilibrium values in a specific extremal equilibrium even though multiple equilibria may exist.

Proposition 7'. Assume $\Delta > 0$ but drop the assumption that $E_{P'}(Q) < (1 + \lambda + \Delta(Q,Q_{-i})/n)/H_n$ on L. Let the same type of extremal equilibrium (i.e. minimum or maximum) be played in the pricing stage of the free entry equilibrium and the planner's solution. Then,

(i) if $q_{n^o(\lambda)-1} \geq q_{n^o(\lambda)}$, then $n^*(\lambda) \geq n^o(\lambda) - 1$.

(ii) if $q_{n^o(\lambda)+1} \geq q_{n^o(\lambda)}$, then $n^*(\lambda) \leq n^o(\lambda)$.

Remark: Proposition 7' and part (ii) of Proposition 7 extend the results of Amir et al. (2014) to the case of post-entry overlapping ownership.

Under $\Delta > 0$, when competition is locally business-stealing, equilibrium entry is not

insufficient by more than one firm as in the case without overlapping ownership. On the other hand, if competition is locally business-enhancing, entry is not excessive.

5 Free entry under pre-entry overlapping ownership

In the last section overlapping develops only after entry, and thus does not directly affect the incentives of firms to enter. The only channel through which it affects entry is by increasing profits in the post entry game. Firms expect this and therefore entry increases with overlapping ownership. Even in an industry with pre-entry overlapping ownership, this channel is still present. Therefore, the effect on entry will depend on the relative magnitude of the different channels. This section studies the main concern about the anti-competitive effects that overlapping ownership can have through entry: that it may suppress entry by inducing firms to internalize the effect their entry would have on other firms' profits.

5.1 The entry stage

Assume that the potential entrants are established firms that already have overlapping ownership with an Edgeworth coefficient of effective sympathy $\lambda \in [0,1]$. Given that $n-1$ firms enter, it is optimal for an n-th firm to enter if and only if $(1+\lambda(n-1))\left(\Pi\left(n,\lambda\right)-f\right) \geq \lambda(n-1)\left(\Pi\left(n-1,\lambda\right)-f\right)$. This can equivalently be written as

$$\Psi(n,\lambda) := \overbrace{\Pi\left(n,\lambda\right)}^{\text{own profit from entry}} - \lambda \overbrace{(n-1)\left(\Pi\left(n-1,\lambda\right)-\Pi\left(n,\lambda\right)\right)}^{=:\Xi(n,\lambda)>0\text{, entry externality on other firms}} \geq \overbrace{f}^{\text{cost of entry}}, \qquad (2)$$

where $\Xi(n,\lambda)$ denotes the externality that the entry of the n-th firm poses on the other firms, that is, the absolute value of the reduction in the aggregate profits of all other firms caused by the entry of the n-th firm.[24] $\Psi(n,\lambda)$ is a firm's own profit from entry minus the part of the entry externality that is internalized by the firm (*i.e.*, the entry externality

[24]This externality can be further decomposed into two effects:

$$\Xi(n,\lambda) = \overbrace{(n-1)\frac{\Pi\left(n-1,\lambda\right)}{n}}^{\text{profit-stealing}} + \frac{n-1}{n}\overbrace{\left[(n-1)\Pi\left(n-1,\lambda\right)-n\Pi\left(n,\lambda\right)\right]}^{\text{change in aggregate industry profits}}$$

Even if the entry of the n-th firm did not affect aggregate industry profits, the firm still steals $1/n$-th of the profit of each of the other $n-1$ firms; this corresponds to the profit-stealing effect. At the same time, the n-th firm's entry affects aggregate industry profits—share $(n-1)/n$ of which is earned by the other $n-1$ firms—as shown in Proposition 12 in Appendix B.

multiplied by λ, the degree to which a firm internalizes this externality). For $\lambda = 0$ no part of the externality is internalized, for $\lambda \in (0,1)$ the externality is partially internalized, while for $\lambda = 1$ it is fully internalized. We call $\Psi(n,\lambda)$ a firm's "internalized profit" from entry.

In deciding whether to enter a firm compares the profit it will make to the cost of entry and the negative externality its entry will pose to the other firms. Thus, pre-existing overlapping ownership directly alters the incentives of firms to enter in a way additional to its effect on individual profit in the Cournot game.

Then, $\boldsymbol{q}_n$ is a free entry equilibrium if and only if

$$\Psi(n,\lambda) \geq f > \Psi(n+1,\lambda), \tag{3}$$

which for $\lambda = 0$ reduces to the standard condition $\Pi(n,0) \geq f > \Pi(n+1,0)$. For $\lambda = 1$, in the equilibrium of the pricing stage entering firms maximize aggregate industry gross profits, $n\Pi(n,1)$.[25] For $\lambda = 1$, (3) reduces to

$$n\Pi(n,1) - (n-1)\Pi(n-1,1) \geq f > (n+1)\Pi(n,1) - n\Pi(n,1).$$

Firms want to maximize aggregate industry profits, so they enter as long as entry increases aggregate gross profits by enough to cover entry costs. Hence, provided that the savings in variable costs are not large enough to compensate for additional entry costs, only one firm enters given that $\Psi(1,1) = \Pi(1,1) = \Pi(1,0) \geq f$. In the simulation of Figure 3(d) savings in variable costs are large enough compared to the fixed cost to make five firms enter when $\lambda = 1$.

We assume that there exists n (large enough) such that $\Psi(n,\lambda) < f$ for every λ.

5.2 Existence and uniqueness of equilibrium

Define $\Delta\Pi(n,\lambda) := \Pi(n,\lambda) - \Pi(n-1,\lambda) < 0$, the decrease in individual profit caused by the entry of an extra firm, so that the entry externality is $\Xi(n,\lambda) \equiv -(n-1)\Delta\Pi(n,\lambda)$. Given that individual profit is decreasing in the number of firms as shown in Proposition

[25] More precisely, each firm individually chooses its quantity to maximize aggregate industry gross profits *given* the quantities of the other firms. Therefore, there can also be equilibria that do not maximize $n\Pi(n,1)$, since a monopolist could change all firms' quantities to increase $n\Pi(n,1)$. We have already seen that under $\Delta < 0$ there are multiple equilibria in the Cournot game, even though with $C'' < 0$, concentrating production in one firm maximizes $n\Pi(n,1)$ (see Proposition 12 in Appendix B).

4, Proposition 8 identifies a condition under which a unique equilibrium exists.

Proposition 8. Assume that for every $n \in [1, +\infty)$

$$E_{\Delta\Pi,n}(n,\lambda) := -\frac{\frac{\partial\Pi(n,\lambda)}{\partial n} - \frac{\partial\Pi(\nu,\lambda)}{\partial\nu}\Big|_{\nu=n-1}}{\Delta\Pi(n,\lambda)}(n-1) < \frac{(n-1)\left(1+\lambda+\varepsilon(n,\lambda)\right)}{1+\lambda(n-1)}$$

where $E_{\Delta\Pi,n}$ (a measure of) the elasticity with respect to n of the slope of individual profit with respect to n, and $\varepsilon(n,\lambda) := \partial\Pi\left(\nu,\lambda\right)/\partial\nu|_{\nu=n-1}/\Delta\Pi(n,\lambda) - 1$. Then, $\Psi(n,\lambda)$ is decreasing in n, and thus a unique equilibrium with free entry exists.[26]

Remark $\varepsilon(n,\lambda)$ will be close to 0, since the fraction inside it is a derivative divided by a "small" discrete change, and thus close to 1. Put differently, by the mean value theorem $\Delta\Pi(n,\lambda) = \partial\Pi\left(\nu,\lambda\right)/\partial\nu|_{\nu=\nu^*}$ for some real number $\nu^* \in [n-1,n]$.

The condition in Proposition 8 requires that equilibrium profit in the pricing stage be not too convex in n; that is, the rate at which individual profit decreases with n should not decrease (in absolute value) too fast with n. Put differently, the magnitude of the entry externality should not decrease too fast with n. Notice in (2) that an increase in n (i) decreases the first term $\Pi(n,\lambda)$, (ii) tends to increase the entry externality $\Xi(n,\lambda)$ through the increase in $(n-1)$ (as entry affects the profits of more firms), which tends to decrease $\Psi(n,\lambda)$, and (iii) affects the entry externality $\Xi(n,\lambda)$ through its effect on the magnitude of the entry externality $\Pi\left(n-1,\lambda\right) - \Pi\left(n,\lambda\right)$ on a single firm. As long as the per-firm entry externality does not decrease with n too fast, $\Psi(n,\lambda)$ decreases with n.[27]

For example, if $\Pi(n,\lambda)$ is concave in n, so that profit is decreasing in n at an increasing (in absolute value) rate, then the condition is satisfied given $\varepsilon(n,\lambda) \approx 0$. For $\lambda = 0$ the condition reduces to $\Pi(n,\lambda)$ being decreasing in n, which has been shown in Proposition 4. For $\lambda = 1$, the proposition requires that as the number of firms increases, aggregate industry profits increase (e.g. due to variable cost-savings) by less and less.

[26] In section B.4 of Appendix B we present a differential version of (2) to derive analytical results. Propositions 13 and 14 in Appendix B are then differential versions of Propositions 8 and 9, respectively. That is, all discreet jumps from $n-1$ to n are replaced by derivatives, so that evaluation of profits and derivatives thereof in different equilbria is not required. Claim 4 in Appendix B then shows that the corresponding condition in the modified model is satisfied under linear demand and constant marginal costs.

[27] The case $\lambda = 1$ may actually be ill-behaved with $\Psi(n,\lambda)$ being constant in n; see for example Figure 4.

We maintain the assumption that $\Psi(n,\lambda)$ is decreasing in n. Then, $\widehat{n}^*(\lambda)$ is pinned down by $\Psi\left(\widehat{n}^*(\lambda),\lambda\right) = f$, and $n^*(\lambda) = \max\left\{n \in \mathbb{N} : \Psi\left(n,\lambda\right) \geq f\right\}$.

5.3 Overlapping ownership effects on entry

Proposition 8 guarantees that a firm's internalized profit from entry $\Psi(n,\lambda)$ is decreasing in n. Thus, the effect of changes in λ on entry will be determined by the sign of the (partial) derivative of $\Psi(n,\lambda)$ with respect to λ. If $\partial\Psi(n,\lambda)/\partial\lambda$ is positive (resp. negative), then increases in λ should be met with increases (resp. decreases) in n for (3) to continue to hold.

Proposition 9. Equilibrium entry (locally) changes with λ in direction given by[28]

$$\operatorname{sgn}\left\{\frac{d\widehat{n}^*(\lambda)}{d\lambda}\right\} = \operatorname{sgn}\left\{\overbrace{\frac{1}{\lambda}\frac{E_{\Pi,\lambda}(\widehat{n}^*(\lambda),\lambda)}{\widehat{E}_{\Pi,n}(\widehat{n}^*(\lambda),\lambda)}}^{\substack{\text{increase in own}\\ \text{profit from entry}}} - \overbrace{E_{\Xi,\lambda}\left(\widehat{n}^*(\lambda),\lambda\right)}^{\substack{\text{change in magnitude}\\ \text{of entry externality}}} - \overbrace{1}^{\substack{\text{increase in internalization}\\ \text{of entry externality}}}\right\}$$

where

$$E_{\Xi,\lambda}(n,\lambda) := \frac{\partial\Xi(n,\lambda)}{\partial\lambda}\frac{\lambda}{\Xi(n,\lambda)} = -\frac{\frac{\partial\Pi(n,\lambda)}{\partial\lambda} - \frac{\partial\Pi(n-1,\lambda)}{\partial\lambda}}{\Pi\left(n,\lambda\right) - \Pi\left(n-1,\lambda\right)}\lambda$$

is the elasticity of the entry externality with respect to λ, which can also be seen as a measure of the elasticity with respect to λ of the slope of individual profit with respect to n,

$$\widehat{E}_{\Pi,n}(n,\lambda) := -\frac{\Pi\left(n,\lambda\right) - \Pi\left(n-1,\lambda\right)}{\Pi(n,\lambda)}(n-1) > 0$$

is a measure of the elasticity of individual profit with respect to n, and

$$E_{\Pi,\lambda}(n,\lambda) := \frac{\frac{\partial\Pi(n,\lambda)}{\partial\lambda}}{\Pi(n,\lambda)}\lambda > 0$$

the elasticity of individual profit with respect to λ.

Remark 1: With the integer constraint $n^*(\lambda)$ does not change with an infinitesimal change $d\lambda$ in λ unless we are the knife-edge case where $\Psi(n^*(\lambda),\lambda) = f$. Thus, as λ

[28] For $\lambda = 0$, cancel the λ in $1/\lambda$ with the one in $E_{\Pi,\lambda}(n,\lambda)$.

increases everything will behave as in the case with a fixed number of firms, until λ reaches knife-edge cases causing a jump in $n^*(\lambda)$ as implied by Proposition 9.

Remark 2: Evaluating the expressions in Propositions 9 and 8 requires evaluation of profits and derivatives thereof in different equilbria of the pricing stage (with n and $n-1$ firms). This is possible under parametric assumptions with the problem still remaining intractable in general. Therefore, we present numerical and simulation results.[29]

Proposition 9 studies the effects of overlapping ownership on entry. An increase in overlapping ownership affects entry through three separate channels. On the one hand, it increases the degree of internalization of the negative externality of entry on other firms' profits; this increased internalization tends to limit entry. On the other hand, it tends to increase equilibrium profits in the Cournot game, which tends to increase entry. Last, there is a channel with an ambiguous effect on entry: overlapping ownership changes the magnitude of the entry externality; that is, it affects how strongly equilibrium profits in the pricing stage decrease with the number of firms. A high (and positive) elasticity $E_{\Xi,\lambda}$ of the entry externality Ξ with respect to λ tends to make entry decreasing in λ, while $E_{\Xi,\lambda}$ being negative tends to make entry increasing in λ. Indeed, the magnitude of the entry externality $\Xi(n^*(\lambda),\lambda)$ can increase or decrease with λ.[30]

The three channels that we have identified are not specific to our assumption of Cournot competition in the pricing stage. Nevertheless, the direction of the change in the entry externality $\Xi(n,\lambda)$ and the magnitudes of the different channels depend on the market structure. For example, the three channels are also present in the circular-market model with common ownership of Sato and Matsumura (SM; 2020)—although the authors discuss only the first two channels. The direction of the third channel's effect is not ambiguous in their model, where the magnitude of the entry externality monotonically

[29]In section B.4 of Appendix B firms decide whether to enter by examining a differential version of (2). Propositions 13 and 14 in Appendix B are then differential versions of Propositions 8 and 9, respectively. That is, all discreet jumps from $n-1$ to n are replaced by derivatives, so that evaluation of profits and derivatives thereof in different equilibria is not required. For the same reason, analytically examining the effects on quantity and welfare is hard. Corollary 14.1 shows that in the modified model of Appendix B, total quantity decreases with λ under constant marginal costs and general assumptions on demand. Indeed, under non-IRS total quantity decreases with λ in Figure 3.

[30]Under the parametrizations of Figures 3(b) and 3(c), where entry is low, Ξ is decreasing in λ. Ξ being decreasing in λ is expected under low entry given Proposition 15 on the modified model in section B.4 of Appendix B. On the other hand, Ξ is decreasing in λ under the parametrization of Figure 3(a). The results on Ξ under these parametrizations are available from the authors upon request.

increases with the extent of overlapping ownership. Also, they find that entry always decreases with overlapping ownership, while in our case both possibilities are present.

Under the parametrization of Figure 3(a), for λ low, the rise in own profit due to increases in λ dominates the other two channels—in this parametrization the magnitude of the entry externality increases with λ.[31] However, for high λ competition in the product market is already soft enough, so that further increases in λ suppress entry. The result is in disagreement with the theoretical result of the stylized model in Newham et al. (2019), which predicts that common ownership between the brand and generic firms unambiguously reduces entry. This contrast is due to the fact that in their model the level of common ownership is assumed to not affect profits in the pricing stage—and clearly then it cannot affect the magnitude of the negative effect of entry on the brand firm's profits either. Therefore, two of the three channels that we have identified are absent, leaving the increased internalization by generics of the brand's profits to suppress entry.

Within the simple framework of linear demand and CRS or quadratic costs of Figure 3, equilibrium total surplus can behave in multiple different ways as the extent of overlapping ownership changes—contrary to SM's unambiguous inverted-U relationship.[32] (i) It can have a U relationship with λ (Figure 3(a)), so that an intermediate level of overlapping ownership actually minimizes total surplus. (ii) It can have an inverted-U relationship with λ (Figure 3(c)) with an intermediate level of overlapping ownership maximizing total surplus. (iii) It can be monotonically increasing (Figure 3(b)) in λ with complete indexation of the industry being optimal ($\lambda = 1$). (iv) Last, it can be decreasing (Figure 3(d)) in λ with $\lambda = 0$ maximizing total surplus. With CRS, the average variable cost of production is not affected by entry. Thus, the planner has to balance the effect of overlapping ownership—direct (in the Cournot game), and indirect (through its effect on entry)—on the total quantity and price, and on the total entry costs (through its effect of entry). For low entry cost (Figure 3(a)), the planner chooses no overlapping ownership, while for higher entry cost, intermediate values of λ (Figure 3(c)) or even $\lambda = 1$ can be optimal (Figure 3(b)). With IRS (Figure 3(e)), an increase in overlapping ownership can both decrease the price and increase total surplus. Particularly, choosing λ high enough (e.g. $\lambda = 1$)—inducing a monopoly—is socially optimal under both a total surplus and a

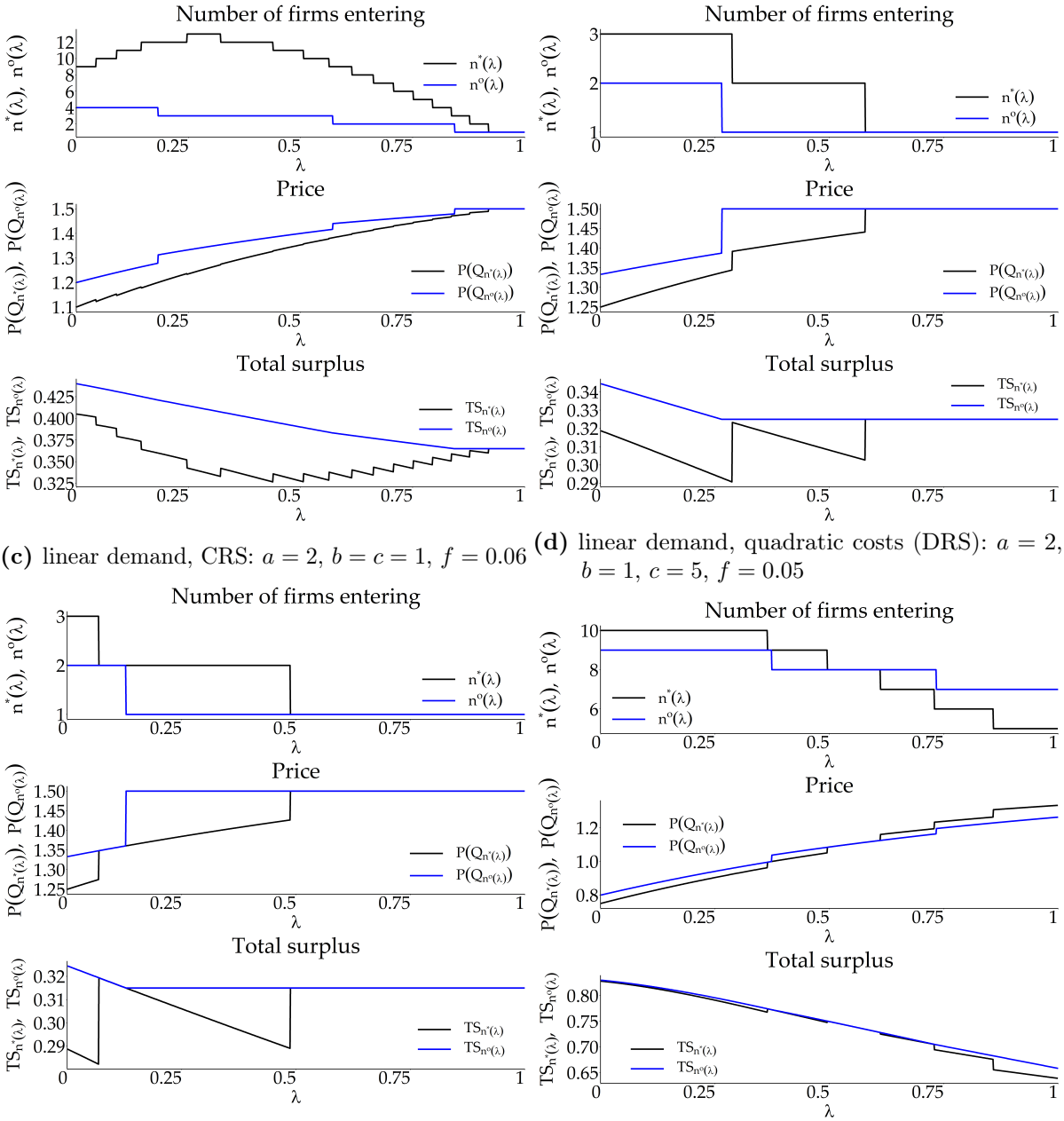
[31] Similarly, under the IRS parametrization of Figure 3(e), the number of firms increases in λ up to a point where it jumps to 1.

[32] In describing the relationship of equilibrium total surplus with λ we ignore the integer constraint on n, which is taken into account in Figure 3. SM always ignore the integer constraint.

consumer surplus standard. Also, notice that a planner that cannot control both entry and overlapping ownership will still achieve what a planner that can control both would (i.e. a monopoly).

Figure 3: Equilibrium with pre-entry overlapping ownership for varying λ

(a) linear demand, CRS: $a = 2$, $b = c = 1$, $f = 0.01$ **(b)** linear demand, CRS: $a = 2$, $b = c = 1$, $f = 0.05$



(c) linear demand, CRS: $a = 2$, $b = c = 1$, $f = 0.06$ **(d)** linear demand, quadratic costs (DRS): $a = 2$, $b = 1$, $c = 5$, $f = 0.05$

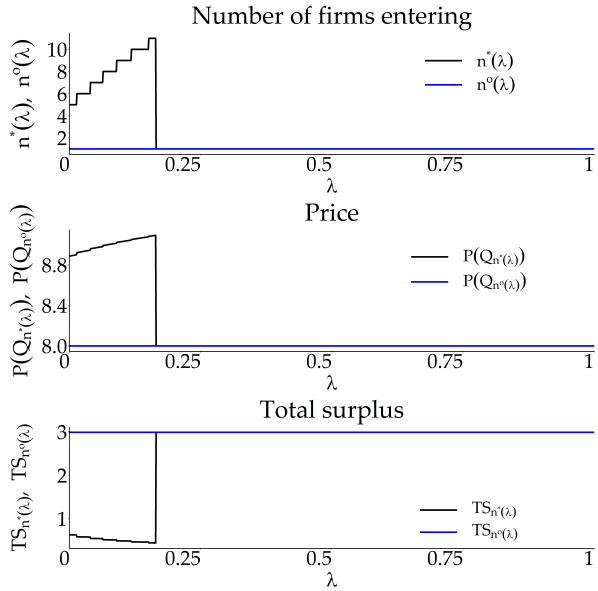
Note: black lines represent values in equilibrium; blue represent values in the planner's solution.

Numerical Result 1 identifies more general sufficient conditions for entry to increase or decrease with λ.[33]

[33]The result that for high λ, further increases in λ reduce entry is in accordance with the simulations in Figure 5 and Corollary 14.1 in the more tractable framework of Appendix B. The same is true for the

Figure 3 (Cont.): Equilibrium with pre-entry overlapping ownership for varying λ

(e) linear demand, linear-quadratic costs (IRS): $a = 10$, $b = 1$, $c_1 = 9$, $c_2 = -3/2$, $f = 0.01$



Note: black lines represent values in equilibrium; blue represent values in the planner's solution.

Numerical Result 1. Under CESL demand, CRS, $\lambda < 1$ and $\widehat{n}^*(\lambda) \geq 2$, it holds that

(i) entry is decreasing in λ if (a) $E \in (1,2)$ and $\lambda \geq 1/2$, or (b) $E < 1$ and $\lambda \geq 2/5$,

(ii) entry is (locally) increasing in λ if $\widehat{n}^*(\lambda) \geq 7$ and (a) $E \in (1,2)$ and $\lambda \leq 1/4$, or (b) $E \in [0,1)$ and $\lambda \leq 1/5$,

(iii) the total quantity is decreasing in λ.

For high enough levels of overlapping ownership further increases in these levels suppress entry. In the U.S. for example, common ownership levels among publicly listed firms have indeed been "high enough" during at least the last decade (e.g. see Azar and Vives, 2021a; Backus et al., 2021a).[34] Thus, if private firms are treated as a competitive fringe that only affects the residual demand in the oligopolies of public firms, then further increases in common ownership among the latter are likely to reduce entry by public firms in product markets where other public firms are already present.

result that for low λ, and not too low entry and E, entry is increasing in λ.

[34]This means that the average value of λ (across pairs of firms) has surpassed $0.4 - 0.5$ in recent years. Clearly, the average λ estimate (and thus the point in time when it surpasses a threshold) depends on the particular assumptions on the γ's in the O'Brien and Salop (2000) model. Also, notice that we compare the average value of λ to the threshold of λ in our model of symmetric firms and overlapping ownership structure.

Under $\Delta > 0$, when entry is decreasing in λ, the price will be increasing in λ, since both the increase in λ and the resulting decrease in entry tend to decrease the total quantity. On the other hand, for low levels of overlapping ownership and not too low entry, overlapping ownership spurs entry (up to the point where λ is too high and then entry decreases with it). However, Numerical Result 1 asserts that with CRS the direct effect of λ on the total quantity dominates, so that the price always increases with λ, as in the case of post-entry overlapping ownership.

These results can be loosely interpreted as follows. For λ low and entry high, competition is intense, so that there is a lot of room for an increase in λ to soften it and increase individual profit in the Cournot game. For λ high, pricing competition is already soft enough so that the increase in the internalization of the entry externality (due to an increase in λ) dominates and entry decreases with λ.

Finally, a few words on the interpretation of this comparative statics exercise on a change in λ are in place. Strictly speaking, this exercise amounts to changing the level of overlapping ownership *before* firms make their entry decisions. Therefore, it can be thought of as a counterfactual or a comparison of otherwise similar markets that have different levels of overlapping ownership (before firms enter). When interpreting changes in λ in a market where firms have already entered, one should consider the following. If our model predicts that a change in λ will cause the number of firms to fall, whether incumbent firms will indeed exit can depend on the extent to which the entry cost f is a sunk cost or a fixed operating cost that they can avoid by exiting.

5.4 Entry cost effect on entry

Proposition 10 below studies the effect of the entry cost on entry, as well as how this effect depends on the extent of overlapping ownership (with the level of entry held fixed). It mirrors Proposition 6 with the role of profit $\Pi(n,\lambda)$ now assumed by the internalized profit $\Psi(n,\lambda)$.

Proposition 10. Ignore the integer constraint on n (so that entry is given by $\widehat{n}^*(\lambda)$). Then

(i) entry is decreasing in the entry cost,

(ii) if λ and other parameters x (e.g. demand, cost parameters) change infinitesimally

so that $\widehat{n}^*(\lambda)$ stays fixed and $\partial^2\Psi(n,\lambda)/(\partial x\partial n) = 0$ (e.g. (f,λ) infinitesimally changes in direction $\mathbf{v} := (-(d\widehat{n}^*(\lambda)/d\lambda)/(d\widehat{n}^*(\lambda)/df),1))$, then $|d\widehat{n}^*(\lambda)/df|$ changes in direction given by sgn $\left\{ \frac{\partial^2\Psi(n,\lambda)}{\partial\lambda\partial n}\Big|_{n=\widehat{n}^*(\lambda)} \right\}$.

Numerical result 2 provides conditions under which the cross derivative of $\Psi(n,\lambda)$ is positive, which by Proposition 10 implies that overlapping ownership exacerbates the negative effect of the entry cost on entry.

Numerical Result 2. Under CESL demand and CRS, $\partial^2\Psi\left(n,\lambda\right)/(\partial\lambda\partial n) > 0$ if (i) $E \in (1,1.7]$ and $n \in [2,7]$, or (ii) $E < 1$ and $n \in [2,8]$.

Under CESL demand and CRS, markets with low entry are particularly susceptible to further decreases in entry when there is overlapping ownership. In such markets, apart from the direct effect it has on entry, overlapping ownership also makes entry more strongly decreasing in the entry cost. This means that overlapping ownership could exacerbate the negative macroeconomic implications of rising entry costs documented by Gutiérrez et al. (2021) in the U.S. over the past 20 years.

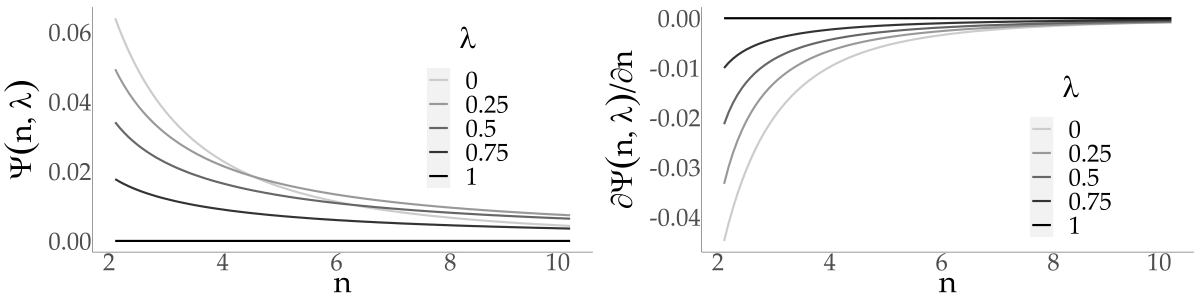
The conditions of Numerical result 2 overlap with those of part (i) of Claim 1, which deals with the case of post-entry overlapping ownership. Thus, under the same parametrization, whether overlapping ownership exacerbates or alleviates the negative effect of the entry cost on entry will depend on the form of overlapping ownership. If overlapping ownership is present prior to entry thus making firms internalize the entry externality, then it exacerbates the effect. However, if it develops after entry thus only affecting entry through its effect on product market competition, then it alleviates the effect.

The result can also be seen in Figure 4. $\partial\Pi(n,\lambda)/\partial n$ increases (*i.e.* it decreases in absolute value) with λ.[35]

[35]Notice that $\Psi(n,1) = 0$ for every $n \geq 2$. This is because (i) $\Psi(n,1)$ is equal to the change in aggregate industry profits when an additional firm enters and (ii) under CRS the equilibrium price in the Cournot game does not change with the number of firms and neither do the aggregate production costs. Therefore, only one firm enters under $\lambda = 1$.

Figure 4: Internalized profit $\Psi(n,\lambda)$ under CESL demand and CRS

(a) Internalized profit **(b)** Slope of internalized profit with respect to n



Note: $E = 0.5$, $a = 2$, $b = 1$, $c = 1$.

5.5 Equilibrium entry versus the socially optimal level of entry

Assume that TS_n is globally concave in n and ignore the integer constraint on n.[36] The internalization of the entry externality makes the analysis significantly different from the one under pre-entry overlapping ownership. Now we have that for $\lambda > 0$

$$\left.\frac{d\,\mathrm{TS}_n}{dn}\right|_{n=\widehat{n}^*(\lambda)} = \Pi\left(\widehat{n}^*(\lambda),\lambda\right) - f - \left(1+\lambda(n-1)\right) Q_n P'(Q_n) \left.\frac{\partial q_n}{\partial n}\right|_{n=\widehat{n}^*(\lambda)}$$

$$\propto \underbrace{\lambda \frac{\Xi\left(\widehat{n}^*(\lambda),\lambda\right)}{\Pi\left(\widehat{n}^*(\lambda),\lambda\right)}}_{\substack{>0 \\ \text{internalized} \\ \text{normalized} \\ \text{entry externality}}} + \underbrace{\left(1 + \frac{E_C\left(q_{\widehat{n}^*(\lambda)}\right) - 1}{\dfrac{P\left(Q_{\widehat{n}^*(\lambda)}\right)}{AC\left(q_{\widehat{n}^*(\lambda)}\right)} - 1}\right)^{-1}}_{\substack{>0 \\ \text{if DRS, then } < 1 \\ \text{if CRS, then } = 1 \\ \text{if IRS, then } > 1;}} \underbrace{\left.\frac{\partial q_n}{\partial n}\frac{n}{q_n}\right|_{n=\widehat{n}^*(\lambda)}}_{\substack{<0 \text{ if } E_{P'}\left(Q_{\widehat{n}^*(\lambda)}\right)<2 \\ \text{and } \widehat{n}^*(\lambda)\geq 2; \\ \text{if negative, the higher} \\ \text{in absolute value,} \\ \text{the stronger the} \\ \text{business-stealing effect}}},$$

where we have used the $\Psi\left(\widehat{n}^*(\lambda),\lambda\right) = f$ entry condition and the pricing formula (1). Let us have a closer look at the two terms in the above expression.

$$\frac{\Xi\left(n,\lambda\right)}{\Pi\left(n,\lambda\right)} \equiv (n-1)\frac{\Pi\left(n-1,\lambda\right) - \Pi\left(n,\lambda\right)}{\Pi\left(n,\lambda\right)}$$

is the normalized entry externality; it is $(n-1)$ times the percentage increase in the profit of each of the $n-1$ other firms when the n-th firm decides *not* to enter compared to the

[36] Under pre-entry overlapping ownership, ignoring the integer constraint is not as important, since as we will see both cases of excessive and insufficient entry are possible (and by more than one firm under the integer constraint as can be seen in Figure 3).

case where it did enter.

$$1+\frac{E_C(q)-1}{\frac{P(nq)}{AC(q)}-1}=\frac{P(nq)-AC(q)}{P(nq)-C'(q)}>0$$

is a (coarse) measure of the elasticity of the cost function, and thus of the extent to which returns to scale are decreasing. It is positive by the pricing formula and since profits are positive in the free entry equilibrium for $\lambda>0$. Also, $(E_C(q)-1)\,/\,(P(nq)/(AC(q))-1)$ and $(E_C(q_n)-1)$ have the same sign, so that for example under constant elasticity costs and for any q, $(E_C(q)-1)\,/\,(P(nq)/(AC(q))-1)$ is higher than (resp. lower than/equal to) 0 if and only if returns to scale are decreasing (resp. increasing/constant).

We see then that whether entry is excessive or insufficient will depend on (i) the level of overlapping ownership λ, (ii) the magnitude of the entry externality $\widetilde{\Xi}$, (iii) whether returns to scale are increasing or decreasing and to what extent, and (iv) whether competition is business-stealing or business-enhancing, and to what degree $\left|\frac{\partial q_n}{\partial n}\frac{n}{q_n}\right|$.

Under business-stealing competition and all else constant, we distinguish the following forces. Increases in the level of overlapping ownership or the magnitude of the entry externality, tend to make entry insufficient; these forces are complements in inducing insufficient entry. Also, DRS tend to make entry insufficient, since the planner takes advantage of variable-cost savings due to entry to a greater extent than firms do. Firms do not fully internalize the variable cost-savings of entry (except in the case of complete indexation) and at the same time more strongly consider the effect of entry on profits through the price. On the other hand, increases in the magnitude of the business-stealing effect and IRS tend to make entry excessive.

Under business-enhancing competition, entry is always insufficient. This can be seen as a generalization of the result of Amir et al. (2014), who prove that entry is insufficient under business-enhancing competition and $\lambda=0$.

We now formally compare equilibrium entry $\widehat{n}^*(\lambda)$ to the socially optimal level of entry $\widehat{n}^o(\lambda)$. First, define

$$\phi\left(n,\lambda\right):=\frac{(n-1)\left(\Pi\left(n,\lambda\right)-\Pi\left(n-1,\lambda\right)\right)}{n\partial\Pi\left(n,\lambda\right)/\partial n}\approx 1,$$

which is non-negative, since the numerator is non-positive and the denominator is negative. It is close to 1, since (i) by the mean value theorem $\Pi\left(n,\lambda\right)-\Pi\left(n-1,\lambda\right)=$

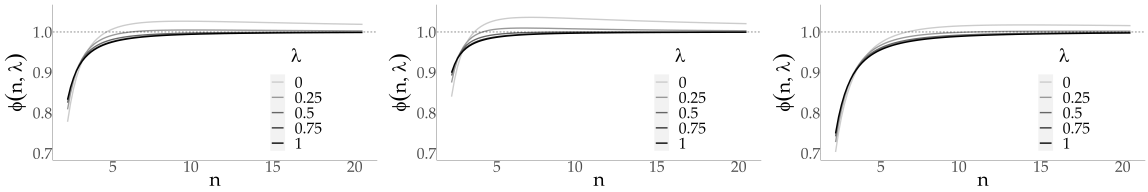
$\partial\Pi(\nu,\lambda)/\partial\nu|_{\nu=\nu^*}$ for some real number $\nu^* \in [n-1,n]$, so that

$$(\Pi(n,\lambda) - \Pi(n-1,\lambda))/(\partial\Pi(n,\lambda)/\partial n) \approx 1$$

and (ii) $(n-1)/n \approx 1$ for n not too small.[37] The numerical results of Figure 5 verify that $\phi(n,\lambda)$ is close to 1, especially for $n \geq 3$.

Figure 5: $\phi(n,\lambda)$ under linear demand and quadratic costs

(a) $b = 1$, $c = 1$ **(b)** $b = 2$, $c = 1$ **(c)** $b = 1$, $c = 2$



Note: it can be checked that $\phi(n,\lambda)$ is invariant to the demand parameter a.

Proposition 11. Assume that TS_n is single-peaked in n, $1-\lambda\phi(\widehat{n}^*(\lambda),\lambda) > 0$ and $\lambda < 1$. Then $\widehat{n}^*(\lambda) \overset{(<)}{>} \widehat{n}^o(\lambda)$ if and only if

$$E_{P'}\left(Q_{\widehat{n}^*(\lambda)}\right) \overset{(>)}{<} H_n^{-1}\left[1+\lambda\left(1-\frac{\phi(n,\lambda)}{1-\lambda\phi(n,\lambda)}\frac{\Delta(Q_n,(n-1)q_n)}{\Lambda_n}\right)\right]\Bigg|_{n=\widehat{n}^*(\lambda)}.$$

(i) If $\Delta(Q_{\widehat{n}^*(\lambda)},(n-1)q_{\widehat{n}^*(\lambda)}) \leq \frac{\Lambda_{\widehat{n}^*(\lambda)}[1-\lambda\phi(\widehat{n}^*(\lambda),\lambda)]}{\phi(\widehat{n}^*(\lambda),\lambda)}$ and $E_{P'}\left(Q_{\widehat{n}^*(\lambda)}\right) < H^{-1}_{\widehat{n}^*(\lambda)}$, then $\widehat{n}^*(\lambda) > \widehat{n}^o(\lambda)$.

(ii) For $\phi(\widehat{n}^*(\lambda),\lambda) = 1$, $C''(q_{\widehat{n}^*(\lambda)}) \leq 0$, if $\widehat{n}^*(\lambda) \geq 3/2$ and $E_{P'}\left(Q_{\widehat{n}^*(\lambda)}\right) < 2-(\widehat{n}^*(\lambda))^{-1}$, then $\widehat{n}^*(\lambda) > \widehat{n}^o(\lambda)$.

Proposition 11 asserts that that unless there are strong DRS (Δ high), entry is excessive under standard assumptions on demand.[38] Without overlapping ownership ($\lambda = 0$), entry is excessive if and only if $E_{P'}\left(Q_{\widehat{n}^*(\lambda)}\right) < \widehat{n}^*(\lambda)$. Thus, under the $E_{P'}\left(Q_{\widehat{n}^*(\lambda)}\right) < 2$ condition and $\widehat{n}^*(\lambda) \geq 2$, there is excessive entry.[39] On the other hand, under DRS and high levels

[37] For $\Pi(n,\lambda)$ strictly convex in n, in which case individual profit decreases with n at a decreasing (in absolute value) rate, $(\Pi(n,\lambda) - \Pi(n-1,\lambda))/(\partial\Pi(n,\lambda)/\partial n) > 1$, which counterbalances $(n-1)/n < 1$.

[38] Notice that the right-hand side in Proposition 11 is decreasing in $C''(q_{\widehat{n}^*(\lambda)})$. The results of Proposition 11 closely resemble those of Proposition 16—which compares $\widehat{n}^*(\lambda)$ and $\widehat{n}^o(\lambda)$ in the model where firms' entry decisions are based on a differential version of (2)—in section B.4 of Appendix B. The difference is that Proposition 11 requires the correction term ϕ, which accounts for the fact that in the pricing stage the slope of profits with respect to n is not constant. ϕ is replaced with exactly 1 in Proposition 16.

[39] Indeed, under these conditions and $\Delta > 0$, Proposition 3 asserts that the total quantity in the

of overlapping ownership, entry is insufficient. Corollary 11.1 shows this result in the case of quadratic costs and linear demand. The numerical simulations in Figures 3(d) and 6 verify the result. Also, higher costs (due to higher cost parameter c) tend to make entry insufficient.

Corollary 11.1. Assume TS_n is single-peaked in n, quadratic costs, linear demand, $1-\lambda\phi\left(\widehat{n}^*(\lambda),\lambda\right)>0$, $\lambda\in(0,1)$ and $\widehat{n}^*(0)>1$. Then, $\widehat{n}^*(\lambda)\overset{(<)}{>}\widehat{n}^o(\lambda)$ if and only if

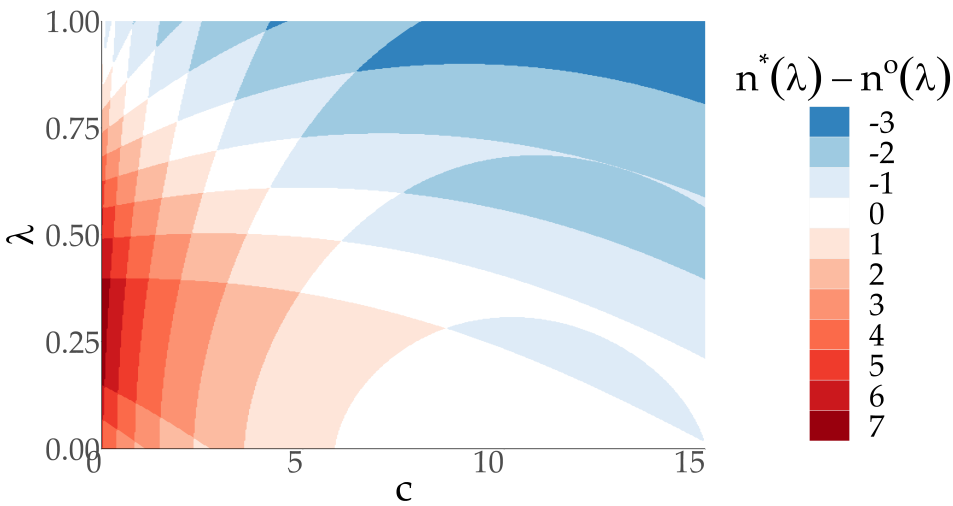
$$\frac{c}{b}\overset{(>)}{<}\zeta(\lambda):=\frac{(1+\lambda)\left(1-\lambda\phi\left(\widehat{n}^*(\lambda),\lambda\right)\right)\Lambda_{\widehat{n}^*(\lambda)}}{\lambda\phi\left(\widehat{n}^*(\lambda),\lambda\right)}-(1-\lambda),$$

so if $\zeta(\lambda)$ is decreasing in λ, then $\widehat{n}^*(\lambda)\overset{(<)}{>}\widehat{n}^o(\lambda)$ if and only if $\lambda\overset{(>)}{<}\zeta^{-1}(c/b)$.

Remark: For instance, if we let for simplicity $\phi\left(\widehat{n}^*(\lambda),\lambda\right)=1$ for every λ, then indeed $\zeta(\lambda)$ is decreasing in λ with $\lim_{\lambda\downarrow 0}\zeta(\lambda)=+\infty$ and $\zeta(1)=0$, so that $\zeta^{-1}(c/b)\in(0,1)$.

On the one hand, the planner combats DRS by having many firms enter. On the other hand, while with positive overlapping ownership this also affects firms' considerations, fewer of them enter the market thereby increasing the price.

Figure 6: Equilibrium versus socially optimal entry under linear demand and quadratic costs



Note: $a=2$, $b=1$, $f=0.05$.

pricing stage is increasing in n and competition is business-stealing, which are the conditions under which Mankiw1986 show excessive entry. However, we see that $\Delta>0$ is not necessary, consistent with Amir et al. (2014), who show excessive entry under business-stealing competition and $\Delta>0$ or $\Delta<0$.

5.6 Entry under the presence of maverick firms

We have examined the effects of overlapping ownership under a symmetric overlapping ownership structure. In that context, overlapping ownership can suppress entry by inducing firms to internalize the negative externality that their entry would have on other firms. However, if there are also potential entrants without ownership ties—which we call maverick firms, then limited entry by commonly-owned firms may spur entry by maverick ones. This could diminish the incentives of a commonly-owned firm to not enter.

For simplicity, model the maverick firms as a competitive fringe that in the first stage (where oligopolists enter) submit an aggregate supply schedule $S : \mathbb{R}_+ \to \mathbb{R}_+$ with $S(p) = 0$ for every $p \in [0,\underline{p}]$ and $S'(p) > 0$ for every $p > \underline{p}$ where $\underline{p} \geq 0$. $S(p)$ gives the total supply of the maverick firms as a function of the market price p. Thus, the price $p > 0$ in the competitive equilibrium among the maverick firms will be implicitly given by $P^{-1}(p) = Q + S(p)$, where Q the total quantity produced by the oligopolists.[40] This means that in the second stage the oligopolists are essentially faced with inverse demand $\widetilde{P} : \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$\widetilde{P}(Q) = \begin{cases} P\left(Q + \omega^{-1}(Q)\right) \in \left(\underline{p}, P(Q)\right) & \text{if } P(Q) > \underline{p} \\ P(Q) & \text{if } P(Q) \leq \underline{p} \end{cases}$$

where $\omega : \mathbb{R}_{++} \to \mathbb{R}_+$ is given by $\omega(y) := P^{-1} \circ S^{-1}(y) - y$.[41] $\omega^{-1}(Q)$ gives the quantity supplied in the competitive equilibrium among the maverick firms when the oligopolists produce Q. For example, in the case of (i) linear demand $P(Q) = \max\{a - bQ,0\}$, (ii) linear maverick aggregate supply schedule $S(p) = \max\{(p - \underline{p})/b_m, 0\}$ with $b_m > 0$ and $\underline{p} \geq 0$, and (iii) CRS (for the oligopolists), $C(q) = cq$, with $a > c \geq \underline{p}$,[42] for any

[40] We assume that $S(p) > P^{-1}(p)$ for p large enough.

[41] To see this substitute $p = P\left(Q + \omega^{-1}(Q)\right)$ in $P^{-1}(p) = Q + S(p)$, which gives

$$Q + \omega^{-1}(Q) = Q + S \circ P\left(Q + \omega^{-1}(Q)\right) \Longleftrightarrow P^{-1} \circ S^{-1} \circ \omega^{-1}(Q) - \omega^{-1}(Q) = Q,$$

which is true by definition of ω.

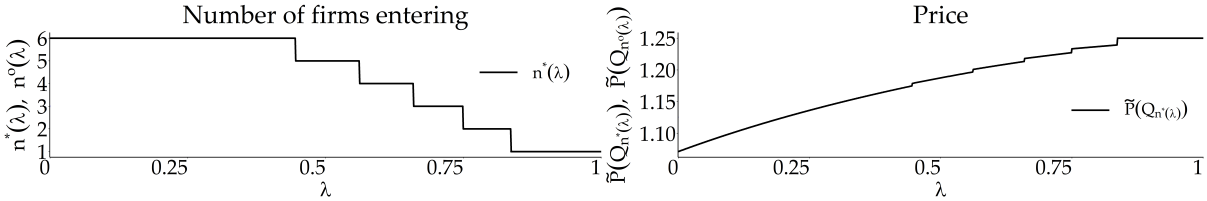
[42] For $c = \underline{p}$, the most efficient maverick firms is as efficient as the oligopolists.

$Q \in [0, (a-c)/b]$, $\widetilde{P}$ is given by[43]

$$\widetilde{P}(Q) = \overbrace{a - \frac{a-\underline{p}}{1+b_m/b}}^{<a} - \overbrace{\frac{b}{1+b/b_m}}^{<b} Q.$$

The (prospect of) entry by maverick firms essentially changes the demand faced by the commonly-owned firms by depressing it and making it more elastic. If wherever P we read $\widetilde{P}$, the results of the previous sections on the effects of overlapping ownership on entry and the price continue to hold (with the number of firms n not counting maverick firm entry). Since demand is depressed, we expect lower levels of entry by commonly-owned firms. Also, given that higher (resp. lower) entry by commonly-owned firms leads to lower (resp. higher) entry by maverick firms, we expect entry to be less sensitive to overlapping ownership due to the presence of the maverick firms. This is indeed the case in Figure 7, which uses the same parametrization as Figure 3(a) but with maverick firms added.

Figure 7: Equilibrium with pre-entry overlapping ownership under the presence of maverick firms for varying λ



Note: lines represent values in equilibrium; linear demand, CRS: $a = 2$, $b = c = 1$, $f = 0.01$; linear maverick aggregate supply schedule: $b_m = \underline{p} = 1$.

Last, the total surplus $\widetilde{\mathrm{TS}}(\boldsymbol{q})$ now includes the maverick firms' surplus, where $\boldsymbol{q}$ still is the quantity profile of the oligopolists. Denote by $\widetilde{\mathrm{TS}}_n$ the pricing stage equilibrium total surplus when n commonly-owned firms enter. As in the previous section, for $\lambda > 0$ we get that[44]

$$\left.\frac{d\widetilde{\mathrm{TS}}_n}{dn}\right|_{n=\widehat{n}^*(\lambda)} \propto \lambda \frac{\widetilde{\Xi}\left(\widehat{n}^*(\lambda),\lambda\right)}{\widetilde{\Pi}\left(\widehat{n}^*(\lambda),\lambda\right)} + \left(1 + \frac{E_C\left(q_{\widehat{n}^*(\lambda)}\right) - 1}{\widetilde{P}\left(Q_{\widehat{n}^*(\lambda)}\right)/AC\left(q_{\widehat{n}^*(\lambda)}\right) - 1}\right)^{-1} \left.\frac{\partial q_n}{\partial n}\frac{n}{q_n}\right|_{n=\widehat{n}^*(\lambda)}$$

where $\widetilde{\Xi}(n,\lambda) := (n-1)\left(\widetilde{\Pi}(n-1,\lambda) - \widetilde{\Pi}(n,\lambda)\right)$, $\widetilde{\Pi}(n,\lambda) := \widetilde{P}(Q_n)q_n - C(q_n)$, and Q_n, q_n are still the quantities produced by the commonly-owned firms in the pricing stage

[43]The inverse demand $\widetilde{P}$ for higher Q does not play a role since the commonly-owned firms will never produce more than $(a-c)/b$. To derive $\widetilde{P}$, solve for it in $(a - \widetilde{P}(Q))/b = Q + (\widetilde{P}(Q) - \underline{p})/b_m$.

[44]Refer to the Online Appendix for a detailed derivation.

equilibrium where n of them enter, and $\widehat{n}^*(\lambda)$ is now pinned down by $\widetilde{\Pi}(\widehat{n}^*(\lambda),\lambda) - \lambda\widetilde{\Xi}(\widehat{n}^*(\lambda),\lambda) = f$.

Whether there is excessive or insufficient entry by commonly-owned firms will depend on the same forces identified in the previous section but with adjusted magnitude since P is replaced by $\widetilde{P}$. Notice that excessive or insufficient entry is based on a planner that controls the entry of oligopolists (but not overlapping ownership) and allows them and the maverick firms to produce freely. Importantly, given the production decisions of the oligopolists, the maverick firms' production level will maximize total surplus since the maverick firms compete perfectly.

6 Conclusion

In this paper we have studied the effects of overlapping ownership in a Cournot oligopoly with free entry. First, we examine the scenario where firms develop overlapping ownership only after entering; this for example will be the case in a novel market with mostly start-up firms. Profits in the pricing stage rise in response to an increase in the extent of overlapping ownership. Although this causes more firms to enter, the anti-competitive effect of overlapping ownership dominates with the price increasing and total surplus falling. Also, entry is never insufficient by more than one firm as in the case without overlapping ownership.

We then move on to examine the scenario where potential entrants are already established and have overlapping ownership before entering. We derive four main results.

First, an increase in overlapping ownership affects entry through three separate channels. It increases the degree of internalization of the negative externality of entry on other firms' profits, which tends to limit entry. However, it also increases equilibrium profits in the pricing stage, which tends to increase entry. Last, it changes the magnitude of the entry externality on other firms' profits; this channel can affect entry in either direction. With Cournot competition in the pricing stage an increase in the degree of overlapping ownership can limit or—in contrast to prior theoretical and empirical work—spur entry.

The three channels through which overlapping ownership affects entry are not specific to the assumption of Cournot competition. Nevertheless, their direction (especially of the ambiguous channel) and magnitudes can vary with the form of competition. For example, (keeping the number of firms fixed) overlapping ownership may increase profits

in a differentiated products market by less than it does in the Cournot game, since with differentiated products competition is already less intense. At the same time, the magnitude of the entry externality may also be diminished. In the extreme case of independent monopolies, overlapping ownership would not affect profits at all.

Second, entry will tend to be insufficient in industries with (i) pre- rather than post-entry overlapping ownership, (ii) high levels of pre-entry overlapping ownership, (iii) strong (negative) effects of entry on other firms' profits, (iv) DRS, (v) business-enhancing competition. In the opposite cases, entry will tend to be excessive.

Third, in industries with significant barriers to entry—so that the number of firms is fixed—overlapping ownership damages welfare when firms are symmetric (with symmetric firms no efficiency gains due to production shifting to more efficient firms are possible). However, in industries with free entry and symmetric firms a total welfare-maximizing planner that can only regulate overlapping ownership and then allow firms to freely enter may choose any level of overlapping ownership, from none at all to complete indexation of the industry. For example, when the entry cost is high, a higher level of overlapping ownership suppresses entry and reduces total entry costs; this effect can be strong enough to make high levels of overlapping ownership welfare-maximizing. Particularly, under increasing returns to scale, complete indexation of the industry will lead to a monopoly, which can be optimal not only under a total welfare standard but also under a consumer surplus standard. Therefore, regulation of overlapping ownership should take into account the latter's effect not only on product market competition but also on entry.

Last, given the negative macroeconomic implications of rising entry costs documented by Gutiérrez et al. (2021) in the U.S. economy over the past 20 years, we are interested in how overlapping ownership mediates the negative effect of entry costs on entry. We find that under common assumptions, post-entry overlapping ownership alleviates the negative impact of an increase of entry costs on entry, while the opposite is true with pre-entry overlapping ownership. This again highlights the importance of accounting for whether overlapping ownership makes firm internalize their entry externality or not.

Further, we derive the following testable implications for markets that existing firms with overlapping ownership consider entering. First, for low levels of overlapping ownership, an increase in overlapping ownership will (i) increase entry if there are many firms in the market already (e.g. because demand is high relative to costs), but (ii) it will decrease entry

if there are only few firms in the industry. Second, for high levels of overlapping ownership, further increases in it will suppress entry. Thus, entry will either depend negatively on overlapping ownership or have an inverted-U relationship with it. Third, unless there are increasing returns to scale, an increase in the extent of overlapping ownership will increase the price. Fourth, increases in the entry cost can suppress entry more in industries with higher levels of overlapping ownership. Fifth, entry by commonly-owned firms is more responsive to the level of overlapping ownership in industries where the prospect of entry by maverick firms (i.e. firms without ownership ties to incumbents) is less salient.

Finally, given that whether and the extent to which ownership ties affect firm conduct is an open empirical question, our results on post-entry overlapping ownership can also be interpreted to address pre-entry overlapping ownership but with overlapping ownership not directly affecting entry behavior. Then, our results suggest a test of the (pre-entry) common ownership hypothesis. If the common ownership hypothesis fails completely—so that common ownership neither affects pricing decisions nor causes a firm to internalize its entry externality, then entry (and any other market outcome) should be independent of common ownership. If the common ownership hypothesis is only *partially* correct in the sense that common ownership influences pricing behavior but does *not* cause the entry externality to be internalized, then entry is expected to increase with the level of common ownership. Finally, if the common ownership hypothesis is correct (*i.e.* common ownership affects firm conduct in both ways), then entry is expected to either depend negatively on common ownership or have an inverted-U relationship with it, as already described.

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A Appendix

A.1 Individual firm's objective function under overlapping ownership

Here we briefly describe settings of common and cross ownership which can give rise to the Cournot-Edgeworth λ oligopoly model that we study.

A.1.1 A model of corporate control under common ownership

There is a finite set $\mathcal{J}$ of investors. For each $j \in \mathcal{J}$, β_{ji} denotes investor j's share of firm i, γ_{ji} captures the extent of her control over firm i, and $u_j(\boldsymbol{q}) := \sum_{i\in\mathcal{F}} \beta_{ji}\pi_i(\boldsymbol{q})$ is her total portfolio profit, where π_i firm i's profit function. O'Brien and Salop (2000) assume that the manager of firm i maximizes a weighted average of the shareholders' portfolio profits; that is, given $\boldsymbol{q}_{-i}$ she maximizes

$$\sum_{j\in\mathcal{J}} \gamma_{ji} u_j(\boldsymbol{q}) \propto \pi_i(\boldsymbol{q}) + \sum_{k\in\mathcal{F}\setminus\{i\}} \lambda_{ik}\pi_k(\boldsymbol{q}),$$

where $\lambda_{ik} := \sum_{j\in J} \gamma_{ji}\beta_{jk} \Big/ \sum_{j\in J} \gamma_{ji}\beta_{ji}$. A common assumption on γ is proportional control, that is $\gamma_{ji} = \beta_{ji}$ for every $j \in \mathcal{J}$ and every $i \in \mathcal{F}$. For appropriate ownership and control structures β and γ it will be that $\lambda_{ik} = \lambda$, fixed for every pair of firms i,k. One such ownership and control structure (β,γ) is described in section 4.

A.1.2 Firm objectives under cross ownership

Firm objectives under cross ownership are also described in Gilo et al. (2006) and López and Vives (2019). Assume that we start with each firm i being held by shareholders who do not hold shares of any of the other firms. Then, each *firm* i buys share $\alpha \in [0, 1/(N-1))$ of every other firm $k \in \mathcal{F} \setminus \{i\}$ *without control rights*. In other words, each firm i acquires a claim to share α of the *total earnings* of every other firm. The total earnings of each firm i now include the profit directly generated by firm i and firm i's earnings from its claims over the other firms' total earnings.

We end up with each firm i being controlled by its initial shareholders, each of whom only hold claims to firm i's total earnings. The controlling shareholders collectively hold a claim to share $(1-(N-1)\alpha)$ of firm i's total earnings. Clearly, all controlling shareholders of firm i agree that firm i should seek to maximize its total earnings.

For every $\boldsymbol{q}$, the total earnings $\widetilde{\pi}_i(\boldsymbol{q})$ of each firm i are then given by the solution to the system of equations

$$\widetilde{\pi}_i(\boldsymbol{q}) = \overbrace{\pi_i(\boldsymbol{q})}^{\substack{\text{firm } i\text{'s earnings from}\\ \text{the profit directly}\\ \text{generated by firm } i}} + \overbrace{\alpha \sum_{k\in\mathcal{F}\setminus\{i\}} \widetilde{\pi}_k(\boldsymbol{q})}^{\substack{\text{firm } i\text{'s earnings from its}\\ \text{claims over each firm } k\text{'s,}\\ k\neq i\text{, total earnings}}}, \quad \text{for each } i \in \mathcal{F}.$$

Solving the system of equations we find that each firm i's objective is to maximize

$$\widetilde{\pi}_i(\boldsymbol{q}) \propto \pi_i(\boldsymbol{q}) + \lambda \sum_{k\in\mathcal{F}\setminus\{i\}} \pi_k(\boldsymbol{q}) \qquad \text{where } \lambda := \alpha/[1-(N-2)\alpha] \in [0,1).$$

A.1.3 An example of post-entry overlapping ownership

Post-entry overlapping ownership can for example arise in the form of common ownership as described below. Let all firms be newly-established and the set of investors $\mathcal{J}$ be partitioned into $\{J_0\} \cup \cup_{i\in\mathcal{F}} \{J_i\}$ with $|J_i| = |J_0| = m$ for every $i \in \mathcal{F}$. Before entry each firm i is (exclusively) held by the set J_i of entrepreneurs with $\beta_{ji} = 1/m$ for every $j \in J_i$; there is no common ownership before entry, so when considering entry, the entrepreneurs of each firm unanimously agree to maximize their own firm's profit.[45] After entry, the set J_0 of investors, who previously held no shares of any firm, buy firm shares. Each investor $j \in J_0$ now holds share $\beta'_{ji} = \sigma/m$ of each firm i that has entered, and each entrepreneur $j \in J_i$ holds share $\beta'_{ji} = (1-\sigma)/m$ of her firm for some $\sigma \in [0,1]$. That is, after entry each entrepreneur sells the same amount of shares to the investors, who are now uniformly invested in all firms in the industry. Consider the O'Brien and Salop (2000) model and for every firm i that has entered let $\gamma'_{ji} = \widetilde{\gamma}/m$ be the control each investor $j \in J_0$ has over firm i for some $\widetilde{\gamma} \in [0,1]$, and $\gamma'_{ji} = (1-\widetilde{\gamma})/m$ the control each entrepreneur $j \in J_i$ has over her firm i.[46] After entry, the manager of each firm i maximizes

$$\pi_i(\boldsymbol{q}) + \lambda \sum_{k\neq i} \pi_k(\boldsymbol{q}), \quad \text{where } \lambda = \frac{\widetilde{\gamma}\sigma}{\widetilde{\gamma}\sigma + (1-\widetilde{\gamma})(1-\sigma)} = \frac{1}{1 + (\widetilde{\gamma}^{-1} - 1)\,(\sigma^{-1} - 1)} \in [0,1].$$

Here λ is increasing in the common owners' level of holdings σ and control $\widetilde{\gamma}$. Under proportional control $\sigma = \widetilde{\gamma}$, and $\lambda = \left[1 + (\sigma^{-1} - 1)^2\right]^{-1}$.

A.2 Pricing-stage equilibria under parametric assumptions

CESL demand is of the form

$$P(Q) = \begin{cases} a + bQ^{1-E} & \text{if } E > 1 \\ \max\{a - b\ln Q, 0\} & \text{if } E = 1 \\ \max\left\{a - bQ^{1-E}, 0\right\} & \text{if } E < 1 \end{cases}$$

for parameters $a \geq 0$ and $b > 0$. For $E = 0$ this reduces to linear demand, while for $a = 0$ and $E > 1$ it reduces to constantly elastic demand with elasticity $\eta = (E-1)^{-1}$. For $Q < \overline{Q}$ the elasticity $E_{P''}(Q)$ of the curvature is then given by

$$E_{P''}(Q) \equiv \frac{QP'''(Q)}{P''(Q)} = \begin{cases} \frac{Qb(E+1)E(1-E)Q^{-(E+2)}}{-bE(1-E)Q^{-(E+1)}} = -(E+1) & \text{if } E \neq 1 \\ \frac{-Q2b/Q^3}{b/Q^2} = -2 & \text{if } E = 1 \end{cases},$$

so $E_{P''}(Q) = -(E+1)$.

Define $\Lambda_n := 1 + \lambda(n-1)$, sometimes written simply Λ. We will use this notation throughout the appendix.

Claim 2 provides the equilibria under parametric assumptions on the demand and cost functions. The total quantity is decreasing in the level of overlapping ownership, λ.

[45] This relies on the fact that a firm's entrepreneurs only hold shares of their own firm both before and after entry. Common ownership develops after entry not through a firm's entrepreneurs investments in other firms but because outside investors invest in multiple firms.

[46] For every other pair of entrepreneur j and firm i, $\beta'_{ji} = \gamma'_{ji} = 0$.

Claim 2. Under CESL demand and constant returns to scale the total equilibrium quantity in the pricing stage is

$$Q_n = \begin{cases} \left[\frac{b(1-H_n(E-1))}{c-a}\right]^{\frac{1}{E-1}} & \text{if } E \in (1,2) \text{ and } c > a \\ e^{\frac{a-c-bH_n}{b}} & \text{if } E = 1 \\ \left[\frac{a-c}{b(1+H_n(1-E))}\right]^{\frac{1}{1-E}} & \text{if } E < 1 \text{ and } a > c, \end{cases}$$

where $H_n := \Lambda_n/n$, $\Lambda_n := 1 + \lambda(n-1)$. Under linear demand and quadratic costs it is $Q_n = \frac{a}{b(1+H_n)+c/n}$.

Proof of Claim 2 Under CESL demand and constant marginal costs the pricing formula $P(Q_n) - C'(q_n) = -H_n Q_n P'(Q_n)$ gives

$$\begin{aligned} a + b(Q_n)^{1-E} - c &= H_n b(E-1)(Q_n)^{1-E} && \text{if } E > 1 \\ a - b\ln Q_n - c &= H_n b && \text{if } E = 1 \\ a - b(Q_n)^{1-E} - c &= H_n b(1-E)(Q_n)^{1-E} && \text{if } E < 1 \end{aligned}$$

and the result follows. In the case $E > 1$, $E < 2$ and $c > a$ guarantee that there is an interior equilibrium. Notice that if $a > c$, then the profit per unit $P(Q) - AC(q) \geq a - c > 0$ is positive and bounded away from zero for every $Q \geq q \geq 0$, and thus there is no equilibrium. In the case $E < 1$, if $a \leq c$, then in the unique equilibrium $Q_n = 0$.

For linear demand and quadratic costs the pricing formula $P(Q_n) - C'(q_n) = -H_n Q_n P'(Q_n)$ gives $a - bQ_n - c(Q_n/n) = H_n b Q_n$, and the result follows. **Q.E.D.**

A.3 Some commonly used conditions

Lemma 1 below provides necessary and sufficient conditions for some of our standard assumptions. The proof is elementary and therefore omitted.

Lemma 1. The following hold:

(i) $\Delta(Q,Q_{-i}) > 0$ on L for every $\lambda \in [0,1)$ if and only if $C''(q) \geq 0$ for every $q < \overline{Q}$.

(ii) $E_{P'}(Q) < (1+\lambda)/H_n$ for every $n \in [2,+\infty)$ and every $\lambda \in [0,1]$ if and only if $E_{P'}(Q) < 2$.

(iii) $E_{P'}(Q) < (1+\lambda)/H_n$ for every $n \in [1,2]$ and every $\lambda \in [0,1]$ if and only if $E_{P'}(Q) < 1$.

In the proofs to come, it will also be useful to remember that if $\Delta > 0$ (resp. $\Delta < 0$), then

$$\begin{aligned} (1+\lambda+\Delta/n)/H_n &= 1 + H_n^{-1} - \Lambda_n^{-1} C''(q)/P'(Q) \\ &\overset{\text{(resp. } \geq)}{\leq} (1+\lambda+\Delta/\Lambda_n)/H_n \\ &= 1 + H_n^{-1} + [(1-\lambda)(1-H_n) - C''(q)/P'(Q)]/(\Lambda_n H_n) \\ &\overset{\text{(resp. } \geq)}{\leq} (1+\lambda+\Delta)/H_n \\ &= (2 - C''(q)/P'(Q))/H_n. \end{aligned}$$

Also, $E_{P'}(Q) < \frac{1+\lambda+\Delta(Q,Q_{-i})}{1-(1-\lambda)(1-s_i)}$ on L implies that for any $n \in [1,+\infty)$ (i.e. with n continuous) and any $Q < \overline{Q}$

$$E_{P'}(Q) < \frac{1+\lambda+\Delta\left(Q,(n-1)Q/n\right)}{1-(1-\lambda)(1-1/n)} = \frac{1+\lambda+\Delta\left(Q,(n-1)Q/n\right)}{H_n}.$$

Thus, part (ii) of the maintained assumption implies that when $\Delta < 0$, $E_{P'}(Q)$ is also lower than $\left(1+\lambda+\Delta/\Lambda_n\right)/H_n$ and $\left(1+\lambda+\Delta/n\right)/H_n$ in the symmetric equilibrium for any $n \in [1,+\infty)$.

A.4 Concavity of total surplus in the number of firms

Lemma 2. TS_n is globally strictly concave in n if for every n

$$\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\left[1-\lambda-H_n\left(\left(\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}-1\right)\left(1-E_{P'}(Q_n)\right)+\frac{\partial^2 Q_n}{(\partial n)^2}\left(\frac{\partial Q_n}{\partial n}\right)^{-1}n-1\right)\right] > \frac{1-\lambda}{n}.$$

Under constant marginal costs and $E_{P'}(Q_n) < 2$ for every n, this is true if $E'_{P'}(Q) \equiv \partial E_{P'}(Q)/\partial Q$ is not too high; particularly, $E'_{P'} \leq 0$ is sufficient, and thus so is CESL demand.

Remark: More generally, all else constant, the condition of Lemma 2 is satisfied if the elasticity of the slope of Q_n with respect to n, $\frac{\partial^2 Q_n}{(n)^2}\left(\frac{\partial Q_n}{\partial n}\right)^{-1}n$, is not too high. Also, remember that $\frac{\partial Q_n}{\partial n}\frac{n}{Q_n} \in (0,1)$ under the assumptions of Proposition 3(iii-a), so all else constant, in that case the condition is satisfied if $E_{P'}(Q)$ is not too high.

Proof of Lemma 2 We have seen that the first derivative of equilibrium total surplus with respect to n is given by

$$\frac{d\,\mathrm{TS}_n}{dn} = \Pi(n,\lambda) - f - \Lambda_n Q_n P'(Q_n)\frac{\frac{\partial Q_n}{\partial n}-q_n}{n},$$

so if we denote $\Pi_n(n,\lambda) \equiv \partial\Pi(n,\lambda)/\partial n$, the second derivative is given by

$$\begin{aligned}
\frac{d^2\,\mathrm{TS}_n}{(dn)^2} =& \Pi_n(n,\lambda) - \lambda Q_n P'(Q_n)\frac{\frac{\partial Q_n}{\partial n}-q_n}{n} - \Lambda_n\frac{\partial Q_n}{\partial n}P'(Q_n)\frac{\frac{\partial Q_n}{\partial n}-q_n}{n} \\
&- \Lambda_n Q_n P''(Q_n)\frac{\partial Q_n}{\partial n}\frac{\frac{\partial Q_n}{\partial n}-q_n}{n} - \Lambda_n Q_n P'(Q_n)\frac{\left(\frac{\partial^2 Q_n}{(\partial n)^2}-\frac{dq_n}{\partial n}\right)n-\frac{\partial Q_n}{\partial n}+q_n}{n^2} \\
=& P'\left(Q_n\right)\left(\frac{Q_n}{n}\right)^2\left[\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\left(1-H_n\right)+H_n\right] - \lambda P'(Q_n)\left(\frac{Q_n}{n}\right)^2\left(\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}-1\right) \\
&- H_n P'(Q_n)\left(\frac{Q_n}{n}\right)^2\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\left(\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}-1\right) \\
&+ H_n P'(Q_n)\left(\frac{Q_n}{n}\right)^2 E_{P'}(Q_n)\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\left(\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}-1\right) \\
&- \Lambda_n P'(Q_n)\left(\frac{Q_n}{n}\right)^2\left[\frac{\partial^2 Q_n}{(\partial n)^2}\frac{n}{Q_n}-\left(\frac{\partial Q_n}{\partial n}-q_n\right)\frac{1}{Q_n}-\frac{\partial Q_n}{\partial n}\frac{1}{Q_n}+\frac{1}{n}\right]
\end{aligned}$$

$$\propto -\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\left[1-\lambda-H_n\left(\left(\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}-1\right)(1-E_{P'}(Q_n))+\frac{\partial^2 Q_n}{(\partial n)^2}\left(\frac{\partial Q_n}{\partial n}\right)^{-1}n-1\right)\right]$$
$$+\frac{1-\lambda}{n}.$$

Under constant marginal costs

$$\frac{\partial Q_n}{\partial n}=\frac{1-\lambda}{n+\Lambda-\Lambda E_{P'}(Q_n)}\frac{Q_n}{n}\implies$$
$$\frac{\partial^2 Q_n}{(\partial n)^2}=(1-\lambda)\left(-\frac{1+\lambda-\lambda E_{P'}(Q_n)-\Lambda E'_{P'}(Q_n)\frac{\partial Q_n}{\partial n}}{\left(n+\Lambda-\Lambda E_{P'}(Q_n)\right)^2}\frac{Q_n}{n}+\frac{\frac{\frac{\partial Q_n}{\partial n}n-Q_n}{n^2}}{n+\Lambda-\Lambda E_{P'}(Q_n)}\right),$$
$$\frac{\partial^2 Q_n}{(\partial n)^2}\left(\frac{\partial Q_n}{\partial n}\right)^{-1}n=-n\frac{1+\lambda-\lambda E_{P'}(Q_n)-\Lambda E'_{P'}(Q_n)\frac{\partial Q_n}{\partial n}}{n+\Lambda-\Lambda E_{P'}(Q_n)}+\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}-1$$

so that

$$\frac{d^2\,\mathrm{TS}_n}{(dn)^2}\propto-\left\{\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\left[1-\lambda-H_n\left(\begin{array}{r}\left(\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}-1\right)(2-E_{P'}(Q_n))\\ -n\frac{1+\lambda-\lambda E_{P'}(Q_n)-\Lambda E'_{P'}(Q_n)\frac{\partial Q_n}{\partial n}}{n+\Lambda-\Lambda E_{P'}(Q_n)}-1\end{array}\right)\right]-\frac{1-\lambda}{n}\right\}$$
$$\propto-\left\{1-\lambda-H_n\left(\begin{array}{r}-\frac{n+\Lambda-(1-\lambda)-\Lambda E_{P'}(Q_n)}{n+\Lambda-\Lambda E_{P'}(Q_n)}(2-E_{P'}(Q_n))\\ -n\frac{1+\lambda-\lambda E_{P'}(Q_n)-\Lambda E'_{P'}(Q_n)\frac{\partial Q_n}{\partial n}}{n+\Lambda-\Lambda E_{P'}(Q_n)}-1\end{array}\right)\right.$$
$$\left.-\frac{n+\Lambda_n-\Lambda_n E_{P'}(Q_n)}{n}\right\}$$
$$\propto-\left\{\begin{array}{r}n(1-\lambda)\left(n+\Lambda_n-\Lambda_n E_{P'}(Q_n)\right)-\left(n+\Lambda_n-\Lambda_n E_{P'}(Q_n)\right)^2\\ -\left(n+\Lambda_n-(1-\lambda)-\Lambda_n E_{P'}(Q_n)\right)(2-E_{P'}(Q_n))\\ -\Lambda_n\left(-n\left(1+\lambda-\lambda E_{P'}(Q_n)-\Lambda_n E'_{P'}(Q_n)\frac{\partial Q_n}{\partial n}\right)-\left(n+\Lambda_n-\Lambda_n E_{P'}(Q_n)\right)\right)\end{array}\right\}.$$

The partial derivative of the expression in the brackets with respect to $E_{P'}(Q_n)$ is given by

$$\begin{aligned}&-\Lambda n(1-\lambda)+2\Lambda\left(n+\Lambda-\Lambda E_{P'}(Q_n)\right)\\&-\Lambda\left(\Lambda\left(2-E_{P'}(Q_n)\right)+n+\Lambda-(1-\lambda)-\Lambda E_{P'}(Q_n)+\Lambda-(1-\lambda)+\Lambda\right)\\&\propto-n(1-\lambda)+2\left(n+\Lambda-\Lambda E_{P'}(Q_n)\right)-\left(\Lambda\left(2-E_{P'}(Q_n)\right)+n+3\Lambda-2(1-\lambda)-\Lambda E_{P'}(Q_n)\right)\\&=\lambda n-(3\Lambda-2(1-\lambda))=-\left(2\Lambda-(1-\lambda)\right)<0,\end{aligned}$$

so that, given $E_{P'}(Q_n)<2$, for $d^2\,\mathrm{TS}_n/(dn)^2$ to be negative it is sufficient that

$$n(1-\lambda)\left(n+\Lambda-2\Lambda\right)-\left(n+\Lambda-2\Lambda\right)^2$$
$$-\Lambda\left(-n\left(1+\lambda-2\lambda-\Lambda E'_{P'}(Q_n)\frac{\partial Q_n}{\partial n}\right)-\left(n+\Lambda-2\Lambda\right)\right)\geq 0\iff$$

$$(n(1-\lambda)+\Lambda-(n-\Lambda))\,(n-\Lambda)+\Lambda n\left(1-\lambda-\Lambda E'_{P'}(Q_n)\frac{\partial Q_n}{\partial n}\right)\geq 0 \Longleftrightarrow$$
$$1-\lambda-\frac{\Lambda}{n}E'_{P'}(Q_n)Q_n\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\geq -\frac{(\Lambda+1-\lambda)\,(n-\Lambda)}{\Lambda n},$$

which is true for $E'_{P'}$ not too high. **Q.E.D.**

A.5 Proofs of section 3

Where clear we may simplify notation, for example omitting the subscript n,λ for equilibrium objects.

Proof of Proposition 1 Since for $q_i > \overline{q}$ profit becomes negative, wlog we can constrain each firm to choose quantity $q_i \in [0,\overline{q}]$. Also, for any $\boldsymbol{q} \in \mathbb{R}^n_+$ such that $Q \geq \overline{Q}$, $P(Q) = 0$ while for at least some firm i, $q_i > 0$ so that $\pi_i(q_i,\boldsymbol{q}_{-i}) = -C(q_i) < 0 = \pi_i(0,\boldsymbol{q}_{-i})$ (this is relevant for the case where $\overline{Q} < +\infty$). Therefore, wlog we can constrain attention to quantity profiles $\boldsymbol{q}$ that lie in the set

$$\left\{\boldsymbol{x} \in [0,\overline{q}]^n : \sum_{i\in\mathcal{F}} x_i \leq \overline{Q}\right\}.$$

Given that the the Edgeworth coefficient λ that firm j assigns to each firm is constant across firms, the best response of firm i depends on $\boldsymbol{q}_{-i}$ only through Q_{-i}; the pricing stage game is an aggregative game. Denote by $r(Q_{-i})$ the best response correspondence of a firm (same across firms). If it is a differentiable function, its slope is given by

$$\begin{aligned}
r'_i(Q_{-i}) &= -\frac{\partial^2\left(\pi_i+\lambda\sum_{j\neq i}\pi_j\right)\Big/\left(\partial q_i\partial Q_{-i}\right)}{\partial^2\left(\pi_i+\lambda\sum_{j\neq i}\pi_j\right)\Big/\left(\partial q_i\right)^2} = -\frac{P'(Q)+q_iP''(Q)+\lambda P'(Q)+\lambda P''(Q)Q_{-i}}{2P'(Q)-C''(q_i)+q_iP''(Q)+\lambda P''(Q)Q_{-i}}\\
&= -1+\frac{1-\lambda-C''(q_i)/P'(Q)}{2-C''(q_i)/P'(Q)-(s_i+\lambda(1-s_i))E_{P'}(Q)}\\
&= -1+\frac{\Delta(Q,Q_{-i})}{1+\lambda+\Delta(Q,Q_{-i})-(s_i+\lambda(1-s_i))E_{P'}(Q)}
\end{aligned}$$

where $q_i = r(Q_{-i})$.

The proof is similar to that of Theorem 2.1 in Amir and Lambson (AL; 2000) (except for the proof of uniqueness under $\Delta > 0$, which is not considered in AL but is also an extension of standard results). We will refer to standard lattice-theoretic theorems that are collectively presented in AL (see also Chapter 2 in Vives, 1999).

Case $\boldsymbol{\Delta} > \mathbf{0}$: We first prove statement (a) of this case, which itself contains two parts.

Existence of at least one symmetric equilibrium: Firm i's problem is equivalent to the firm choosing the total quantity to be given by the correspondence $R : [0,\overline{Q}] \to [0,\overline{Q}]$ defined as

$$R(Q_{-i}) := \underset{Q\in[Q_{-i},Q_{-i}+\overline{q}]}{\arg\max}\left\{P\left(Q\right)\left[Q-(1-\lambda)Q_{-i}\right]-C(Q-Q_{-i})\right\} = r(Q_{-i})+Q_{-i}.$$

taking Q_{-i} as given. The maximand above is strictly supermodular since $\Delta(Q,Q_{-i}) > 0$, so by Theorem A.1 in AL every selection from $R(Q_{-i})$ is nondecreasing in Q_{-i}. It follows

then that every selection of the correspondence $B : [0,(n-1)\overline{q}] \rightrightarrows [0,(n-1)\overline{q}]$ given by $B(Q_{-i}) \coloneqq (n-1)R(Q_{-i})/n$ is also nondecreasing in Q_{-i}. By Tarski's fixed point theorem (Theorem A.3 in AL), B has a fixed point, which can readily be checked to be a symmetric equilibrium.

Non-existence of asymmetric equilibria: Suppose by contradiction that an asymmetric equilibrium exists, and denote it by $\widetilde{\boldsymbol{q}}$. Then, by symmetry of the firms any permutation of q should also be an equilibrium, and since $\widetilde{\boldsymbol{q}}$ is asymmetric there exists a permutation $\widehat{\boldsymbol{q}}$ such that there exists a firm i that produces a different quantity in the permutation compared to its quantity in the original equilibrium, w.l.o.g. say $\widehat{q}_i > \widetilde{q}_i$. But the total quantity is the same in the two permutations, that is $\widetilde{Q} = \widehat{Q}$ and so $\widehat{Q}_{-i} < \widetilde{Q}_{-i}$. It must then be that

$$R\left(\widehat{Q}_{-i}\right) = R\left(\widetilde{Q}_{-i}\right) = \widetilde{Q} \geq \widetilde{Q}_{-i} > \widehat{Q}_{-i}.$$

The inequalities in the last expression show that $R\left(\widehat{Q}_{-i}\right) > \widehat{Q}_{-i}$, so $R\left(\widehat{Q}_{-i}\right)$ makes the first derivative of the firm's objective non-negative, that is

$$P\left(\widetilde{Q}\right) + P'\left(\widetilde{Q}\right)\left[Q - (1-\lambda)\widehat{Q}_{-i}\right] - C'(Q - \widehat{Q}_{-i}) \geq 0.$$

Also, since the firm's action space is not bounded from above, it trivially holds that[47]

$$P\left(\widetilde{Q}\right) + P'\left(\widetilde{Q}\right)\left[Q - (1-\lambda)\widetilde{Q}_{-i}\right] - C'(Q - \widetilde{Q}_{-i}) \leq 0.$$

Combining the last two inequalities we get that

$$-(1-\lambda)P'\left(\widetilde{Q}\right)\widetilde{Q}_{-i} - C'(Q - \widetilde{Q}_{-i}) \leq -(1-\lambda)P'\left(\widetilde{Q}\right)\widehat{Q}_{-i} - C'(Q - \widehat{Q}_{-i}) \implies$$

$$-(1-\lambda)P'\left(\widetilde{Q}\right) - \frac{C'(Q - \widetilde{Q}_{-i}) - C'(Q - \widehat{Q}_{-i})}{\widetilde{Q}_{-i} - \widehat{Q}_{-i}} \leq 0. \tag{4}$$

Last, since as we have shown every selection from $R(\widetilde{Q}_{-i})$ is nondecreasing in $\widetilde{Q}_{-i}$, it follows from $R\left(\widehat{Q}_{-i}\right) = R\left(\widetilde{Q}_{-i}\right) = Q$ that $R(X) = Q$ for all $X \in [\widehat{Q}_{-i},\widetilde{Q}_{-i}]$. Therefore, in (4) we can let $\widehat{Q}_{-i} \to \widetilde{Q}_{-i}$, which gives

$$-(1-\lambda)P'\left(\widetilde{Q}\right) + C''(Q - \widetilde{Q}_{-i}) \leq 0 \implies \Delta(Q,\widetilde{Q}_{-i}) \leq 0,$$

a contradiction. This completes the proof of part (a).

We now turn to part (b). It suffices to show that at most one symmetric equilibrium exists under the additional assumption.

Existence of at most one symmetric equilibrium: $E_{P'} < (1+\lambda+\Delta)/H_n$ on L—which holds given that $E_{P'} < (1+\lambda+\Delta/n)/H_n$ and $\Delta > 0$ on L—implies that $\partial^2\left(\pi_i + \lambda\sum_{j\neq i}\pi_j\right)\Big/(\partial q_i)^2 < 0$, so that $r(Q_{-i})$ is indeed a differentiable function. At a

[47] Under a capacity constraint $\overline{q}$, the following would still hold since we would have $\widetilde{Q}_{-i} + \overline{q} > \widehat{Q}_{-i} + \overline{q} \geq R\left(\widehat{Q}_{-i}\right) = R\left(\widetilde{Q}_{-i}\right)$.

symmetric quantity profile we have

$$r'(Q_{-i}) = -1 + \frac{\Delta(Q,Q_{-i})}{1+\lambda+\Delta(Q,Q_{-i}) - H_n E_{P'}(Q)}.$$

Symmetric equilibria are solutions to $g(q) \equiv r((n-1)q) - q = 0$. Thus, there will be at most one symmetric equilibrium if $g' < 0$, that is, if for any $q \in [0,\overline{Q}/n)$, $r_i'((n-1)q) < (n-1)^{-1}$, or equivalently

$$-\frac{1+\lambda - H_n E_{P'}(nq)}{1+\lambda+\Delta(nq,(n-1)q) - H_n E_{P'}(nq)} < \frac{1}{n-1} \iff E_{P'}(nq) < \frac{1+\lambda+\Delta(nq,(n-1)q)/n}{H_n}$$

which is true, since by assumption it is true on L.

Case $\boldsymbol{\Delta < 0}$: We first prove part (a) for $m = n$, that is, we prove that there is a unique symmetric equilibrium. $\Delta(Q,Q_{-i}) < 0$ and $E_{P'}(Q) < \frac{2-C''(Q-Q_{-i})/P'(Q)}{1-(1-\lambda)(1-s_i)}$ implies that the objective function of each firm is strictly concave in its quantity (in the part where $P(Q) > 0$). Thus, for Q_{-i} such that $r(Q_{-i}) > 0$, $r(Q_{-i})$ is a differentiable function with slope

$$r_i'(Q_{-i}) = -1 + \frac{1-\lambda - C''(q_i)/P'(Q)}{2 - C''(q_i)/P'(Q) - (s_i + \lambda(1-s_i))E_{P'}(Q)} < -1$$

given $\Delta < 0$. Thus, again $g' < 0$ since $r' < -1 < (n-1)^{-1}$ for every $n \geq 2$. Also, $g(0) \geq 0$ and $\lim_{q\to\infty} g(q) = -\infty$, so by continuity of g there exists a unique symmetric equilibrium.

We know prove part (a) for $m < n$. Let q_m be the symmetric equilibrium quantity produced by each firm when m firms are in the market, which as we have seen exists and is unique. The m firms are clearly best-responding by producing q_m each, so it remains to show that $r(mq_m) = 0$, so that the non-producing firms are also best-responding. By definition of q_m, $r((m-1)q_m) = q_m$. Indeen, the fact that $r'(Q_{-i}) < -1$ (when $r(Q_{-i}) > 0$) implies that

$$r(mq_m) = r((m-1)q_m + q_m) \leq \max\{r((m-1)q_m) - q_m, 0\} = 0.$$

It remains to show part (b), that no other equilibria exist. Assume by contradiction that there is an equilibrium $\widetilde{\boldsymbol{q}}$ of a different type. Then there exist firms i and j such that $\widetilde{q}_i \neq \widetilde{q}_j$, $\widetilde{q}_i > 0$, $\widetilde{q}_j > 0$ in that equilibrium. Wlog let $\widetilde{q}_i > \widetilde{q}_j$. Given that $R'(Q_{-i}) = r'(Q_{-i}) + 1 < 0$ (when $R(Q_{-i}) > Q_{-i}$) it follows that

$$R(\widetilde{Q}_{-i}) = R(\widetilde{Q}_{-j}) \implies \widetilde{Q}_{-i} = \widetilde{Q}_{-j} \implies \widetilde{q}_i = \widetilde{q}_j,$$

a contradiction. **Q.E.D.**

Proof of Corollary 1.1 $\Delta(Q,Q_{-i}) = 1 - \lambda + c_2/b$, constant over L. $E_{P'}(Q) = 0$, also constant. Last, we have that $1 + \lambda + \Delta(Q,Q_{-i}) = 2 + c_2/b$. Proposition 1 then gives that

(i) if $c_2 > -b(1-\lambda)$, $\Delta > 0$ on L and a unique and symmetric equilibrium exists,

(ii) if $c_2 < -b(1-\lambda)$, then $\Delta < 0$ on L and a unique (in the class of symmetric equilibria) symmetric equilibrium exists.

Under linear demand and linear-quadratic costs the pricing formula $P(Q_n) - C'(q_n) = -H_n Q_n P'(Q_n)$ gives

$$a - bQ_n - \left(c_1 + c_2 \frac{Q_n}{n}\right) = bH_n Q_n \Longleftrightarrow Q_n = \frac{a - c_1}{b(H_n + 1) + c_2/n},$$

which is positive since $a > c_1$ and $c_2 > -2bc_1/a > -2b$. Also, we get that

$$\begin{aligned}\Pi(n,\lambda) &= \left(P(Q_n) - c_1 - c_2 \frac{q_n}{2}\right) q_n = \left(a - \frac{a - c_1}{H_n + 1 + c_2/(bn)} - c_1 - \frac{c_2}{2bn} \frac{a - c_1}{H_n + 1 + c_2/(bn)}\right) q_n \\ &= \frac{(a - c_1)(H_n + c_2/(2bn))}{H_n + 1 + c_2/(bn)} q_n = \frac{(a - c_1)^2 (bnH_n + c_2/2)}{[bn(H_n + 1) + c_2]^2},\end{aligned}$$

which is also positive given that $c_2 > -2b$, so the symmetric equilibrium is interior.

Last, notice that

$$\begin{aligned}C'(q_n) &= c_1 + \frac{c_2}{n} \frac{a - c_1}{b(H_n + 1) + c_2/n} = \frac{1}{n} \frac{nc_1 [b(H_n + 1) + c_2/n] + c_2(a - c_1)}{b(H_n + 1) + c_2/n} \\ &= \frac{bc_1(H_n + 1) + ac_2/n}{b(H_n + 1) + c_2/n},\end{aligned}$$

which is positive given $c_2 > -2bc_1/a$, so in equilibrium marginal cost is positive. **Q.E.D.**

Proof of Propositions 2 and 2' For simplicity, we use the notation Q_n and q_n to refer to values in a specific equilibrium even if that equilibrium is not unique.

Let $\widetilde{\pi}_i(\boldsymbol{q}) := \pi_i(\boldsymbol{q}) + \lambda \sum_{j \neq i} \pi_j(\boldsymbol{q})$. A linearization of the adjustment process around an equilibrium production profile $\boldsymbol{g}_n$ gives

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix} = \overbrace{\begin{bmatrix} k_1 \frac{\partial^2 \widetilde{\pi}_1(\boldsymbol{q})}{(\partial q_1)^2} & k_1 \frac{\partial^2 \widetilde{\pi}_1(\boldsymbol{q})}{\partial q_1 \partial q_2} & \cdots & k_1 \frac{\partial^2 \widetilde{\pi}_1(\boldsymbol{q})}{\partial q_1 \partial q_n} \\ k_2 \frac{\partial^2 \widetilde{\pi}_2(\boldsymbol{q})}{\partial q_2 \partial q_1} & k_2 \frac{\partial^2 \widetilde{\pi}_2(\boldsymbol{q})}{(\partial q_2)^2} & \cdots & k_2 \frac{\partial^2 \widetilde{\pi}_2(\boldsymbol{q})}{\partial q_2 \partial q_n} \\ \vdots & \vdots & \cdots & \vdots \\ k_n \frac{\partial^2 \widetilde{\pi}_n(\boldsymbol{q})}{\partial q_n \partial q_1} & k_n \frac{\partial^2 \widetilde{\pi}_n(\boldsymbol{q})}{\partial q_n \partial q_2} & \cdots & k_n \frac{\partial^2 \widetilde{\pi}_n(\boldsymbol{q})}{(\partial q_n)^2} \end{bmatrix}}^{=:A} \begin{bmatrix} q_1 - q_n \\ q_2 - q_n \\ \vdots \\ q_n - q_n \end{bmatrix},$$

where for $i \neq j$

$$\begin{aligned}\frac{\partial^2 \widetilde{\pi}_i(\boldsymbol{q})}{\partial q_i \partial q_j} &= P'(Q) + q_i P''(Q) + \lambda P'(Q) + \lambda P''(Q) Q_{-i} = P'(Q)\left(1 + \lambda - E_{P'}(Q)\left((1 - \lambda)s_i + \lambda\right)\right), \\ \frac{\partial^2 \widetilde{\pi}_i(\boldsymbol{q})}{(\partial q_i)^2} &= 2P'(Q) - C_i''(q_i) + q_i P''(Q) + \lambda P''(Q) Q_{-i} \\ &= P'(Q)\left(2 - \frac{C_i''(q_i)}{P'(Q)} - E_{P'}(Q)\left((1 - \lambda)s_i + \lambda\right)\right) < 0\end{aligned}$$

are evaluated at the equilibrium production profile $\boldsymbol{q}_n$. The second derivative with respect to q_i evaluated at $\boldsymbol{q}_n$ is negative given that $E_{P'}(Q_n) < (1 + \lambda + \Delta(Q_n, (n-1)q_n))/H_n$.

Notice that $\frac{\partial^2 \widetilde{\pi}_i(\boldsymbol{q})}{\partial q_i \partial q_j}$ does not depend on the identity of firm j as long as $i \neq j$, so that the off-diagonal elements in each row are equal. From Theorem 2(i) in al Nowaihi and

Levine (1985)—which also follows from Hosomatsu (1969)—it follows that all eigenvalues of A are real.

Also, we have that

$$\left.\frac{\partial^2 \widetilde{\pi}_i(\boldsymbol{q})}{\partial q_i \partial q_j}\right|_{\boldsymbol{q}=\boldsymbol{q}_n} = P'(Q_n)\left(1+\lambda - H_n E_{P'}(Q_n)\right).$$

We distinguish two cases.

Case 1: If $E_{P'}(Q_n) \leq (1+\lambda)/H_n$, then that combined with $\Delta(Q_n,(n-1)q_n) > 0$ imply

$$\begin{aligned} E_{P'}(Q_n) \leq \frac{1+\lambda}{H_n} \quad \wedge \quad 1-\lambda-\frac{C''(q_n)}{P'(Q_n)} > 0 \implies \\ 0 \leq 1+\lambda - H_n E_{P'}(Q_n) < 2 - \frac{C_i''(q_n)}{P'(Q_n)} - H_n E_{P'}(Q_n) \implies \\ k_i P'(Q_n)\left(2-\frac{C_i''(q_n)}{P'(Q_n)} - H_n E_{P'}(Q_n)\right) < k_i P'(Q_n)\left(1+\lambda - H_n E_{P'}(Q_n)\right) \leq 0 \implies \\ \left.\frac{\partial^2 \widetilde{\pi}_i(q)}{(\partial q_i)^2}\right|_{\boldsymbol{q}=\boldsymbol{q}_n} < \left.\frac{\partial^2 \widetilde{\pi}_i(q)}{\partial q_i \partial q_j}\right|_{\boldsymbol{q}=\boldsymbol{q}_n} \leq 0, \end{aligned}$$

for every $i \neq j$, and it follows from Theorem 2(ii-a) in al Nowaihi and Levine (1985)—also in Hosomatsu (1969)— that all eigenvalues of A are negative. From standard stability theory we then have that the equilibrium is locally stable.

Case 2: For $E_{P'}(Q_n) > (1+\lambda)/H_n$ we get

$$\left.\frac{\partial^2 \widetilde{\pi}_i(q)}{\partial q_i \partial q_j}\right|_{\boldsymbol{q}=\boldsymbol{q}_n} > 0$$

and it follows from Theorem 2(ii-b) in al Nowaihi and Levine (1985) that all eigenvalues of A are negative it and only if

$$\sum_{i=1}^{n} \frac{\left.\frac{\partial^2 \widetilde{\pi}_i(q)}{\partial q_i \partial q_j}\right|_{\boldsymbol{q}=\boldsymbol{q}_n}}{\left.\frac{\partial^2 \widetilde{\pi}_i(q)}{\partial q_i \partial q_j}\right|_{\boldsymbol{q}=\boldsymbol{q}_n} - \left.\frac{\partial^2 \widetilde{\pi}_i(q)}{(\partial q_i)^2}\right|_{\boldsymbol{q}=\boldsymbol{q}_n}} < 1,$$

or equivalently

$$-n\frac{1+\lambda - H_n E_{P'}(Q_n)}{1-\lambda-\frac{C''(q_n)}{P'(Q_n)}} < 1 \iff -\left[1+\lambda - H_n E_{P'}(Q_n)\right] < \Delta(Q_n,(n-1)q_n)/n,$$

Again the result follows from standard stability theory. **Q.E.D.**

Proof of Proposition 3 By totally differentiating (1) at the symmetric equilibrium we get that

$$\left[P'(Q) - \frac{C''(Q/n)}{n} + \frac{\Lambda P''(Q)Q}{n} + \frac{\Lambda P'(Q)}{n}\right]\frac{dQ}{d\lambda} + \frac{(n-1)P'(Q)Q}{n} = 0 \iff$$

$$P'(Q)\left\{\left[n+\Lambda-C''(Q/n)/P'(Q)-\Lambda E_{P'}(Q)\right]\frac{dQ}{d\lambda}+(n-1)Q\right\}=0 \implies$$
$$\frac{dQ}{d\lambda}=-\frac{(n-1)Q}{n+\Lambda-C''(Q/n)/P'(Q)-\Lambda E_{P'}(Q)}<0.$$

Similarly, we have that

$$0=\left[P'(Q)-\frac{C''(Q/n)}{n}+\frac{\Lambda P''(Q)Q}{n}+\frac{\Lambda P'(Q)}{n}\right]\frac{dQ}{dn}+\frac{C''(Q/n)Q}{n^2}-\frac{(1-\lambda)P'(Q)Q}{n^2} \implies$$
$$\frac{dQ}{dn}\frac{n}{Q}=\frac{1-\lambda-C''\left(q\right)/P'\left(Q\right)}{n+\Lambda-C''\left(q\right)/P'\left(Q\right)-\Lambda E_{P'}\left(Q\right)}=\frac{\Delta(Q,Q_{-i})}{\Lambda_n(1+\lambda+\Delta/n)/H_n}.$$

It follows then also that $\operatorname{sgn}\left\{\frac{d^2Q}{d\lambda dn}\right\}$ is equal to

$$=-\operatorname{sgn}\left\{\begin{array}{c}\left[Q+(n-1)\frac{dQ}{dn}\right]\left[n+\Lambda-C''(Q/n)/P'(Q)-\Lambda E_{P'}(Q)\right]-(n-1)Q\times\\ \left[1+\lambda-\frac{C'''(Q/n)\left(\frac{dQ}{dn}-\frac{Q}{n}\right)\frac{P'(Q)}{n}-C''(Q/n)P''(Q)\frac{dQ}{dn}}{\left(P'(Q)\right)^2}-\lambda E_{P'}(Q)-\Lambda E'_{P'}(Q)\frac{dQ}{dn}\right]\end{array}\right\}$$
$$=-\operatorname{sgn}\left\{\begin{array}{c}\left[\frac{n}{n-1}+\frac{dQ}{dn}\frac{n}{Q}\right]\left[n+\Lambda-C''(Q/n)/P'(Q)-\Lambda E_{P'}(Q)\right]\\ -\left[\begin{array}{c}n\left[1+\lambda\left(1-E_{P'}(Q)\right)\right]-\Lambda E'_{P'}(Q)Q\frac{dQ}{dn}\frac{n}{Q}\\ -\frac{C'''(Q/n)\left(\frac{dQ}{dn}\frac{n}{Q}-1\right)\frac{P'(Q)Q}{n}-C''(Q/n)P''(Q)Q\frac{dQ}{dn}\frac{n}{Q}}{\left(P'(Q)\right)^2}\end{array}\right]\end{array}\right\}$$
$$=-\operatorname{sgn}\left\{\begin{array}{c}\frac{n}{n-1}\left[n+\Lambda-C''(Q/n)/P'(Q)-\Lambda E_{P'}(Q)-(n-1)\left[1+\lambda\left(1-E_{P'}(Q)\right)\right]\right]\\ -\frac{C'''(Q/n)Q/n}{P'(Q)}+\frac{dQ}{dn}\frac{n}{Q}\left[\begin{array}{c}n+\Lambda-C''(Q/n)/P'(Q)-\Lambda E_{P'}(Q)\\ +\Lambda E'_{P'}(Q)Q+\frac{C'''(Q/n)\frac{P'(Q)Q}{n}-C''(Q/n)P''(Q)Q}{\left(P'(Q)\right)^2}\end{array}\right]\end{array}\right\}$$
$$=-\operatorname{sgn}\left\{\begin{array}{c}\frac{n}{n-1}\left[2-C''(Q/n)/P'(Q)-E_{P'}(Q)\right]-\frac{C'''(Q/n)Q/n}{P'(Q)}\\ +\frac{dQ}{dn}\frac{n}{Q}\left[n+\Lambda-\Lambda E_{P'}(Q)\left[1-\frac{E'_{P'}(Q)Q}{E_{P'}(Q)}\right]+\frac{C'''(Q/n)Q/n+C''(Q/n)\left[E_{P'}(Q)-1\right]}{P'(Q)}\right]\end{array}\right\}$$
$$=-\operatorname{sgn}\left\{\begin{array}{c}\frac{n}{n-1}\left[2-E_{P'}(Q)\right]+\frac{dQ}{dn}\frac{n}{Q}\left[n+\Lambda+\Lambda E_{P'}(Q)\left[E_{P'}(Q)+E_{P''}(Q)\right]\right]\\ +\left(\frac{dQ}{dn}\frac{n}{Q}-1\right)\frac{C'''(Q/n)Q/n}{P'(Q)}-\frac{C''(Q/n)}{P'(Q)}\left[\frac{n}{n-1}+\frac{dQ}{dn}\frac{n}{Q}\left(1-E_{P'}(Q)\right)\right]\end{array}\right\}.$$

If also $C'',C'''\geq 0$ and $E_{P'}(Q)\left[E_{P'}(Q)+E_{P''}(Q)\right]\geq -2$, then $\operatorname{sgn}\left\{d^2Q/(d\lambda dn)\right\}=-$ given that $n\geq$ and $E_{P'}(Q)<2$; the latter two imply $\frac{dQ}{dn}\frac{n}{Q}\in(0,1)$.

For individual output we have

$$\frac{dq}{dn}=\frac{d(Q/n)}{dn}=\frac{dQ}{dn}\frac{1}{n}-\frac{Q}{n^2}=-\frac{1}{n}\left[\frac{\left[C''(q)/P'(Q)-(1-\lambda)\right]q}{n+\Lambda-C''(q)/P'(Q)-\Lambda E_{P'}(Q)}+q\right]$$

$$
\begin{aligned}
&\propto -q\left[C''(q)/P'(Q)-(1-\lambda)+n+\Lambda-C''(q)/P'(Q)-\Lambda E_{P'}(Q)\right]\\
&=-q\left[n(1+\lambda)-\Lambda E_{P'}(Q)\right],
\end{aligned}
$$

Notice then that $E_{P'}(Q)<2$ (resp. $E_{P'}(Q)>(1+\lambda)/\lambda$) on L implies that $E_{P'}(Q)<(1+\lambda)/H_n$ (resp. $E_{P'}(Q)>(1+\lambda)/H_n$) for every $n\geq 2$.

Also,

$$
\begin{aligned}
\frac{d^2q}{d\lambda dn}&=\frac{d\left(\frac{dq}{d\lambda}\right)}{dn}=\frac{d\left[\left(\frac{dQ}{d\lambda}\right)\frac{1}{n}\right]}{dn}=\frac{1}{n}\left[\frac{d^2Q}{d\lambda dn}-\frac{1}{n}\frac{dQ}{d\lambda}\right]\\
&\propto\frac{1}{n}\left\{\begin{array}{r}-\dfrac{Q(n-1)}{n}\left[\begin{array}{r}\left(\dfrac{dQ}{dn}\dfrac{n}{Q}-1\right)\dfrac{C'''(q)q}{P'(Q)}-\dfrac{C''(q)}{P'(Q)}\left[\dfrac{n}{n-1}+\dfrac{dQ}{dn}\dfrac{n}{Q}\left(1-E_{P'}(Q)\right)\right]\\+\dfrac{n}{n-1}\left[2-E_{P'}(Q)\right]+\dfrac{dQ}{dn}\dfrac{n}{Q}\left[n+\Lambda+\Lambda E_{P'}(Q)\left[E_{P'}(Q)+E_{P''}(Q)\right]\right]\end{array}\right]\\-\dfrac{\left(n+\Lambda-C''(q)/P'(Q)-\Lambda E_{P'}(Q)\right)^2}{n}\dfrac{dQ}{d\lambda}\end{array}\right\}\\
&\propto-\frac{Q(n-1)}{n}\left[\begin{array}{r}\left(\dfrac{dQ}{dn}\dfrac{n}{Q}-1\right)\dfrac{C'''(q)q}{P'(Q)}-\dfrac{C''(q)}{P'(Q)}\left[\dfrac{n}{n-1}+\dfrac{dQ}{dn}\dfrac{n}{Q}\left(1-E_{P'}(Q)\right)\right]\\+\dfrac{n}{n-1}\left[2-E_{P'}(Q)\right]+\dfrac{dQ}{dn}\dfrac{n}{Q}\left[n+\Lambda+\Lambda E_{P'}(Q)\left[E_{P'}(Q)+E_{P''}(Q)\right]\right]\end{array}\right]\\
&\quad+\frac{\left(n+\Lambda-C''(q)/P'(Q)-\Lambda E_{P'}(Q)\right)^2}{n}\frac{(n-1)Q}{n+\Lambda-C''(q)/P'(Q)-\Lambda E_{P'}(Q)}\\
&\propto-\left(\frac{dQ}{dn}\frac{n}{Q}-1\right)\frac{C'''(q)q}{P'(Q)}+\frac{C''(q)}{P'(Q)}\left[\frac{1}{n-1}+\frac{dQ}{dn}\frac{n}{Q}\left(1-E_{P'}(Q)\right)\right]-\frac{n}{n-1}\left[2-E_{P'}(Q)\right]\\
&\quad-\frac{dQ}{dn}\frac{n}{Q}\left[n+\Lambda+\Lambda E_{P'}(Q)\left[E_{P'}(Q)+E_{P''}(Q)\right]\right]+n+\Lambda-\Lambda E_{P'}(Q).
\end{aligned}
$$

If marginal costs are linear and demand is CESL, we get

$$
\begin{aligned}
\frac{d^2q}{d\lambda dn}&\propto\frac{n(n-3)}{n-1}+\Lambda-\left(\Lambda-\frac{n}{n-1}\right)E-\frac{1-\lambda}{n+\Lambda-\Lambda E}\left[n+\Lambda-\Lambda E\right]\\
&\propto n-3+\lambda(n-1)-\left(\frac{\Lambda(n-1)}{n}-1\right)E=n+\Lambda-4-\left(\frac{\Lambda(n-1)}{n}-1\right)E,
\end{aligned}
$$

Total surplus changes with λ in the same direction as the total quantity, since with fixed n

$$
d\,\mathrm{TS}=P(Q)dQ-\sum_{i=1}^{n}C'(q)dq=\left(P(Q)-C'(q)\right)dQ.
$$

Q.E.D.

Proof of Proposition 3' We show each of parts (i) and (ii) separately.

Part (i): consider $R(Q_{-i})$ as defined in the proof of Proposition 1. Notice that for any Q_{-i} the maximand satisfies

$$
\frac{\partial^2\left\{P\left(Q\right)\left[Q-(1-\lambda)Q_{-i}\right]-C(Q-Q_{-i})\right\}}{\partial\lambda\partial Q}=P'(Q)Q_{-i}\leq 0
$$

Thus, by Topkis' (1978) Monotonicity Theorem, for any fixed Q_{-i}, $R(Q_{-i})$ is nonincreasing in λ in the strong set order, and thus, so is $B(Q_{-i})$ as defined in the proof of Proposition 1. It follows then by Corollary 1 in Milgrom and Roberts (1994) that the extreme fixed points of $B(Q_{-i})$ (i.e. the total quantity produced by $n-1$ firms in extremal equilibria) are nonincreasing in λ, and the result follows.

Part (ii): notice that for any fixed Q_{-i}, $B(Q_{-i})$ is increasing (and thus nondecreasing) in n, so by the same Corollary it follows that the total quantity produced by $n-1$ firms in an extremal equilibrium is nondecreasing in n. We have also seen in the proof of Proposition 1 that when $\Delta > 0$, $R(Q_{-i})$ is nondecreasing in Q_{-i} and the result follows. **Q.E.D.**

A.6 Proofs of section 4

Where clear we may simplify notation, and write for example Q_n instead of $Q_{n^*(\lambda)}$, n instead of $n^*(\lambda)$.

Proof of Proposition 4 Differentiating equilibrium profits in the Cournot game with respect to n we get

$$\begin{aligned}
\frac{\partial \Pi\left(n,\lambda\right)}{\partial n} &= P'(Q_n)\frac{Q_n}{n}\frac{\partial Q_n}{\partial n} + \left(P(Q_n) - C'(q_n)\right)\frac{n\frac{\partial Q_n}{\partial n} - Q_n}{n^2} \\
&\overset{(1)}{=} P'(Q_n)\frac{Q_n}{n}\frac{\partial Q_n}{\partial n} - Q_n P'(Q_n) H_n \frac{n\frac{\partial Q_n}{\partial n} - Q_n}{n^2} \\
&= P'(Q_n)\left(\frac{Q_n}{n}\right)^2\left[\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\left(1-H_n\right) + H_n\right] \\
&\propto -\left[\frac{1-\lambda - C''\left(q_n\right)/P'\left(_n\right)}{n+\Lambda - C''\left(q_n\right)/P'\left(Q_n\right) - \Lambda E_{P'}\left(Q_n\right)}\left(n-\Lambda\right) + \Lambda\right] \\
&\propto -\left[\left(1-\lambda\right)\left(n-\Lambda\right) + \Lambda(n+\Lambda) - nC''\left(q_n\right)/P'\left(Q_n\right) - \Lambda_n^2 E_{P'}\left(Q_n\right)\right] \\
&\propto -\left[\left(1-\lambda\right)\left(H_n^{-1}-1\right) + n + \Lambda_n - H_n^{-1}C''\left(q_n\right)/P'\left(Q_n\right) - \Lambda_n E_{P'}\left(Q_n\right)\right] < 0,
\end{aligned}$$

where the inequality follows from what we have seen in section A.3. This also means that $E_{Q_n,n} \equiv \frac{\partial Q_n}{\partial n}\frac{n}{Q_n} > -\left(H_n^{-1}-1\right)^{-1}$ for every $n \in [1,+\infty)$. The result follows given that $\Pi(n,\lambda) < f$ for n large. **Q.E.D.**

Proof of Proposition 4' Let q_n denote the individual quantity in an extremal equilibrium with n firms. We have then that

$$\begin{aligned}
\pi(q_n) &= q_n P(q_n + (n-1)q_n) - C(q_n) \\
&\geq q_{n+1}P(q_{n+1} + (n-1)q_n) - C(q_{n+1}) \\
&\geq q_{n+1}P(q_{n+1} + nq_{n+1}) - C(q_{n+1}) = \pi(q_{n+1}),
\end{aligned}$$

where the first inequality follows since producing q_n is a best response quantity of an individual firm, and the second inequality follows from the fact that $(n-1)q_n \leq nq_{n+1}$ by part (a) of Proposition 3'. We conclude that individual profit in extremal equilibria is nonincreasing in n, and given that $\Pi(n,\lambda) < f$ for n large, a free entry equilibrium where in the pricing stage firms play an extremal equilibrium exists. **Q.E.D.**

Proof of Proposition 5 (i) Differentiating equilibrium profits in the Cournot game with respect to λ we get

$$\begin{aligned}\frac{\partial \Pi\left(n,\lambda\right)}{\partial \lambda} &= P'(Q_n)\frac{Q_n}{n}\frac{\partial Q_n}{\partial \lambda} + \left(P(Q_n) - C'(q_n)\right)\frac{\partial Q_n}{\partial \lambda}\frac{1}{n} \\ &\overset{(1)}{=} P'(Q_n)\frac{Q_n}{n}\frac{\partial Q_n}{\partial \lambda} - Q_n P'(Q_n)\frac{\Lambda_n}{n}\frac{\partial Q_n}{\partial \lambda}\frac{1}{n} = P'(Q_n)\frac{Q_n}{n}\frac{\partial Q_n}{\partial \lambda}\frac{n-\Lambda_n}{n},\end{aligned}$$

which is positive for $\lambda < 1$. Also, as in Proposition 4, profits in the Cournot game are decreasing in n.

Therefore, if we ignore the integer constraint on n, as λ increases, n will increase so that the zero-profit condition $\Pi\left(\widehat{n}^*(\lambda),\lambda\right) = f$ is satisfied, that is

$$\begin{aligned}0 = d\Pi\left(\widehat{n}^*(\lambda),\lambda\right) =& P'(Q_n)\frac{Q_n}{n}\frac{\partial Q_n}{\partial \lambda}\frac{n-\Lambda_n}{n}d\lambda + \frac{P'(Q_n)}{n}\left(\frac{Q_n}{n}\right)^2 \times \\ & \left[\frac{(1-\lambda)(n-\Lambda) + \Lambda(n+\Lambda) - nC''(q_n)/P'(Q_n) - \Lambda_n^2 E_{P'}(Q_n)}{n+\Lambda_n - C''(q_n)/P'(Q_n) - \Lambda_n E_{P'}(Q_n)}\right] d\widehat{n}^*(\lambda) \iff \\ \frac{d\widehat{n}^*(\lambda)}{d\lambda} &= -\frac{\frac{\partial Q_n}{\partial \lambda}(n-\Lambda_n)}{\frac{Q_n}{n}\left[\frac{(1-\lambda)(n-\Lambda)+\Lambda(n+\Lambda)-nC''(q_n)/P'(Q_n)-\Lambda_n^2 E_{P'}(Q_n)}{n+\Lambda_n-C''(q_n)/P'(Q_n)-\Lambda_n E_{P'}(Q_n)}\right]} \\ &= \frac{\frac{(n-1)(n-\Lambda_n)}{n+\Lambda_n-C''(q_n)/P'(Q_n)-\Lambda_n E_{P'}(Q_n)}}{\frac{(1-\lambda)(1-H_n)+\Lambda(1+H_n)-C''(q_n)/P'(Q_n)-\Lambda_n^2 E_{P'}(Q_n)/n}{n+\Lambda_n-C''(q_n)/P'(Q_n)-\Lambda_n E_{P'}(Q_n)}} \\ &= \frac{(n-1)(H_n^{-1}-1)}{1+H_n+\Lambda_n^{-1}\left[(1-\lambda)(1-H_n) - C''(q_n)/P'(Q_n)\right] - H_n E_{P'}(Q_n)} > 0\end{aligned}$$

evaluated in equilibrium, where the inequality follows from what we have seen in section A.3.

(ii) The total derivative of the total quantity is then

$$\begin{aligned}\frac{dQ_{\widehat{n}^*(\lambda)}}{d\lambda} &= \frac{\partial Q_n}{\partial \lambda}\frac{\Lambda_n(1+\lambda)+1-\lambda-C''(q_n)/P'(Q_n)-\Lambda_n^2 E_{P'}(Q_n)/n}{(n-1)(n-\Lambda_n)} + \frac{\partial Q_n}{\partial n} \\ &= \frac{Q_n\begin{bmatrix}-\left(\Lambda_n(1+\lambda)+1-\lambda-C''(q_n)/P'(Q_n)-\Lambda_n^2 E_{P'}(Q_n)/n\right) \\ +(n-\Lambda_n)\left(1-\lambda-C''(q_n)/P'(Q_n)\right)/n\end{bmatrix}}{(n-\Lambda_n)\left(n+\Lambda_n-C''(q_n)/P'(Q_n)-\Lambda_n E_{P'}(Q_n)\right)} \\ &= -\frac{\Lambda_n Q_n\left[n+\lambda n+1-\lambda-C''(q_n)/P'(Q_n)-\Lambda_n E_{P'}(Q_n)\right]}{n(n-\Lambda_n)\left(n+\Lambda_n-C''(q_n)/P'(Q_n)-\Lambda_n E_{P'}(Q_n)\right)} = -\frac{\Lambda_n Q_n}{n(n-\Lambda_n)} < 0,\end{aligned}$$

so total quantity decreases with λ, and thus so does individual quantity since the number of firms increases with λ. Also, since the number of firms increases with λ, HHI decreases with λ. The total derivative of the total surplus is

$$\begin{aligned}\frac{d\,\mathrm{TS}_{\widehat{n}^*(\lambda)}}{d\lambda} &= P(Q_n)\frac{dQ_{\widehat{n}^*(\lambda)}}{d\lambda} - \frac{d\widehat{n}^*(\lambda)}{d\lambda}C(q_n) - nC'(q_n)\left(\frac{dQ_{\widehat{n}^*(\lambda)}/d\lambda}{n} - \frac{q_n}{n}\frac{d\widehat{n}^*(\lambda)}{d\lambda}\right) - \frac{d\widehat{n}^*(\lambda)}{d\lambda}f \\ &= -\frac{\Lambda_n Q_n}{n(n-\Lambda_n)}\left(P(Q_n)-C'(q_n)\right) - \left(C(q_n)-C'(q_n)q_n+f\right)\frac{d\widehat{n}^*(\lambda)}{d\lambda} \\ &\overset{\Pi(\widehat{n}^*(\lambda),\lambda)=0}{=} -\frac{\Lambda_n Q_n}{n(n-\Lambda_n)}\left(P(Q_n)-C'(q_n)\right) - \left(P(Q_n)-C'(q_n)\right)q_n\frac{d\widehat{n}^*(\lambda)}{d\lambda} < 0.\end{aligned}$$

(iii) Last, the total derivative of $\text{MHHI}^* = H_{n^*}$ is

$$\begin{aligned}
\frac{d\,\text{MHHI}\left(\boldsymbol{q}_{\widehat{n}^*(\lambda)}\right)}{d\lambda} &= \frac{n-1}{n} + \left(\frac{\lambda}{n} - \frac{\Lambda_n}{n^2}\right)\frac{d\widehat{n}^*(\lambda)}{d\lambda} \propto \frac{n-1}{n}\left(\frac{d\widehat{n}^*(\lambda)}{d\lambda}\right)^{-1} + \frac{\lambda n - \Lambda_n}{n^2} \\
&= \frac{\Lambda_n(1+\lambda) + 1 - \lambda - C''\left(q_n\right)/P'\left(Q_n\right) - \Lambda_n^2 E_{P'}\left(Q_n\right)/n}{n(n-\Lambda_n)} - \frac{1-\lambda}{n^2} \\
&= \frac{\Lambda_n(1+H_n) - C''\left(q_n\right)/P'\left(Q_n\right) - \Lambda_n^2 E_{P'}\left(Q_n\right)/n}{n(n-\Lambda_n)} \\
&\propto \frac{1 + H_n^{-1} - C''\left(q_n\right)/\left(H_n\Lambda_n P'\left(Q_n\right)\right) - E_{P'}\left(Q_n\right)}{n(n-\Lambda_n)} \\
&= \frac{1 + H_n^{-1} + \left(\Delta(Q_n,(n-1)q_n) - (1-\lambda)\right)/\left(H_n\Lambda_n\right) - E_{P'}\left(Q_n\right)}{n(n-\Lambda_n)} \\
&= \frac{\left[1 + H_n - (1-\lambda)/\Lambda_n + \left(\Delta(Q_n,(n-1)q_n)/\Lambda_n\right)\right]/H_n - E_{P'}\left(Q_n\right)}{n(n-\Lambda_n)} \\
&= \frac{\left[1 + \lambda - (1-\lambda)(1-H_n)/\Lambda_n + \Delta(Q_n,(n-1)q_n)/\Lambda_n\right]/H_n - E_{P'}\left(Q_n\right)}{n(n-\Lambda_n)} \\
&= \frac{\left[1 + \lambda + \Delta(Q_n,(n-1)q_n)/n + \left(\frac{1}{n} - \frac{1}{\Lambda_n}\right)\frac{C''(q_n)}{P'(Q_n)}\right]/H_n - E_{P'}\left(Q_n\right)}{n(n-\Lambda_n)} > 0,
\end{aligned}$$

where the inequality is implied by $C'' \geq 0$ combined with the maintained assumption (ii) that requires $E_{P'}(Q_n) < \left(1 + \lambda + \Delta(Q_n,(n-1)q_n)/n\right)/H_n$.

(iv) If we take into account the integer constraint, then as λ increases no more firms will enter until profits increase by the amount such that one more firm enters so that $\pi\left(\boldsymbol{q}_{n^*(\lambda)}\right) = f > \pi\left(\boldsymbol{q}_{n^*(\lambda)+1}\right)$ holds. Since Cournot profits fall below f for n large enough, there will thus be a finite set of thresholds $L := \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$, $\lambda_1 < \cdots < \lambda_k$, such that as λ increases, one more firm enters when it reaches a threshold in L. Between the thresholds the total quantity will now decrease faster compared to the case where we ignored the integer constraint, as now n does not increase between the thresholds. On the other hand, for each value of λ not in L individual quantity will be higher when we take into account the integer constraint than when we ignore it, since individual quantity is decreasing in n.[48] **Q.E.D.**

Proof of Proposition 6 We have that

$$\frac{d\widehat{n}^*(\lambda)}{df} = \left(\frac{\partial\Pi\left(n,\lambda\right)}{\partial n}\right)^{-1}\Bigg|_{n=\widehat{n}^*(\lambda)},$$

and part (ii) follows if we differentiate $d\widehat{n}^*(\lambda)/df$ in the direction in which λ and x change. **Q.E.D.**

Proof of Claim 1 We have that

$$\frac{\partial\Pi\left(n,\lambda\right)}{\partial n} = P'(Q_n)\left(\frac{Q_n}{n}\right)^2\left[\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\left(1 - \frac{\Lambda}{n}\right) + \frac{\Lambda}{n}\right] < 0,$$

[48] For a similar argument spelled out in more detail see the next paragraph.

so that

$$\frac{\partial^2\Pi(n,\lambda)}{\partial n\partial\lambda} = \left[\begin{array}{c} \dfrac{P''(Q_n)\frac{\partial Q_n}{\partial\lambda}}{P'(Q_n)} + \dfrac{\partial Q_n}{\partial\lambda}\dfrac{2}{Q_n} \\ + \dfrac{\frac{\partial^2 Q_n}{\partial n\partial\lambda}\frac{n}{Q_n}\frac{n-\Lambda_n}{n} - \frac{\partial Q_n}{\partial n}\frac{n}{(Q_n)^2}\frac{\partial Q_n}{\partial\lambda}\frac{n-\Lambda_n}{n} - \frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\frac{n-1}{n} + \frac{n-1}{n}}{\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\frac{n-\Lambda_n}{n} + \frac{\Lambda_n}{n}} \end{array}\right]\frac{\partial\Pi(n,\lambda)}{\partial n}$$

$$\propto -\frac{\partial Q_n}{\partial\lambda}\frac{1}{Q_n}\left(2 - E_{P'}(Q_n)\right) - \frac{\frac{\partial^2 Q_n}{\partial n\partial\lambda}\frac{n}{Q_n} - \frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\frac{\partial Q_n}{\partial\lambda}\frac{1}{Q_n} + \frac{n-1}{n-\Lambda_n}\left(1 - \frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\right)}{\frac{\partial Q_n}{\partial n}\frac{n}{Q_n} + \frac{\Lambda_n}{n-\Lambda_n}}$$

$$= -\frac{\frac{\partial^2 Q_n}{\partial n\partial\lambda}\frac{n}{Q_n} + \frac{n-1}{n-\Lambda_n}\left(1 - \frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\right)}{\frac{\partial Q_n}{\partial n}\frac{n}{Q_n} + \frac{\Lambda_n}{n-\Lambda_n}} - \left[1 + \frac{\frac{\Lambda_n}{n-\Lambda_n}}{\frac{\partial Q_n}{\partial n}\frac{n}{Q_n} + \frac{\Lambda_n}{n-\Lambda_n}} - E_{P'}(Q_n)\right]\frac{\partial Q_n}{\partial\lambda}\frac{1}{Q_n}$$

$$\propto \left\{\begin{array}{c} \left[-\left(1 - E_{P'}(Q_n)\right)\dfrac{\partial Q_n}{\partial n}\dfrac{n}{Q_n} - \left(2 - E_{P'}(Q_n)\right)\dfrac{\Lambda_n}{n-\Lambda_n}\right]\dfrac{\partial Q_n}{\partial\lambda}\dfrac{1}{Q_n} \\ -\dfrac{\partial^2 Q_n}{\partial n\partial\lambda}\dfrac{n}{Q_n} - \dfrac{n-1}{n-\Lambda_n}\left(1 - \dfrac{\partial Q_n}{\partial n}\dfrac{n}{Q_n}\right) \end{array}\right\},$$

since $\frac{\partial Q_n}{\partial n}\frac{n}{Q_n} + \frac{\Lambda_n}{n-\Lambda_n} > 0$. Denote $E'_{P'}(Q) \equiv \partial E_{P'}(Q)/\partial Q$. Under constant marginal costs

$$\frac{\partial^2 Q_n}{\partial n\partial\lambda} = \frac{\left\{\begin{array}{c} \left(-Q_n + (1-\lambda)\dfrac{\partial Q_n}{\partial\lambda}\right)\left(n + \Lambda - \Lambda E_{P'}\left(Q_n\right)\right) \\ -(1-\lambda)Q_n\left[n - 1 - (n-1)E_{P'}\left(Q_n\right) - \Lambda E'_{P'}\left(Q_n\right)\dfrac{\partial Q_n}{\partial\lambda}\right] \end{array}\right\}}{\left(n + \Lambda - \Lambda E_{P'}\left(Q_n\right)\right)^2}\frac{1}{n}$$

so that

$$\frac{\partial^2\Pi(n,\lambda)}{\partial n\partial\lambda} \propto \left\{\begin{array}{c} \left[\left(1 - E_{P'}(Q_n)\right)\dfrac{1-\lambda}{n + \Lambda - \Lambda E_{P'}\left(Q_n\right)} + \left(2 - E_{P'}(Q_n)\right)\dfrac{\Lambda}{n-\Lambda}\right]\dfrac{n-1}{n + \Lambda - \Lambda E_{P'}\left(Q_n\right)} \\ + \dfrac{\left\{\begin{array}{c} \left(1 + (1-\lambda)\dfrac{n-1}{n + \Lambda - \Lambda E_{P'}\left(Q_n\right)}\right)\left(n + \Lambda - \Lambda E_{P'}\left(Q_n\right)\right) \\ +(1-\lambda)\left[(n-1)\left(1 - E_{P'}\left(Q_n\right)\right) - \dfrac{\Lambda(n-1)Q_n E'_{P'}\left(Q_n\right)}{n + \Lambda - \Lambda E_{P'}\left(Q_n\right)}\right] \end{array}\right\}}{\left(n + \Lambda - \Lambda E_{P'}\left(Q_n\right)\right)^2} \\ -\dfrac{n-1}{n-\Lambda}\left(1 - \dfrac{1-\lambda}{n + \Lambda - \Lambda E_{P'}\left(Q_n\right)}\right) \end{array}\right\}$$

$$\propto \left\{\begin{array}{c} \dfrac{\left(1 - E_{P'}(Q_n)\right)(1-\lambda)}{n + \Lambda - \Lambda E_{P'}\left(Q_n\right)} + \dfrac{\Lambda\left(2 - E_{P'}(Q_n)\right) - n(1+\lambda) - \Lambda E_{P'}\left(Q_n\right)}{n-\Lambda} \\ + \dfrac{2n - \Lambda E_{P'}\left(Q_n\right) + (n-\Lambda)\left[1 - E_{P'}\left(Q_n\right) - \frac{\Lambda Q_n E'_{P'}(Q_n)}{n+\Lambda-\Lambda E_{P'}(Q_n)}\right]}{(n-1)\left(n + \Lambda - \Lambda E_{P'}\left(Q_n\right)\right)} \end{array}\right\}$$

$$= \left\{\begin{array}{c} \dfrac{2\left(1 - E_{P'}(Q_n)\right)(1-\lambda)}{n + \Lambda - \Lambda E_{P'}\left(Q_n\right)} + \dfrac{\Lambda + 1 - \lambda - n - 2\Lambda E_{P'}\left(Q_n\right)}{n-\Lambda} \\ + \dfrac{2n - \Lambda E_{P'}\left(Q_n\right) - (n-\Lambda)\frac{\Lambda Q_n E'_{P'}(Q_n)}{n+\Lambda-\Lambda E_{P'}(Q_n)}}{(n-1)\left(n + \Lambda - \Lambda E_{P'}\left(Q_n\right)\right)} \end{array}\right\}$$

$$
\begin{aligned}
&\propto \left\{\begin{array}{r} 2\left(1-E_{P'}(Q_n)\right)(n-\Lambda)^2-\left((n-\Lambda)(n-2)+2\Lambda E_{P'}\left(Q_n\right)\right)\left((n+\Lambda-\Lambda E_{P'}\left(Q_n\right)\right)) \\ +(n-\Lambda)\left(2n-\Lambda E_{P'}\left(Q_n\right)\right)-\dfrac{\Lambda(n-\Lambda)^2 Q_n E'_{P'}\left(Q_n\right)}{n+\Lambda-\Lambda E_{P'}\left(Q_n\right)} \end{array}\right\} \\
&= \left\{\begin{array}{r} 2\Lambda^2\left(E_{P'}(Q_n)\right)^2+\left[-2\Lambda(n+\Lambda)+\Lambda(n-\Lambda)(n-2)-\Lambda(n-\Lambda)-2(n-\Lambda)^2\right]E_{P'}(Q_n) \\ +2(n-\Lambda)^2-(n-\Lambda)(n-2)(n+\Lambda)+2n(n-\Lambda)-\dfrac{\Lambda(n-\Lambda)^2 Q_n E'_{P'}\left(Q_n\right)}{n+\Lambda-\Lambda E_{P'}\left(Q_n\right)} \end{array}\right\} \\
&= \left\{\begin{array}{r} 2\Lambda^2\left(E_{P'}(Q_n)\right)^2+\left[n\Lambda(n-\Lambda-1)-2n^2-\Lambda^2\right]E_{P'}(Q_n) \\ -n(n-\Lambda)(n+\Lambda-6)-\dfrac{\Lambda(n-\Lambda)^2 Q_n E'_{P'}\left(Q_n\right)}{n+\Lambda-\Lambda E_{P'}\left(Q_n\right)} \end{array}\right\} \\
&\overset{E'_{P'}\geq 0}{\leq} \left\{\begin{array}{r} 2\Lambda^2\left(E_{P'}(Q_n)\right)^2+\left[n\Lambda(n-\Lambda-1)-2n^2-\Lambda^2\right]E_{P'}(Q_n) \\ -n(n-\Lambda)(n+\Lambda-6) \end{array}\right\} \\
&= \left\{\begin{array}{r} 2\Lambda^2\left(E_{P'}(Q_n)\right)^2-\left(n\Lambda+2n^2+\Lambda^2\right)E_{P'}(Q_n) \\ -n(n-\Lambda)\left(n+\Lambda\left(1-E_{P'}(Q_n)\right)-6\right) \end{array}\right\} \\
&\overset{n\geq 5+E_{P'}(Q_n),\lambda\in(0,1),E_{P'}(Q_n)\leq 1}{<} E_{P'}(Q_n)\left[2\Lambda^2 E_{P'}(Q_n)-n\Lambda-2n^2-\Lambda^2\right] \overset{0\leq E_{P'}(Q_n)\leq 1}{\leq} 0.
\end{aligned}
$$

Q.E.D.

Proof of Proposition 7 We deal with the two parts separately.

(i) If $n^o(\lambda)\leq 2$, we are done since $n^*(\lambda)\geq 1$ given that monopoly profit is positive. For $n^o(\lambda)\geq 3$ keep in mind that $E_{P'}(Q)<2$ on L implies that for every $n\in[2,+\infty)$, $E_{P'}(Q)<(1+\lambda)/H_n$ on L. The proof follows the proof of part (a) of Proposition 1 in Amir et al. (ACK; 2014). By definition $\mathrm{TS}_{n^o(\lambda)}\geq \mathrm{TS}_{n^o(\lambda)-1}$, which implies

$$
\int_{Q_{n^o(\lambda)-1}}^{Q_{n^o(\lambda)}} P(X)dX-n^o(\lambda)C\left(q_{n^o(\lambda)}\right)+\left(n^o(\lambda)-1\right)C\left(q_{n^o(\lambda)-1}\right)\geq f,
$$

which then gives

$$
\begin{aligned}
\Pi\left(n^o(\lambda)-1,\lambda\right)-f &\geq P\left(Q_{n^o(\lambda)-1}\right)q_{n^o(\lambda)-1} \\
-\int_{Q_{n^o(\lambda)-1}}^{Q_{n^o(\lambda)}} P(X)dX &+n^o(\lambda)\left(C\left(q_{n^o(\lambda)}\right)-C\left(q_{n^o(\lambda)-1}\right)\right),
\end{aligned}
$$

which given $P'<0$ and that in the Cournot game total quantity is increasing in n, implies that

$$
\begin{aligned}
\Pi\left(n^o(\lambda)-1,\lambda\right)-f &> P\left(Q_{n^o(\lambda)-1}\right)\left(q_{n^o(\lambda)-1}+Q_{n^o(\lambda)-1}-Q_{n^o(\lambda)}\right) \\
&\quad +n^o(\lambda)\left(C\left(q_{n^o(\lambda)}\right)-C\left(q_{n^o(\lambda)-1}\right)\right) \implies \\
\Pi\left(n^o(\lambda)-1,\lambda\right)-f &> n^o(\lambda)\left(P\left(Q_{n^o(\lambda)-1}\right)-C'\left(\widetilde{q}\right)\right)\left(q_{n^o(\lambda)-1}-q_{n^o(\lambda)}\right),
\end{aligned}
$$

for some $\widetilde{q}\in\left[q_{n^o(\lambda)-1}-q_{n^o(\lambda)}\right]$, where the last implication follows by the mean value theorem. Since $R(Q_{-i})$ is nondecreasing in Q_{-i}, it follows as in the proof in ACK that there exists $\widetilde{Q}_{-i}\in\left[(n^o(\lambda)-2)q_{n^o(\lambda)-1},(n^o(\lambda)-1)q_{n^o(\lambda)}\right]$ such that $\widetilde{q}\in r\left(\widetilde{Q}_{-i}\right)$ with $R\left(\widetilde{Q}_{-i}\right)\geq Q_{n^o(\lambda)-1}$ and $P\left(R(\widetilde{Q}_{-i})\right)\geq C'(\widetilde{q})$, so that $P\left(Q_{n^o(\lambda)-1}\right)\geq P\left(R(\widetilde{Q}_{-i})\right)\geq$

$C'(\widetilde{q})$.

Given $\Delta > 0$ and $E_{P'} < (1+\lambda)/H_n$, Proposition 3 implies that $q_{n^o(\lambda)-1} > q_{n^o(\lambda)}$, which combined with the above gives $\Pi\left(n^o(\lambda)-1,\lambda\right) - f \geq 0$. Also, given $\Delta > 0$ and $E_{P'}(Q_n) < (1+\lambda)/H_n$, Proposition 4 implies that $\Pi(n,\lambda)$ is decreasing in n, so it must be $n^*(\lambda) \geq n^o(\lambda) - 1$ for the entry condition to be satisfied.

(ii) Since $\Pi(1,\lambda) > f$, $n^*(\lambda) \geq 1$. Also, $\Delta < 0$ on L implies that $C''(q) < 0$ for every $q < \overline{Q}$. By Proposition 3 the price $P(Q_n)$ is increasing in n. Also, we have that

$$n\Pi(n,\lambda) \equiv P(Q_n)Q_n - nC(q_n) \overset{C''<0}{<} P(Q_n)Q_n - C(Q_n) \leq P(q_1)q_1 - C(q_1) = \Pi(1,\lambda),$$

that is aggregate industry profits are maximized for $n = 1$. Thus, both consumer surplus and industry profits are maximized for $n = 1$, so $n^o(\lambda) = 1$.

(iii) Given that $\widehat{n}^o(\lambda)$ is interior so that $\partial\,\mathrm{TS}_n/\partial n = 0$ at $n = \widehat{n}^o(\lambda)$, we get

$$\frac{d\,\mathrm{TS}_{\widehat{n}^o(\lambda)}}{d\lambda} = P(Q_n)\frac{\partial Q_n}{\partial\lambda} - nC'(q_n)\frac{\partial q_n}{\partial\lambda} = -\frac{(n-1)\left(P(Q_n) - C'(q_n)\right)Q_n}{n + \Lambda_n - C''(q_n)/P'(Q_n) - \Lambda_n E_{P'}(Q_n)}.$$

Given also what we have seen about $d\,\mathrm{TS}_{\widehat{n}^*(\lambda)}/d\lambda$ in the proof of Proposition 5, we conclude then that

$$\begin{aligned}\frac{d\,\mathrm{WL}(\lambda)}{d\lambda} \propto & \frac{\widehat{n}^o(\lambda) + \Lambda_{\widehat{n}^o(\lambda)} - C''\left(q_{\widehat{n}^o(\lambda)}\right)/P'\left(Q_{\widehat{n}^o(\lambda)}\right) - \Lambda_{\widehat{n}^o(\lambda)}E_{P'}\left(Q_{\widehat{n}^o(\lambda)}\right)}{(n-1)\left(\widehat{n}^*(\lambda) - \Lambda_{\widehat{n}^*(\lambda)}\right)} \\ & - \frac{\left(P\left(Q_{\widehat{n}^o(\lambda)}\right) - C'\left(q_{\widehat{n}^o(\lambda)}\right)\right)Q_{\widehat{n}^o(\lambda)}}{\left(P(Q_{\widehat{n}^*(\lambda)}) - C'(q_{\widehat{n}^*(\lambda)})\right)Q_{\widehat{n}^*(\lambda)}}.\end{aligned}$$

Under constant marginal costs

$$\begin{aligned}\frac{d\,\mathrm{WL}(\lambda)}{d\lambda} &\propto \frac{\widehat{n}^o(\lambda) + \Lambda_{\widehat{n}^o(\lambda)} - \Lambda_{\widehat{n}^o(\lambda)}E_{P'}\left(Q_{\widehat{n}^o(\lambda)}\right)}{(\widehat{n}^*(\lambda)-1)\left(\widehat{n}^*(\lambda) - \Lambda_{\widehat{n}^*(\lambda)}\right)} - \frac{\widehat{n}^o(\lambda)\Pi\left(\widehat{n}^o(\lambda),\lambda\right)}{\widehat{n}^*(\lambda)\Pi\left(\widehat{n}^*(\lambda),\lambda\right)} \\ &\propto \frac{1 + H_{\widehat{n}^o(\lambda)} - H_{\widehat{n}^o(\lambda)}E_{P'}\left(Q_{\widehat{n}^o(\lambda)}\right)}{(\widehat{n}^*(\lambda)-1)\left(1 - H_{\widehat{n}^*(\lambda)}\right)} - \frac{\Pi\left(\widehat{n}^o(\lambda),\lambda\right)}{\Pi\left(\widehat{n}^*(\lambda),\lambda\right)}.\end{aligned}$$

Q.E.D.

Proof of Proposition 7' For simplicity, we use the notation Q_n, q_n, TS_n and $\Pi(n,\lambda)$ to refer to values in a specific equilibrium even if that equilibrium is not unique. We deal with the two parts separately.

(i) The proof works like that of part (i) of Proposition 7. The only differences are that

(a) in the Cournot game the total quantity in extremal equilibria is non-decreasing in n (instead of increasing in n),

(b) $q_{n^o(\lambda)-1} \geq q_{n^o(\lambda)}$ by assumption (instead of $q_{n^o(\lambda)-1} > q_{n^o(\lambda)}$ implied by conditions on the primitives)

(c) $\Pi(n,\lambda)$ is non-increasing in n in extremal equilibria (instead of decreasing).

Still, the weak inequality $\Pi\left(n^o(\lambda)-1,\lambda\right) - f \geq 0$ must hold and given that $\Pi(n,\lambda)$ is non-increasing in n, it must be that $n^*(\lambda) \geq n^o(\lambda) - 1$.

(ii) The proof follows the proof of part (b) of Proposition 1 in ACK. Since P is decreasing, we have

$$q_{n^o(\lambda)+1} P\left(Q_{n^o(\lambda)+1}\right) < \int_{n^o(\lambda) q_{n^o(\lambda)+1}}^{(n^o(\lambda)+1) q_{n^o(\lambda)+1}} P(Q) dQ. \tag{5}$$

Also, notice that $V_n(q) := \int_0^{nq} P(Q)dQ - nC(q)$ is concave in q (for every fixed n), since

$$V_n'(q) = n\left(P(nq) - C'(q)\right),$$

so that

$$V_n''(q) = nP'(nq)\left(n - \frac{C''(q)}{P'(nq)}\right) = nP'(nq)\left(\Delta(nq,(n-1)q) + n - 1 + \lambda\right) < 0.$$

Since $V_n(q)$ is concave in n, it follows then that for any n and any q,q' such that $q' \geq q$ it holds that

$$V_n(q) - V_n(q') \leq V_n'(q)(q - q') = n\left(P(nq') - C'(q')\right)(q - q'). \tag{6}$$

Also, by definition $\mathrm{TS}_{n^o(\lambda)} \geq \mathrm{TS}_{n^o(\lambda)+1}$, which implies

$$\begin{aligned}
\Pi\left(n^o(\lambda)+1, \lambda\right) - f &\leq P\left(Q_{n^o(\lambda)+1}\right) q_{n^o(\lambda)+1} - \int_{Q_{n^o(\lambda)}}^{Q_{n^o(\lambda)+1}} P(X)dX + n^o(\lambda)\left(C\left(q_{n^o(\lambda)+1}\right) - C\left(q_{n^o(\lambda)}\right)\right) \\
&\overset{(5)}{<} \int_0^{Q_{n^o(\lambda)}} P(Q)dQ - n^o(\lambda)C\left(q_{n^o(\lambda)}\right) - \left[\int_0^{Q_{n^o(\lambda)+1}} P(Q)dQ - n^o(\lambda)C\left(q_{n^o(\lambda)+1}\right)\right] \\
&= V_{n^o(\lambda)}\left(q_{n^o(\lambda)}\right) - V_{n^o(\lambda)}\left(q_{n^o(\lambda)+1}\right) \\
&\overset{(6), q_{n^o(\lambda)+1} \geq q_{n^o(\lambda)}}{\leq} n^o(\lambda)\left(P\left(n^o(\lambda) q_{n^o(\lambda)+1}\right) - C'\left(q_{n^o(\lambda)+1}\right)\right)\left(q_{n^o(\lambda)} - q_{n^o(\lambda)+1}\right) \leq 0,
\end{aligned}$$

where the last inequality follows from $q_{n^o(\lambda)+1} \geq q_{n^o(\lambda)+1}$ and

$$P\left(n^o(\lambda) q_{n^o(\lambda)+1}\right) \geq P\left(\left(n^o(\lambda)+1\right) q_{n^o(\lambda)+1}\right) = P\left(Q_{n^o(\lambda)+1}\right) \geq C'\left(q_{n^o(\lambda)+1}\right)$$

by the pricing formula (1). Thus, $\Pi\left(n^o(\lambda)+1, \lambda\right) < f$, and given that $\Pi(n,\lambda)$ is non-increasing in n, we conclude that $n^*(\lambda) \leq n^o(\lambda)$. **Q.E.D.**

A.7 Proofs of section 5

Proof of Proposition 8 $\Psi(1,1) = \Pi(1,1) = \Pi(1,0) \geq f$. Also, the derivative of $\Psi(n,\lambda)$ with respect to n is equal to

$$\begin{aligned}
\frac{\partial \Psi(n,\lambda)}{\partial n} &= \lambda\left(\Pi(n,\lambda) - \Pi(n-1,\lambda)\right) + \Lambda_n \frac{\partial \Pi(n,\lambda)}{\partial n} - \left(\Lambda_n - 1\right) \left.\frac{\partial \Pi(\nu,\lambda)}{\partial \nu}\right|_{\nu=n-1} \\
&= \Lambda_n\left(\Pi(n,\lambda) - \Pi(n-1,\lambda)\right)\left[\frac{\lambda}{\Lambda_n} + \frac{\frac{\partial \Pi(n,\lambda)}{\partial n} - \left(1 - \frac{1}{\Lambda_n}\right) \left.\frac{\partial \Pi(\nu,\lambda)}{\partial \nu}\right|_{\nu=n-1}}{\Pi(n,\lambda) - \Pi(n-1,\lambda)}\right]
\end{aligned}$$

$$\propto E_{\Delta\Pi,n} - \left(\frac{\Lambda_n - 1}{\Lambda_n} + \frac{n-1}{\Lambda_n} \frac{\left.\frac{\partial \Pi(\nu,\lambda)}{\partial \nu}\right|_{\nu=n-1}}{\Pi(n,\lambda) - \Pi(n-1,\lambda)} \right) < 0,$$

and the result follows given Proposition 1. **Q.E.D.**

Proof of Proposition 9 The derivative of $\Psi(n,\lambda)$ with respect to λ is given by

$$\begin{aligned}
\frac{\partial \Psi(n,\lambda)}{\partial \lambda} &= (n-1)\left(\Pi(n,\lambda) - \Pi(n-1,\lambda)\right) + \Lambda_n \frac{\partial \Pi(n,\lambda)}{\partial \lambda} - (\Lambda_n - 1)\frac{\partial \Pi(n-1,\lambda)}{\partial \lambda} \quad (7) \\
&= (n-1)\left(\Pi(n,\lambda) - \Pi(n-1,\lambda)\right)\left[1 + \frac{\lambda\left(\frac{\partial \Pi(n,\lambda)}{\partial \lambda} - \frac{\partial \Pi(n-1,\lambda)}{\partial \lambda}\right) + \frac{1}{n-1}\frac{\partial \Pi(n,\lambda)}{\partial \lambda}}{\Pi(n,\lambda) - \Pi(n-1,\lambda)}\right] \\
&\propto -\frac{\lambda\left(\frac{\partial \Pi(n,\lambda)}{\partial \lambda} - \frac{\partial \Pi(n-1,\lambda)}{\partial \lambda}\right)}{\Pi(n,\lambda) - \Pi(n-1,\lambda)} - \frac{1}{\lambda}\frac{\lambda \frac{\partial \Pi(n,\lambda)}{\partial \lambda} \Big/ \Pi(n,\lambda)}{\frac{\Pi(n,\lambda) - \Pi(n-1,\lambda)}{\Pi(n,\lambda)}(n-1)} - 1
\end{aligned}$$

where the last line follows given $\Pi(n,\lambda) < \Pi(n-1,\lambda)$. The result follows by the Implicit Function Theorem given Proposition 8. **Q.E.D.**

Proof of Proposition 10 We have that

$$\frac{d\widehat{n}^*(\lambda)}{df} = \left.\left(\frac{\partial \Psi(n,\lambda)}{\partial n}\right)^{-1}\right|_{n=\widehat{n}^*(\lambda)}$$

and part (ii) follows if we differentiate $d\widehat{n}^*(\lambda)/df$ in the direction in which λ and x change. **Q.E.D.**

Proof of Proposition 11 The derivative of equilibrium total surplus (in the Cournot game with a fixed number of firms) with respect to n is given by

$$\frac{\partial \,\mathrm{TS}_n}{\partial n} = \Pi(n,\lambda) - f - \Lambda_n Q_n P'(Q_n)\frac{\partial q_n}{\partial n}.$$

Given $\Psi(\widehat{n}^*(\lambda),\lambda) = f$ we then have that (denote $\Pi_n(n,\lambda) \equiv \partial \Pi(n,\lambda)/\partial n$)

$$\begin{aligned}
\left.\frac{d\,\mathrm{TS}_n}{dn}\right|_{n=\widehat{n}^*(\lambda)} &= -\phi(\widehat{n}^*(\lambda),\lambda)\,\lambda\widehat{n}^*(\lambda)\Pi_n(\widehat{n}^*(\lambda),\lambda) - \Lambda_{\widehat{n}^*(\lambda)} Q_{\widehat{n}^*(\lambda)} P'\left(Q_{\widehat{n}^*(\lambda)}\right)\left.\frac{\partial q_n}{\partial n}\right|_{n=\widehat{n}^*(\lambda)} \\
&= -\phi(\widehat{n}^*(\lambda),\lambda)\,\lambda\widehat{n}^*(\lambda)P'\left(Q_{\widehat{n}^*(\lambda)}\right)\left(\frac{Q_{\widehat{n}^*(\lambda)}}{\widehat{n}^*(\lambda)}\right)^2 \times \\
&\quad \left[\left.\frac{\partial Q_n}{\partial n}\right|_{n=\widehat{n}^*(\lambda)} \frac{\widehat{n}^*(\lambda)}{Q_{\widehat{n}^*(\lambda)}}\left(1 - \frac{\Lambda_{\widehat{n}^*(\lambda)}}{\widehat{n}^*(\lambda)}\right) + \frac{\Lambda_{\widehat{n}^*(\lambda)}}{\widehat{n}^*(\lambda)}\right] \\
&\quad - \Lambda_{\widehat{n}^*(\lambda)} P'\left(Q_{\widehat{n}^*(\lambda)}\right)\left(\frac{Q_{\widehat{n}^*(\lambda)}}{\widehat{n}^*(\lambda)}\right)^2\left[\left.\frac{dQ^n}{\partial n}\right|_{n=\widehat{n}^*(\lambda)} \frac{\widehat{n}^*(\lambda)}{Q_{\widehat{n}^*(\lambda)}} - 1\right] \\
&\propto \left.\frac{\partial Q_n}{\partial n}\right|_{n=\widehat{n}^*(\lambda)} \frac{\widehat{n}^*(\lambda)}{Q_{\widehat{n}^*(\lambda)}}\left[\phi(\widehat{n}^*(\lambda),\lambda)\,\lambda\widehat{n}^*(\lambda) + \Lambda_{\widehat{n}^*(\lambda)}\left(1 - \phi(\widehat{n}^*(\lambda),\lambda)\,\lambda\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&- \Lambda_{\widehat{n}^*(\lambda)} \left(1 - \phi\left(\widehat{n}^*(\lambda),\lambda\right)\lambda\right) \\
=&\frac{\left[1 - \lambda - \frac{C''\left(q_{\widehat{n}^*(\lambda)}\right)}{P'\left(Q_{\widehat{n}^*(\lambda)}\right)}\right]\left[\phi\left(\widehat{n}^*(\lambda),\lambda\right)\lambda\widehat{n}^*(\lambda) + \Lambda_{\widehat{n}^*(\lambda)}\left(1 - \phi\left(\widehat{n}^*(\lambda),\lambda\right)\lambda\right)\right]}{\widehat{n}^*(\lambda) + \Lambda_{\widehat{n}^*(\lambda)} - \frac{C''\left(q_{\widehat{n}^*(\lambda)}\right)}{P'\left(Q_{\widehat{n}^*(\lambda)}\right)} - \Lambda_{\widehat{n}^*(\lambda)}E} \\
&- \Lambda_{\widehat{n}^*(\lambda)} \left(1 - \phi\left(\widehat{n}^*(\lambda),\lambda\right)\lambda\right) \\
\propto&(1-\lambda)\left\{\begin{array}{r}\phi\lambda\widehat{n}^*(\lambda) - \Lambda_{\widehat{n}^*(\lambda)}\frac{1-\phi\lambda}{1-\lambda}\left[\widehat{n}^*(\lambda) + \Lambda_{\widehat{n}^*(\lambda)} - (1-\lambda)\right] \\ +\frac{1-\phi\lambda}{1-\lambda}\Lambda^2_{\widehat{n}^*(\lambda)}E_{P'}\left(Q_{\widehat{n}^*(\lambda)}\right)\end{array}\right\} \\
&- \phi\lambda\widehat{n}^*(\lambda)\frac{C''\left(q_{\widehat{n}^*(\lambda)}\right)}{P'\left(Q_{\widehat{n}^*(\lambda)}\right)} \\
\propto&E_{P'}\left(Q_{\widehat{n}^*(\lambda)}\right) - \frac{\widehat{n}^*(\lambda)}{\Lambda_{\widehat{n}^*(\lambda)}}\left\{1 + \lambda\left[1 - \frac{\phi}{1-\phi\lambda}\frac{1}{\Lambda_{\widehat{n}^*(\lambda)}}\left(1 - \lambda - \frac{C''(q_{\widehat{n}^*(\lambda)})}{P'\left(Q_{\widehat{n}^*(\lambda)}\right)}\right)\right]\right\},
\end{aligned}
$$

and the result follows from global concavity of total surplus in n.

Part (i) follows easily.

For part (ii) notice that for $\phi\left(\widehat{n}^*(\lambda),\lambda\right) = 1$, if marginal costs are constant, then

$$
\left.\frac{d\,\mathrm{TS}(\boldsymbol{q}_n)}{dn}\right|_{n=\widehat{n}^*(\lambda)} \propto E_{P'}\left(Q_{\widehat{n}^*(\lambda)}\right) - \frac{\widehat{n}^*(\lambda)}{\Lambda_{\widehat{n}^*(\lambda)}}\left[1 + \lambda\left(1 - \frac{1}{\Lambda_{\widehat{n}^*(\lambda)}}\right)\right].
$$

All else constant (also $\widehat{n}^*(\lambda)$ held constant), the expression above is nonincreasing in λ, since

$$
\begin{aligned}
\frac{\partial\left(\frac{\widehat{n}^*(\lambda)}{\Lambda_{\widehat{n}^*(\lambda)}}\left[1 + \lambda\left(1 - \frac{1}{\Lambda_{\widehat{n}^*(\lambda)}}\right)\right]\right)}{\partial\lambda} &\propto -\frac{\widehat{n}^* - 1}{\Lambda^2_{\widehat{n}^*}}\left[1 + \lambda\left(1 - \frac{1}{\Lambda_{\widehat{n}^*}}\right)\right] + \left(1 - \frac{1}{\Lambda_{\widehat{n}^*}} + \frac{\lambda(\widehat{n}^* - 1)}{\Lambda^2_{\widehat{n}^*}}\right)\frac{1}{\Lambda_{\widehat{n}^*}} \\
&\propto -(\widehat{n}^* - 1)\left[\Lambda_{\widehat{n}^*} + \lambda\left(\Lambda_{\widehat{n}^*} - 1\right)\right] + \Lambda_{\widehat{n}^*}(\Lambda_{\widehat{n}^*} - 1) + \lambda(\widehat{n}^* - 1) \\
&= -(\widehat{n}^* - 1)\Lambda_{\widehat{n}^*} + \lambda(\widehat{n}^* - 1) = \Lambda_{\widehat{n}^*}\left(3 - \widehat{n}^*\right) - 2 \leq 0,
\end{aligned}
$$

where the inequality follows given $\widehat{n}^* \geq 3/2$. We conclude that a sufficient condition for excessive entry under constant marginal costs is derived if we set $\lambda = 1$ (but not in $\widehat{n}^*(\lambda)$). Also, the necessary and sufficient condition for excessive entry is relaxed as C'' decreases, and (ii) follows. **Q.E.D.**

Proof of Corollary 11.1 Proposition 11 gives that $\widehat{n}^*(\lambda) \overset{(<)}{>} \widehat{n}^o(\lambda)$ if and only if

$$
0 \overset{(>)}{<} H_n^{-1}\left[1 + \lambda\left(1 - \frac{\phi\left(n,\lambda\right)}{1 - \lambda\phi\left(n,\lambda\right)}\frac{1 - \lambda + c/b}{\Lambda_n}\right)\right]\Bigg|_{n=\widehat{n}^*(\lambda)}.
$$

and the result follows. **Q.E.D.**

B Online appendix

B.1 Aggregate industry profits as function of the number of firms

Define

$$\mu_n := 1 - \frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\frac{1-H_n}{\eta(Q_n)-H_n}.$$

Proposition 12. Let $\lambda < 1$. Then, the following statements hold:

(i) if $\mu_n \leq 0$, aggregate industry profits are decreasing in n,

(ii) if $\mu_n > 0$, aggregate industry profits are decreasing (resp. increasing) in n if $E_C(q_n) \overset{(>)}{<} \mu_n^{-1}$,

(iii) if $C''(q) < 0$ for every $Q \in [0,Q_n]$, then monopoly maximizes aggregate industry profits, $\Pi(1,\lambda) > n\Pi(n,\lambda)$.

Remark 1: If $\Delta < 0$, then $\partial Q_n/\partial n > 0$, so $\mu_n < 1$, and thus, aggregate industry profits are decreasing in n if $E_C(q_n) \leq 1$. If for example $C'' < 0$ globally (consistent with $\Delta < 0$), then indeed $E_C(q_n) < 1$.

Remark 2: If $\lambda = 1$ and $C''(q) > 0$ for every $q \in [0,q_n]$, aggregate industry profits are increasing in n.

Consider the extreme case of $\lambda = 1$ and notice the following. Condition $\Delta > 0$ requires decreasing returns to scale, so that aggregate gross profits increase with n (*i.e.*, $n\Pi(n,1) > (n-1)\Pi(n-1,1)$ for any n) due to savings in variable costs as production is distributed across more firms, even though the total quantity increases (see Proposition 3), and thus price decreases with the number of firms. Intuitively, aggregate gross profits being increasing in n for $\lambda = 1$ is tied to uniqueness of the (symmetric) equilibrium in the pricing stage. Since firms jointly maximize aggregate profits, the latter should increase with n for firms to strictly prefer to spread production evenly. On the other hand, under constant returns to scale aggregate profits are constant in n; increasing the number of firms simply increases the ways in which the firms can jointly produce the fixed level of total output that maximizes joint profits.[49] Last, under increasing returns to scale it is an equilibrium for all production to be concentrated in a single firm, so again aggregate profits are constant in n.

Claim 3. Under linear demand and quadratic costs

$$\frac{\partial\left[n\Pi\left(n,\lambda\right)\right]}{\partial n} = \frac{c}{2bn} - \frac{b(1-\lambda)+c}{b(n+\Lambda_n)+c}\left(1-H_n\right) \quad \text{with} \quad \frac{\partial^2\left[n\Pi\left(n,\lambda\right)\right]}{\partial\lambda\partial n} > 0.$$

(i) for $\lambda = 0$, $\operatorname{sgn}\left\{\partial\left[n\Pi\left(n,0\right)\right]/\partial n\right\} = \operatorname{sgn}\left\{c - b(n-1)\right\}$,

(ii) for $\lambda = 1$, $\partial\left[n\Pi\left(n,1\right)\right]/\partial n > 0$,

[49] As argued already, in this case the are infinitely many equilibria of the pricing stage, all with the same total quantity.

(iii) if $c > b(n-1)$, then $\partial\left[n\Pi\left(n,\lambda\right)\right]/\partial n \overset{(<)}{>} 0$ for every $\lambda \in [0,1]$,

(iv) if $c < b(n-1)$, then there exists $\lambda^* \in (0,1)$ such that $\partial\left[n\Pi\left(n,\lambda\right)\right]/\partial n \overset{(<)}{>} 0$ if and only if $\lambda \overset{(<)}{>} \lambda^*$.

In the decreasing returns to scale case of Claim 3 we see that λ and n are complements in increasing aggregate industry profits. Particularly, for λ high enough aggregate industry profits are increasing in the number of firms. This is because with λ high, entry does not reduce the price as much (see point (iii-b) of Proposition 3), so the cost-saving effect of entry under decreasing returns to scale dominates.

B.2 Derivation of Numerical Results

Under CESL demand and constant returns to scale, given Claim 2 we find that

$$\Pi(n,\lambda) = \begin{cases} \frac{1}{n}\left[a + b\left[\frac{b(1-H_n(E-1))}{c-a}\right]^{\frac{1-E}{E-1}} - c\right]\left[\frac{b(1-H_n(E-1))}{c-a}\right]^{\frac{1}{E-1}} & \text{if } E \in (1,2) \text{ and } c > a \\ \frac{1}{n}\left[a - b\ln\left(e^{\frac{a-c-bH_n}{b}}\right) - c\right]e^{\frac{a-c-bH_n}{b}} & \text{if } E = 1 \text{ and } a > b+c \\ \frac{1}{n}\left[a - b\left[\frac{a-c}{b(1+H_n(1-E))}\right]^{\frac{1-E}{1-E}} - c\right]\left[\frac{a-c}{b(1+H_n(1-E))}\right]^{\frac{1}{1-E}} & \text{if } E < 1 \text{ and } a > c, \end{cases}$$

$$= \begin{cases} \frac{H_n(E-1)b^{\frac{1}{E-1}}}{n}\left[\frac{1-H_n(E-1)}{c-a}\right]^{\frac{2-E}{E-1}} & \text{if } E \in (1,2) \text{ and } c > a \\ \frac{bH_n}{n}e^{\frac{a-c-bH_n}{b}} & \text{if } E = 1 \text{ and } a > b+c \\ \frac{H_n(1-E)}{nb^{\frac{1}{1-E}}}\left[\frac{a-c}{1+H_n(1-E)}\right]^{\frac{2-E}{1-E}} & \text{if } E < 1 \text{ and } a > c, \end{cases}$$

Derivation of Numerical Result 1 Parameters a, b and c only affect the magnitudes of $d\widehat{n}^*(\lambda)/d\lambda$ and $dQ_{\widehat{n}^*(\lambda)}/d\lambda$, and not their signs. The result then obtains in a way analogous to the one described in the Derivation of Numerical Result 2. ■

Derivation of Numerical Result 2 It is easy to see that the signs of derivatives of $\Psi(n,\lambda)$ are independent of a, b and c. Thus, we can wlog set (i) $a = b = 1$ and $c = 2$ for the case $E \in (1,2)$, and (ii) $a = 2$, $b = c = 1$ for the case $E < 1$.

For $E > 1$ we run the following R code:

```
# load packages #
library(Deriv)
library(optimx)

# define functions #
Lambda = function(n,lambda) {1 + lambda*(n-1)}
H = function(n,lambda) {(1 + lambda*(n-1))/n}

Pi = function(n,lambda,E,a,b,c) { H(n,lambda)*(E-1)*b^(1/(E-1))*
( (1-H(n,lambda)*(E-1))/(c-a) )^((2-E)/(E-1))/n }
Psi = function(n,lambda,E,a,b,c) {  Pi(n,lambda,E,a,b,c)-lambda*(n-1)*
( Pi(n-1,lambda,E,a,b,c)-Pi(n,lambda,E,a,b,c)) }

# symbolically differentiate Psi #
```

```
Deriv_wrt_n_Psi = Deriv(Psi,"n")
Deriv_wrt_nlambda_Psi = Deriv(Deriv_wrt_n_Psi,"lambda")

# define function that creates grid of starting points for optimization #
grid = function(density_n,min_n,max_n,density_l,min_l,max_l,
density_E,min_E,max_E) {
  output = matrix(nrow = (density_n+1)*(density_l+1)*(density_E+1), ncol = 3)
  row_number = 1
  for (i in seq(from = min_n, to = max_n, by = (max_n-min_n)/density_n)) {
    for (j in seq(from = min_l, to = max_l, by = (max_l-min_l)/density_l)) {
      for (k in seq(from = min_E, to = max_E, by = (max_E-min_E)/density_E)) {
        output[row_number,] = c(i,j,k)
        row_number = row_number + 1
      }
    }
  }
  return(output)
}

# minimize cross derivative of Psi from multiple starting points #

minima =  multistart(parmat = grid(15,2,7,15,0,1,30,1.001,1.7),
      fn = function(x) {Deriv_wrt_nlambda_Psi(x[1],x[2],x[3],1,1,2)},
      method = c("L-BFGS-B"), lower = c(2,0,1.001), upper = c(7,1,1.7))
```

The code returns that

$$\min_{(n,\lambda,E)\in[2,7]\times[0,1]\times[1.001,1.7]} \frac{\Psi(n,\lambda)}{\partial\lambda\partial n} \approx 2.31\cdot 10^{-6} > 0,$$

which is reached for $n = 7$, $\lambda = 0$ and $E = 1.001$.

In the case of $E < 1$ we similarly find that

$$\min_{(n,\lambda,E)\in[2,8]\times[0,1]\times[-1000,0.999]} \frac{\Psi(n,\lambda)}{\partial\lambda\partial n} \approx 1.11\cdot 10^{-7} > 0,$$

which is reached for $n = 8$, $\lambda = 0$ and $E = 0.999$. In additional simulations, allowing E to be even lower than -1000 does not change the result. ■

B.3 Free entry under pre-entry overlapping ownership and the presence of maverick firms

This section presents further details on the model of free entry under pre-entry overlapping ownership and the presence of maverick firms that is studied in the paper.

After the entry and production decisions of the commonly-owned firms, a competitive fringe makes production decisions. Namely, there is a set $\mathcal{F}_m$ of infinitesimal firms. Each firm $i \in \mathcal{F}_m$ chooses to either be inactive or produce one (infinitesimal) unit of the good at cost $\chi(i)$.[50] Thus, the aggregate supply function by the maverick firms in the third stage $S : \mathbb{R}_+ \to \mathbb{R}_+$ is given by $S(p) := \int_{i\in\mathcal{F}_m} \mathbb{1}(\chi(i) \le p)\, di$. As argued in the paper, in

[50] This cost can be thought to include any applicable entry costs. Since maverick firms are infinitesimal and each supply an infinitesimal quantity, their entry cost is also infinitesimal.

the second stage the oligopolists are essentially faced with inverse demand $\widetilde{P} : \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$\widetilde{P}(Q) = \begin{cases} P\left(Q + \omega^{-1}(Q)\right) \in \left(\underline{p}, P(Q)\right) & \text{if } P(Q) > \underline{p} \\ P(Q) & \text{if } P(Q) \leq \underline{p} \end{cases}$$

where $\omega : \mathbb{R}_{++} \to \mathbb{R}_+$ is given by $\omega(y) \coloneqq P^{-1} \circ S^{-1}(y) - y$. The slope of $\widetilde{P}$ is then given by

$$\widetilde{P}'(Q) = \begin{cases} \dfrac{P'\left(Q + \omega^{-1}(Q)\right)}{1 - S'\left(P\left(Q + \omega^{-1}(Q)\right)\right) P'\left(Q + \omega^{-1}(Q)\right)} \in \left(P'\left(Q + \omega^{-1}(Q)\right), 0\right) & \text{if } P(Q) > \underline{p} \\ P'(Q) & \text{if } P(Q) < \underline{p}. \end{cases}$$

Provided $\widetilde{P}(Q) \geq \underline{p}$ or equivalently $P(Q) \geq \underline{p}$,[51] total surplus now includes the maverick firms' surplus and is thus given by

$$\begin{aligned} \widetilde{\mathrm{TS}}(\boldsymbol{q}) \coloneqq & \overbrace{\int_0^{Q+S\left(\widetilde{P}(Q)\right)} \left(P(X) - \widetilde{P}(Q)\right) dX}^{\text{consumer surplus}} + \overbrace{\int_{\underline{p}}^{\widetilde{P}(Q)} S(p)dp}^{\text{maverick firms' surplus}} + \overbrace{\widetilde{P}(Q)Q - \sum_{i=1}^{n} C(q_i) - nf}^{\text{commonly-owned firms' profits}} \\ = & \,\mathrm{TS}(\boldsymbol{q}) + \underbrace{\int_Q^{Q+S\left(\widetilde{P}(Q)\right)} P(X)dX - S\left(\widetilde{P}(Q)\right)\widetilde{P}(Q)}_{\geq 0;\ \text{consumer surplus "due to" maverick firms' production}} + \underbrace{\int_{\underline{p}}^{\widetilde{P}(Q)} S(p)dp}_{\geq 0;\ \text{maverick firms' surplus}} \end{aligned}$$

where $\boldsymbol{q}$ still the quantity profile of the oligopolists and $\mathrm{TS}(\boldsymbol{q}) \equiv \int_0^Q P(X)dX - \sum_{i=1}^n C(q_i) - nf$ the total surplus without maverick firms. For any fixed quantity profile of the oligopolists, total surplus is higher when the maverick firms are present (and produce) compared to when they are not. We have then that

$$\begin{aligned} \frac{d\widetilde{\mathrm{TS}}_n}{dn} = & P(Q_n)\left(n\frac{\partial q_n}{\partial n} + q_n\right) - C(q_n) - nC'(q_n)\frac{\partial q_n}{\partial n} - f \\ & + \begin{bmatrix} \left(1 + S'(\widetilde{P}(Q_n))\widetilde{P}'(Q_n)\right) P\left(Q_n + S(\widetilde{P}(Q_n))\right) - P(Q_n) \\ -S'(\widetilde{P}(Q_n))\widetilde{P}'(Q_n)\widetilde{P}(Q_n) - S(\widetilde{P}(Q_n))\widetilde{P}'(Q_n) + S(\widetilde{P}(Q_n))\widetilde{P}'(Q_n) \end{bmatrix} \frac{\partial Q_n}{\partial n} \\ = & \widetilde{P}(Q_n)\left(n\frac{\partial q_n}{\partial n} + q_n\right) - C(q_n) - nC'(q_n)\frac{\partial q_n}{\partial n} - f + \left(P(Q_n) - \widetilde{P}(Q_n)\right)\frac{\partial Q_n}{\partial n} \\ & + \left[\left(1 + S'(\widetilde{P}(Q_n))\widetilde{P}'(Q_n)\right)\widetilde{P}(Q_n) - P(Q_n) - S'(\widetilde{P}(Q_n))\widetilde{P}'(Q_n)\widetilde{P}(Q_n)\right]\frac{\partial Q_n}{\partial n} \\ = & \widetilde{\Pi}(n,\lambda) - f - \left(1 + \lambda(n-1)\right) Q_n \widetilde{P}'(Q_n)\frac{\partial q_n}{\partial n}, \end{aligned}$$

where $\widetilde{\mathrm{TS}}_n$ is the pricing stage equilibrium total surplus when n commonly-owned firms enter, $\widetilde{\Pi}(n,\lambda) \coloneqq \widetilde{P}(Q_n)q_n - C(q_n)$, and Q_n, q_n are still the quantities produced by the commonly-owned firms in the pricing stage equilibrium where n of them enter.

[51] Otherwise, wherever $\widetilde{P}(Q)$ substitute $\underline{p}$, and the equation reduces to $\widetilde{\mathrm{TS}}(\boldsymbol{q}) = \mathrm{TS}(\boldsymbol{q})$.

B.4 Free entry with pre-entry overlapping ownership: a more tractable framework

In this section we make the model of free entry with pre-entry overlapping ownership more tractable by ignoring the integer constraint on n. The way we do this is *not* just by letting (2) hold with equality. Instead, now each "infinitesimal" firm considers whether to enter or not examining a differential version of (3).[52] Consider firm i of "size" $\varepsilon > 0$ and let $n \in \mathbb{R}_+$ be the number of other firms entering. Firm i's payoff if it enters is $(\varepsilon + \lambda n)\,(\Pi(n+\varepsilon,\lambda) - f)$, while if it does not, it is $\lambda n\,(\Pi(n,\lambda) - f)$. The difference is

$$\varepsilon\Pi(n+\varepsilon,\lambda) + \lambda n\,[\Pi(n+\varepsilon,\lambda) - \Pi(n,\lambda)] - \varepsilon f.$$

Notice that for $\varepsilon = 1$ we recover the case with the integer constraint. Dividing this expression by ε and letting $\varepsilon \to 0$ gives

$$\Pi(n,\lambda) + \lambda n \frac{\partial \Pi(n,\lambda)}{\partial n} - f.$$

Therefore, $\boldsymbol{q}_n$ is a free entry equilibrium if

$$\overbrace{\Pi(n,\lambda)}^{\substack{\text{own profit}\\ \text{from entry}}} + \lambda \overbrace{n\frac{\partial \Pi(n,\lambda)}{\partial n}}^{\substack{\text{entry externality}\\ \text{on other firms}}} = \overbrace{f}^{\substack{\text{entry}\\ \text{cost}}} \quad \text{and} \tag{8}$$

$$(1+\lambda)\frac{\partial \Pi(n,\lambda)}{\partial n} + \lambda n \frac{\partial^2 \Pi(n,\lambda)}{(\partial n)^2} < 0. \tag{9}$$

Naturally, we only consider the free entry equilibrium and planner's solution with $n \in \mathbb{R}_+$; we denote the number of firms in the two solutions by $n^*(\lambda)$ and $n^o(\lambda)$, respectively. The entry externality is now measured by $n\partial\Pi(n,\lambda)/\partial n$. (8) says that the marginal firm entering is exactly indifferent between entering or not. (9) guarantees that an extra infinitesimal firm does not want to enter, and given that $\partial\Pi(n,\lambda)/\partial n < 0$, can equivalently be written as

$$1 + \lambda - \lambda E_{\partial\Pi/\partial n, n}(n,\lambda) > 0, \qquad \text{where} \quad E_{\partial\Pi/\partial n, n}(n,\lambda) := -\frac{\frac{\partial^2 \Pi(n,\lambda)}{(\partial n)^2}}{\frac{\partial \Pi(n,\lambda)}{\partial n}} n$$

is the elasticity of the slope of individual profit with respect to n. Also, given that $\partial\Pi(n,\lambda)/\partial n < 0$, $\lambda > 0$ implies through (8) that the entering firms make positive profits in equilibrium in contrast to the zero-profit condition characterizing the equilibirum of free-entry Cournot without or post-entry overlapping ownership when we ignore the integer constraint. For $\lambda = 0$, (8) reduces to the standard zero profit condition.

Provided that (9) holds for every n, the (unique) equilibrium level of entry $n^*(\lambda)$ is pinned down by

$$\Pi\left(n^*(\lambda),\lambda\right) + \lambda n^*(\lambda) \left.\frac{\partial \Pi(n,\lambda)}{\partial n}\right|_{n=n^*(\lambda)} = f.$$

[52] Of course, the firm is infinitesimal only for the purpose of the algebra. The firm understands the (marginal) effect of its entry on market outcomes, and in the pricing stage firms still complete á la Cournot but with the symmetric equilibrium solution extended to $n \in \mathbb{R}_{++}$.

Assume that $\Pi(1,\lambda) + \lambda\partial\Pi(n,\lambda)/\partial n|_{n=1} > f$ so that more than 1 firm enters, and

$$\lim_{n\to\infty} [\Pi(n,\lambda) + \lambda\partial\Pi(n,\lambda)/\partial n] < f.$$

Proposition 13 guarantees that the left-hand side of (8) is decreasing in n, thus ensuring existence of a unique equilibrium.

Proposition 13. If for every n such that $\Pi(n,\lambda) + \lambda\partial\Pi(n,\lambda)/\partial n \geq f$ it holds that $1 + \lambda - \lambda E_{\partial\Pi/\partial n,n}(n,\lambda) > 0$, then a unique Cournot equilibrium with free entry exists.

Proposition 14. Fix a value for λ and consider the unique symmetric Cournot equilibrium with free entry, where $\partial\Pi(n,\lambda)/\partial n|_{n=n^*(\lambda)} < 0$ and $1 + \lambda - \lambda E_{\partial\Pi/\partial n,n}(n^*(\lambda),\lambda) > 0$.

(i) The number of firms locally changes with λ with direction given by[53]

$$\operatorname{sgn}\left\{\frac{dn^*(\lambda)}{d\lambda}\right\} = \operatorname{sgn}\left\{\overbrace{E_{\partial\Pi/\partial n,\lambda}\left(n^*(\lambda),\lambda\right)}^{\substack{\text{change in magnitude}\\ \text{of entry externality}}} + \overbrace{\frac{1}{\lambda}\frac{E_{\Pi,\lambda}\left(n^*(\lambda),\lambda\right)}{E_{\Pi,n}\left(n^*(\lambda),\lambda\right)}}^{\substack{\text{increase in own}\\ \text{profit from entry}}} - \overbrace{1}^{\substack{\text{increase in internalization}\\ \text{of entry externality}}}\right\}.$$

(ii) The total quantity changes with λ with direction given by

$$\operatorname{sgn}\left\{\frac{dQ_{n^*(\lambda)}}{d\lambda}\right\} = \operatorname{sgn}\left\{\frac{E_{\partial\Pi/\partial n,\lambda}\left(n^*(\lambda),\lambda\right) + \frac{1}{\lambda}\frac{E_{\Pi,\lambda}(n^*(\lambda),\lambda)}{E_{\Pi,n}(n^*(\lambda),\lambda)} - 1}{1 + \lambda - \lambda E_{\partial\Pi/\partial n,n}\left(n^*(\lambda),\lambda\right)}\frac{\Delta(Q_{n^*(\lambda)}, (n-1)\, q_{n^*(\lambda)})}{n^*(\lambda) - 1} - 1\right\}$$

where

$$E_{\partial\Pi/\partial n,\lambda}(n,\lambda) := -\frac{\frac{\partial^2\Pi(n,\lambda)}{\partial\lambda\partial n}}{\frac{\partial\Pi(n,\lambda)}{\partial n}}\lambda, \quad E_{\Pi,n}(n,\lambda) := -\frac{\frac{\partial\Pi(n,\lambda)}{\partial n}}{\Pi(n,\lambda)}n > 0, \quad E_{\Pi,\lambda}(n,\lambda) := \frac{\frac{\partial\Pi(n,\lambda)}{\partial\lambda}}{\Pi(n,\lambda)}\lambda > 0$$

are, respectively, the elasticity with respect to λ of the slope of individual profit with respect to n, the elasticity of profit with respect to n, and the elasticity of profit with respect to λ.

Corollary 14.1. In addition to the assumptions of Proposition 14, assume constant returns to scale. Then

$$\operatorname{sgn}\left\{\frac{dn^*(\lambda)}{d\lambda}\right\} = \operatorname{sgn}\left\{\begin{aligned}&\left(n - 1 + 2\lambda - \frac{\Lambda_n\left(2n - \Lambda_n E_{P'}\left(Q_n\right)\right)}{n - \Lambda_n}\right)\left(n + \Lambda_n - \Lambda_n E_{P'}\left(Q_n\right)\right)\\ &\qquad + \lambda(2n - \Lambda_n)\left(2 - E_{P'}\left(Q_n\right)\right) - \left.\frac{\lambda\Lambda_n(n - \Lambda_n)Q_n E'_{P'}\left(Q_n\right)}{n + \Lambda_n - \Lambda_n E_{P'}\left(Q_n\right)}\right|_{n=n^*(\lambda)}\end{aligned}\right\},$$

(i) for $\lambda = 0$, given $E_{P'}(Q_{n^*(0)}) < 2$, $dn^*(\lambda)/d\lambda \overset{(>)}{<} 0$ if and only if $n^*(0) \overset{(>)}{<} 2 + \sqrt{3 - E_{P'}(Q_{n^*(0)})}$.

(ii) If $E'_{P'}(Q_{n^*(\lambda)}) \leq 0$ and $E_{P'}\left(Q_{n^*(\lambda)}\right) > \left[2n - (H_n^{-1} - 1)\,(n - 1 + 2\lambda)\right]/\Lambda_n$, then $dn^*(\lambda)/d\lambda > 0$.

[53] For $\lambda = 0$ cancel the λ in the second term with the one in $E_{\Pi,\lambda}(n,\lambda)$.

(iii) If $E_{P'}\left(Q_{n^*(\lambda)}\right) < \max\left\{2, \left[2n - \left(H_n^{-1} - 1\right)(n+1+2\lambda)\right]/\Lambda_n\right\}$ and $E'_{P'}(Q_{n^*(\lambda)}) \geq 0$, then $dn^*(\lambda)/d\lambda < 0$.

(iv) If $\lim_{\lambda\to 1^-} E_{P'}(Q_{n^*(\lambda)}) < 2$ (and $E'_{P'}$ bounded), then $dn^*(\lambda)/d\lambda < 0$ for λ close to 1.

(v) Under linear demand

$$\operatorname{sgn}\left\{\frac{dn^*(\lambda)}{d\lambda}\right\} = \operatorname{sgn}\left\{\left(n - 1 + 2\lambda - \frac{2n\Lambda_n}{n-\Lambda_n}\right)(n+\Lambda_n) + 2\lambda(2n-\Lambda_n)\Big|_{n=n^*(\lambda)}\right\}.$$

(vi) If $E_{\partial\Pi/\partial n,n}(n^*(\lambda),\lambda) \leq 2$, $E_{P'}(Q_{n^*(\lambda)}) < 2$, $E'_{P'}(Q_{n^*(\lambda)}) \geq 0$ and $n^*(\lambda) \geq 2$, then the total quantity decreases with λ.

Claim 4. Under linear demand and constant marginal costs $E_{\partial\Pi/\partial n,n}(n,\lambda) \leq 2$ for every $\lambda \in [0,1]$ and $n \geq 1$.

Corollary 14.1 shows that under reasonable assumptions overlapping ownership can spur entry. Proposition 15 shows that the effect of overlapping ownership on the magnitude of the entry externality is ambiguous in our setting. Proposition 16 shows that with pre-entry overlapping ownership both possibilities of excessive and insufficient entry are possible.

Proposition 15. Assume that $\Delta > 0$ and at $n = n^*(\lambda)$, $E_{P'}(Q_n) < 1 + \frac{\Lambda_n}{n-\Lambda_n} \Big/ \left(E_{Q_n,n}(n,\lambda) + \frac{\Lambda_n}{n-\Lambda_n}\right)$. The direction of the change (due to the change in λ) in the magnitude of the entry externality, $\operatorname{sgn}\left\{E_{\partial\Pi/\partial n,\lambda}(n,\lambda)\right\}$, is given by

$$\operatorname{sgn}\left\{\overbrace{\frac{E_{Q_n,\lambda}(n,\lambda)}{E_{Q_n,n}(n,\lambda)}}^{+} - \frac{\overbrace{E_{\partial Q_n/\partial n,\lambda}(n,\lambda)}^{-\text{ under Prop. 3(ii)}} + \overbrace{\frac{n-1}{n-\Lambda_n}\left[\left(E_{Q_n,n}(n,\lambda)\right)^{-1} - 1\right]}^{+}}{\left[E_{Q_n,n}(n,\lambda) + \frac{\Lambda_n}{n-\Lambda_n}\right]\left[1 + \frac{\frac{\Lambda_n}{n-\Lambda_n}}{E_{Q_n,n}(n,\lambda)+\frac{\Lambda_n}{n-\Lambda_n}} - E_{P'}(Q_n)\right]}\Bigg|_{n=n^*(\lambda)}\right\}$$

evaluated at $n = n^*(\lambda)$, where

$$E_{\partial Q_n/\partial n,\lambda}(n,\lambda) := \frac{\frac{\partial^2 Q_n}{\partial\lambda\partial n}}{\frac{\partial Q_n}{\partial n}}\lambda, \quad E_{Q_n,n}(n,\lambda) := \frac{\partial Q_n}{\partial n}\frac{n}{Q_n} > 0, \quad E_{Q_n,\lambda}(n,\lambda) := -\frac{\partial Q_n}{\partial\lambda}\frac{\lambda}{Q_n} > 0.$$

Under constant marginal costs

$$\operatorname{sgn}\left\{E_{\partial\Pi/\partial n,\lambda}(n,\lambda)\right\} = \operatorname{sgn}\left\{\begin{array}{r} 2\Lambda_n^2\left(E_{P'}(Q_n)\right)^2 + \left[n\Lambda_n(n-\Lambda_n-1) - 2n^2 - \Lambda_n^2\right]E_{P'}(Q_n) \\ -n(n-\Lambda_n)(n+\Lambda_n-6) - \left.\dfrac{\Lambda_n(n-\Lambda_n)^2 Q_n E'_{P'}(Q_n)}{n+\Lambda_n-\Lambda_n E_{P'}(Q_n)}\right|_{n=n^*(\lambda)} \end{array}\right\},$$

which can be negative or positive. For linear demand, $\operatorname{sgn}\left\{E_{\partial\Pi/\partial n,\lambda}(n,\lambda)\right\} = \operatorname{sgn}\left\{6 - (n+\Lambda_n)\right\}$.

Proposition 16. Consider the Cournot model with free entry and pre-entry overlapping ownership. Assume that $\mathrm{TS}(\boldsymbol{q}_n)$ is globally concave in n, and $\lambda < 1$. Then in equilibrium there is excessive (insufficient) entry if and only if

$$E_{P'}\left(Q_{n^*(\lambda)}\right) \overset{(>)}{<} \frac{n^*(\lambda)}{\Lambda_{n^*(\lambda)}}\left\{1 + \lambda\left(1 - \frac{\Delta(Q_{n^*(\lambda)}, (n-1)\,q_{n^*(\lambda)})}{(1-\lambda)\Lambda_{n^*(\lambda)}}\right)\right\}.$$

The results of this section very closely resemble the ones we obtain under the integer constraint. Therefore, the gain in tractability from dropping the constraint as described above comes at a minimal cost.

C Proofs of additional results

Where clear we may simplify notation, for example omitting the subscript n,λ for equilibrium objects. We may also write for example Q_n instead of $Q_{n^*(\lambda)}$, n instead of $n^*(\lambda)$. Also, we write $\Pi_n(n,\lambda) \equiv \partial\Pi(n,\lambda)/\partial n$, $\Pi_\lambda(n,\lambda) \equiv \partial\Pi(n,\lambda)/\partial\lambda$, $\Pi_{n\lambda}(n,\lambda) \equiv \partial^2\Pi(n,\lambda)/(\partial n\partial\lambda)$, $\Pi_{nn}(n,\lambda) \equiv \partial^2\Pi(n,\lambda)/(\partial n)^2$.

Proof of Proposition 12 (i-ii) Given what we see in the proof of Proposition 4, for aggregate industry profits we have that

$$\begin{aligned}
\frac{\partial\left[n\Pi\left(n,\lambda\right)\right]}{\partial n} &= P(Q_n)\frac{Q_n}{n} - C(q_n) + nP'(Q_n)\left(\frac{Q_n}{n}\right)^2\left[\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\left(1-H_n\right)+H_n\right] \\
&= nP'(Q_n)\left(\frac{Q_n}{n}\right)^2\left[\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\left(1-H_n\right) - \eta(Q_n) - \frac{C(q_n)}{P'(Q_n)}\frac{n}{Q_n^2} + H_n\right] \\
&\propto -\left[\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\left(1-H_n\right) - \eta(Q_n) + \eta(Q_n)\frac{C(q_n)}{P(Q_n)}\frac{n}{Q_n} + H_n\right] \\
&= -\left[\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\left(1-H_n\right) - \eta(Q_n)\frac{P(Q_n) - C'(q_n) + C'(q_n)\frac{E_C(q_n)-1}{E_C(q_n)}}{P(Q_n)} + H_n\right] \\
&\overset{(1)}{=} -\left[\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\left(1-H_n\right) - H_n - \eta(Q_n)\frac{C'(q_n)}{P(Q_n)}\frac{E_C(q_n)-1}{E_C(q_n)} + H_n\right] \\
&\overset{(1)}{=} -\left[\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\left(1-H_n\right) - \eta(Q_n)\left(1-\frac{H_n}{\eta(Q_n)}\right)\frac{E_C(q_n)-1}{E_C(q_n)}\right] \\
&\propto E_C(q_n)\left(1 - \frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\frac{1-H_n}{\eta(Q_n)-H_n}\right) - 1,
\end{aligned}$$

where $H_n < \eta(Q_n)$ from the pricing formula (1).

(iii) We have that

$$n\Pi(n,\lambda) \equiv P(Q_n)Q_n - nC(q_n) \overset{C''<0}{<} P(Q_n)Q_n - C(Q_n) \leq P(q_1)q_1 - C(q_1) = \Pi(1,\lambda),$$

where the last inequality follows by definition of q_1 being the monopolist's optimal quantity.

To see why Remark 2 holds notice that for $\lambda = 1$

$$\begin{aligned}
\frac{\partial\left[n\Pi\left(n,\lambda\right)\right]}{\partial n} &= P(Q_n)\frac{Q_n}{n} - C(q_n) + nP'(Q_n)\left(\frac{Q_n}{n}\right)^2 \\
&\overset{C''>0}{>} P(Q_n)\frac{Q_n}{n} - C'(q_n)q_n + P'(Q_n)\frac{Q_n^2}{n} \propto \frac{P(Q_n) - C'(q_n)}{P(Q_n)} - \frac{1}{\eta(Q_n)} \overset{(1)}{=} 0.
\end{aligned}$$

Q.E.D.

Proof of Claim 3 From Claim 2 it follows that

$$(Q_n, P(Q_n)) = \left(\frac{a}{b\,(1+H_n)+c/n}, a\left(1-\frac{b}{b\,(1+H_n)+c/n}\right)\right)$$

and

$$\begin{aligned}\frac{\partial\left[n\Pi\left(n,\lambda\right)\right]}{\partial n} =& P(Q_n)\frac{Q_n}{n} - C(q_n) + nP'(Q_n)\left(\frac{Q_n}{n}\right)^2\left[\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\left(1-H_n\right)+H_n\right]\\ =& \left(1-\frac{b}{b\,(1+H_n)+c/n}\right)\frac{a^2}{n\left(b\,(1+H_n)+c/n\right)} - \frac{c}{2n^2}\left(\frac{a}{b\,(1+H_n)+c/n}\right)^2\\ &-\frac{b}{n}\left(\frac{a}{b\,(1+H_n)+c/n}\right)^2\left[\frac{1-\lambda+c/b}{n+\Lambda+c/b}\left(1-H_n\right)+H_n\right]\\ \propto& bH_n + \frac{c}{2n} - b\left[\frac{1-\lambda+c/b}{n+\Lambda+c/b}\left(1-H_n\right)+H_n\right] = \frac{c}{2n} - \frac{b(1-\lambda)+c}{n+\Lambda+c/b}\left(1-H_n\right),\end{aligned}$$

and the rest follow. **Q.E.D.**

Proof of Proposition 13 The LHS of (8) is globally decreasing, so (8) has a unique solution given that for $n=0$ the LHS is at least as high as f and for $n\to\infty$ it is lower than f. Also, (9) is immediately satisfied. **Q.E.D.**

Proof of Proposition 14 and Corollary 14.1 Totally differentiating (8) with respect to λ we get

$$\begin{aligned}&\Pi_n(n^*(\lambda),\lambda)\left(n^*(\lambda)+(1+\lambda)\frac{dn^*(\lambda)}{d\lambda}\right)+\Pi_\lambda(n^*(\lambda),\lambda)\\ &+\lambda n^*(\lambda)\left(\Pi_{nn}(n^*(\lambda),\lambda)\frac{dn^*(\lambda)}{d\lambda}+\Pi_{n\lambda}(n^*(\lambda),\lambda)\right)=0,\end{aligned}$$

which gives

$$\begin{aligned}\frac{dn^*(\lambda)}{d\lambda} &= -\frac{n^*(\lambda)\left(\Pi_n(n^*(\lambda),\lambda)+\lambda\Pi_{n\lambda}(n^*(\lambda),\lambda)\right)+\Pi_\lambda(n^*(\lambda),\lambda)}{(1+\lambda)\Pi_n(n^*(\lambda),\lambda)+\lambda n^*(\lambda)\Pi_{nn}(n^*(\lambda),\lambda)}\\ &= -\frac{n^*(\lambda)\left(1+\frac{\Pi_\lambda(n^*(\lambda),\lambda)}{\Pi(n^*(\lambda),\lambda)}\left(\frac{\Pi_n(n^*(\lambda),\lambda)}{\Pi(n^*(\lambda),\lambda)}n^*(\lambda)\right)^{-1}-E_{\partial\Pi/\partial n,\lambda}\left(n^*(\lambda),\lambda\right)\right)}{1+\lambda-\lambda E_{\partial\Pi/\partial n,n}\left(n^*(\lambda),\lambda\right)}.\end{aligned}$$

Given what we see in the proof of Claim 1, $E_{\partial\Pi/\partial n,\lambda}\left(n,\lambda\right)-\frac{\Pi_\lambda(n,\lambda)}{\Pi_n(n,\lambda)}\frac{1}{n}-1$ is equal to

$$\lambda\frac{\left[-\left(1-E_{P'}(Q_n)\right)\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}-\left(2-E_{P'}(Q_n)\right)\frac{\Lambda_n}{n-\Lambda_n}\right]\frac{\partial Q_n}{\partial\lambda}\frac{1}{Q_n}-\frac{\partial^2 Q_n}{\partial n\partial\lambda}\frac{n}{Q_n}-\frac{n-1}{n-\Lambda_n}\left(1-\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\right)}{\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}+\frac{\Lambda_n}{n-\Lambda_n}}$$
$$-\frac{P'(Q_n)\frac{Q_n}{n}\frac{\partial Q_n}{\partial\lambda}\frac{n-\Lambda_n}{n}}{P'(Q_n)\left(\frac{Q_n}{n}\right)^2\left[\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\left(1-\frac{\Lambda_n}{n}\right)+\frac{\Lambda_n}{n}\right]}\frac{1}{n}-1=$$

$$\lambda\frac{\left[-\left(1-E_{P'}(Q_n)\right)\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}-\left(2-E_{P'}(Q_n)\right)\frac{\Lambda_n}{n-\Lambda_n}\right]\frac{\partial Q_n}{\partial\lambda}\frac{1}{Q_n}-\frac{\partial^2 Q_n}{\partial n\partial\lambda}\frac{n}{Q_n}-\frac{n-1}{n-\Lambda_n}\left(1-\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\right)}{\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}+\frac{\Lambda_n}{n-\Lambda_n}}$$

$$-\frac{\frac{\partial Q_n}{\partial\lambda}\frac{1}{Q_n}}{\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}+\frac{\Lambda_n}{n-\Lambda_n}}-1=$$

$$\frac{-\left[\lambda\left(1-E_{P'}(Q_n)\right)\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}+\lambda\left(2-E_{P'}(Q_n)\right)\frac{\Lambda_n}{n-\Lambda_n}+1\right]\frac{\partial Q_n}{\partial\lambda}\frac{1}{Q_n}-\frac{\partial^2 Q_n}{\partial n\partial\lambda}\frac{\lambda n}{Q_n}+\frac{2\Lambda_n-n-1}{n-\Lambda_n}\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}-\frac{2\Lambda_n-1}{n-\Lambda_n}}{\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}+\frac{\Lambda_n}{n-\Lambda_n}},$$

where for constant marginal costs

$$\frac{\partial Q_n}{\partial n}=\frac{1-\lambda}{n+\Lambda-\Lambda E_{P'}\left(Q_n\right)}\frac{Q_n}{n}\xRightarrow{C\text{ linear}}$$

$$\frac{\partial^2 Q_n}{\partial n\partial\lambda}=\frac{\left\{\begin{array}{c}\left(-Q_n+(1-\lambda)\frac{\partial Q_n}{\partial\lambda}\right)\left(n+\Lambda-\Lambda E_{P'}\left(Q_n\right)\right)\\ -(1-\lambda)Q_n\left[n-1-(n-1)E_{P'}\left(Q_n\right)-\Lambda E'_{P'}\left(Q_n\right)\frac{\partial Q_n}{\partial\lambda}\right]\end{array}\right\}}{\left(n+\Lambda-\Lambda E_{P'}\left(Q_n\right)\right)^2}\frac{1}{n},$$

$$\frac{\partial^2 Q_n}{(\partial n)^2}=\frac{\left\{(1-\lambda)\frac{Q_n}{n}\left[\begin{array}{c}-\left(1+\lambda-\lambda E_{P'}(Q_n)-\Lambda E'_{P'}(Q_n)\frac{\partial Q_n}{\partial n}\right)\\ +\frac{n+\Lambda-\Lambda E_{P'}\left(Q_n\right)}{n}\left(\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}-1\right)\end{array}\right]\right\}}{\left(n+\Lambda-\Lambda E_{P'}\left(Q_n\right)\right)^2}\Longrightarrow$$

$$\frac{\partial^2 Q_n}{(\partial n)^2}=\frac{\partial Q_n}{\partial n}\frac{\left[\begin{array}{c}\frac{n+\Lambda-\Lambda E_{P'}\left(Q_n\right)}{n}\left(\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}-1\right)\\ -\left(1+\lambda-\lambda E_{P'}(Q_n)-\Lambda E'_{P'}(Q_n)\frac{\partial Q_n}{\partial n}\right)\end{array}\right]}{n+\Lambda-\Lambda E_{P'}\left(Q_n\right)}$$

so that $E_{\partial\Pi/\partial n,\lambda}\left(n,\lambda\right)-\frac{\Pi_\lambda(n,\lambda)}{\Pi_n(n,\lambda)}\frac{1}{n}-1$ has the same sign as

$$\left[\lambda\left(1-E_{P'}(Q_n)\right)\frac{1-\lambda}{n+\Lambda-\Lambda E_{P'}\left(Q_n\right)}+\lambda\left(2-E_{P'}(Q_n)\right)\frac{\Lambda}{n-\Lambda}+1\right]\frac{n-1}{n+\Lambda-\Lambda E_{P'}\left(Q_n\right)}$$

$$+\lambda\frac{\left\{\begin{array}{c}\left(1+(1-\lambda)\frac{n-1}{n+\Lambda-\Lambda E_{P'}\left(Q_n\right)}\right)\left(n+\Lambda-\Lambda E_{P'}\left(Q_n\right)\right)\\ +(1-\lambda)\left[(n-1)\left(1-E_{P'}\left(Q_n\right)\right)-\frac{\Lambda(n-1)Q_nE'_{P'}\left(Q_n\right)}{n+\Lambda-\Lambda E_{P'}\left(Q_n\right)}\right]\end{array}\right\}}{\left(n+\Lambda-\Lambda E_{P'}\left(Q_n\right)\right)^2}$$

$$+\frac{2\Lambda-n-1}{n-\Lambda}\frac{1-\lambda}{n+\Lambda-\Lambda E_{P'}\left(Q_n\right)}-\frac{2\Lambda-1}{n-\Lambda}$$

$$\propto\left[\lambda\left(1-E_{P'}(Q_n)\right)(1-\lambda)+\left(\lambda\left(2-E_{P'}(Q_n)\right)\frac{\Lambda}{n-\Lambda}+1\right)\left(n+\Lambda-\Lambda E_{P'}\left(Q_n\right)\right)\right](n-1)$$

$$+\lambda\left\{n+\Lambda-\Lambda E_{P'}\left(Q_n\right)+n-\Lambda\right\}+\lambda\left\{(n-\Lambda)\left(1-E_{P'}\left(Q_n\right)\right)-\frac{\Lambda(n-\Lambda)Q_nE'_{P'}\left(Q_n\right)}{n+\Lambda-\Lambda E_{P'}\left(Q_n\right)}\right\}$$

$$
\begin{aligned}
&+\frac{2(\Lambda_n-n)+n-1}{n-\Lambda}(1-\lambda)\left(n+\Lambda-\Lambda E_{P'}\left(Q_n\right)\right)-\frac{2\Lambda-1}{n-\Lambda}\left(n+\Lambda-\Lambda E_{P'}\left(Q_n\right)\right)^2\\
=&\lambda(n-\Lambda)\left(1-E_{P'}(Q_n)\right)+\left(\left(2-E_{P'}(Q_n)\right)\frac{\Lambda(\Lambda-1)}{n-\Lambda}+n-1\right)\left(n+\Lambda-\Lambda E_{P'}\left(Q_n\right)\right)\\
&+\lambda\left(3n-\Lambda-nE_{P'}\left(Q_n\right)\right)-\frac{\lambda\Lambda(n-\Lambda)Q_nE'_{P'}\left(Q_n\right)}{n+\Lambda-\Lambda E_{P'}\left(Q_n\right)}\\
&+(2\lambda-1)\left(n+\Lambda-\Lambda E_{P'}\left(Q_n\right)\right)-\frac{2\Lambda-1}{n-\Lambda}\left(n+\Lambda-\Lambda E_{P'}\left(Q_n\right)\right)^2\\
=&\left(\frac{\Lambda(\Lambda-1)\left(2-E_{P'}(Q_n)\right)-(2\Lambda-1)\left(n+\Lambda-\Lambda E_{P'}\left(Q_n\right)\right)}{n-\Lambda}+n-1+2\lambda-1\right)\left(n+\Lambda-\Lambda E_{P'}\left(Q_n\right)\right)\\
&+\lambda(n-\Lambda)\left(1-E_{P'}(Q_n)\right)+\lambda\left(3n-\Lambda-nE_{P'}\left(Q_n\right)\right)-\frac{\lambda\Lambda(n-\Lambda)Q_nE'_{P'}\left(Q_n\right)}{n+\Lambda-\Lambda E_{P'}\left(Q_n\right)}\\
=&\left(\frac{2\Lambda(\Lambda-1)-\Lambda(\Lambda-1)E_{P'}(Q_n)-(2\Lambda-1)\left(n+\Lambda\right)+(2\Lambda-1)\Lambda E_{P'}\left(Q_n\right)}{n-\Lambda}+n-2(1-\lambda)\right)\\
&\times\left(n+\Lambda-\Lambda E_{P'}\left(Q_n\right)\right)+\lambda(2n-\Lambda)\left(2-E_{P'}\left(Q_n\right)\right)-\frac{\lambda\Lambda(n-\Lambda)Q_nE'_{P'}\left(Q_n\right)}{n+\Lambda-\Lambda E_{P'}\left(Q_n\right)}\\
=&\left(\frac{n-\Lambda-2\Lambda n+\Lambda^2E_{P'}\left(Q_n\right)}{n-\Lambda}+n-2(1-\lambda)\right)\left(n+\Lambda-\Lambda E_{P'}\left(Q_n\right)\right)\\
&+\lambda(2n-\Lambda)\left(2-E_{P'}\left(Q_n\right)\right)-\frac{\lambda\Lambda(n-\Lambda)Q_nE'_{P'}\left(Q_n\right)}{n+\Lambda-\Lambda E_{P'}\left(Q_n\right)}\\
=&\left(n-1+2\lambda-\frac{\Lambda\left(2n-\Lambda E_{P'}\left(Q_n\right)\right)}{n-\Lambda}\right)\left(n+\Lambda-\Lambda E_{P'}\left(Q_n\right)\right)+\lambda(2n-\Lambda)\left(2-E_{P'}\left(Q_n\right)\right)\\
&-\frac{\lambda\Lambda(n-\Lambda)Q_nE'_{P'}\left(Q_n\right)}{n+\Lambda-\Lambda E_{P'}\left(Q_n\right)},
\end{aligned}
$$

which is positive if $E'_{P'}\leq 0$ and $E_{P'}\left(Q_n\right)>\left[2n-\left(n/\Lambda-1\right)\left(n-1+2\lambda\right)\right]/\Lambda$. On the other hand, given $E_{P'}\left(Q_n\right)<2$, $2(n+\Lambda-\Lambda E_{P'}\left(Q_n\right))>\lambda(2n-\Lambda)\left(2-E_{P'}\left(Q_n\right)\right)$, so if $E'_{P'}\geq 0$ and $E_{P'}\left(Q_n\right)<\left[2n-\left(n/\Lambda-1\right)\left(n+1+2\lambda\right)\right]/\Lambda$, then the expression is negative.

If $dn^*(\lambda)/d\lambda\leq 0$, then $Q_{n^*(\lambda)}$ clearly decreases with λ. If $dn^*(\lambda)/d\lambda>0$, then in equilibrium

$$
E_{\partial\Pi/\partial n,\lambda}\left(n,\lambda\right)-\frac{\Pi_\lambda(n,\lambda)}{\Pi_n(n,\lambda)}\frac{1}{n}-1>0
$$

and the directional derivative of the total quantity when (λ,n) changes in direction $\mathbf{v}\coloneqq(1,dn^*(\lambda)/d\lambda)$ is

$$
\begin{aligned}
\nabla_{\mathbf{v}}Q_n=&\frac{\partial Q_n}{\partial\lambda}+\frac{\partial Q_n}{\partial n}\frac{dn^*(\lambda)}{d\lambda}\\
=&\frac{\partial Q_n}{\partial\lambda}-\frac{n^*(\lambda)\left(1+\frac{\Pi_\lambda(n^*(\lambda),\lambda)}{\Pi_n(n^*(\lambda),\lambda)}\frac{1}{n^*(\lambda)}-E_{\partial\Pi/\partial n,\lambda}\left(n^*(\lambda),\lambda\right)\right)}{1+\lambda-\lambda E_{\partial\Pi/\partial n,n}\left(n^*(\lambda),\lambda\right)}\frac{\partial Q_n}{\partial n}\\
=&-\frac{(n-1)Q}{n+\Lambda-C''(q)/P'(Q)-\Lambda E_{P'}(Q)}\\
&-\frac{n\left(1+\frac{\Pi_\lambda(n,\lambda)}{\Pi_n(n,\lambda)}\frac{1}{n}-E_{\partial\Pi/\partial n,\lambda}\left(n,\lambda\right)\right)}{1+\lambda-\lambda E_{\partial\Pi/\partial n,n}\left(n^*,\lambda\right)}\frac{Q}{n}\frac{1-\lambda-C''\left(q\right)/P'\left(Q\right)}{n+\Lambda-C''\left(q\right)/P'\left(Q\right)-\Lambda E_{P'}\left(Q\right)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(n-1)Q\left[\frac{E_{\partial\Pi/\partial n,\lambda}(n,\lambda)-\frac{\Pi_\lambda(n,\lambda)}{\Pi_n(n,\lambda)}\frac{1}{n}-1}{1+\lambda-\lambda E_{\partial\Pi/\partial n,n}(n,\lambda)}\frac{1-\lambda-C''(q)/P'(Q)}{n-1}-1\right]}{n+\Lambda-C''(q)/P'(Q)-\Lambda E_{P'}(Q)} \\
&\propto \frac{E_{\partial\Pi/\partial n,\lambda}(n,\lambda)-\frac{\Pi_\lambda(n,\lambda)}{\Pi_n(n,\lambda)}\frac{1}{n}-1}{1+\lambda-\lambda E_{\partial\Pi/\partial n,n}(n,\lambda)}\frac{1-\lambda-C''(q)/P'(Q)}{n-1}-1,
\end{aligned}
$$

so that under constant marginal costs, $E_{\partial\Pi/\partial n,n}(n,\lambda) \leq 2$, $E_{P'}(Q)' \geq 0$ and $E_{P'}(Q) < 2$,

$$
\begin{aligned}
\operatorname{sgn}\{\nabla_{\mathbf{v}} Q_n\} &\leq \operatorname{sgn}\left\{E_{\partial\Pi/\partial n,\lambda}(n,\lambda)-\frac{\Pi_\lambda(n,\lambda)}{\Pi_n(n,\lambda)}\frac{1}{n}-1-(n-1)\right\} \\
&= \operatorname{sgn}\left\{\begin{array}{r}\left(n-1+2\lambda-\frac{\Lambda(2n-\Lambda E_{P'}(Q_n))}{n-\Lambda}\right)(n+\Lambda-\Lambda E_{P'}(Q_n)) \\ +\lambda(2n-\Lambda)(2-E_{P'}(Q_n))-\frac{\lambda\Lambda(n-\Lambda)Q_n E'_{P'}(Q_n)}{n+\Lambda-\Lambda E_{P'}(Q_n)} \\ -(n-1)\left(\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}+\frac{\Lambda_n}{n-\Lambda_n}\right)(n+\Lambda-\Lambda E_{P'}(Q_n))^2\end{array}\right\} \\
&= \operatorname{sgn}\left\{\begin{array}{r}\left(n-1+2\lambda-\frac{\Lambda(2n-\Lambda E_{P'}(Q_n))}{n-\Lambda}\right)(n+\Lambda-\Lambda E_{P'}(Q_n)) \\ +\lambda(2n-\Lambda)(2-E_{P'}(Q_n))-\frac{\lambda\Lambda(n-\Lambda)Q_n E'_{P'}(Q_n)}{n+\Lambda-\Lambda E_{P'}(Q_n)} \\ -(n-1)\left(1-\lambda+\frac{\Lambda(n+\Lambda-\Lambda E_{P'}(Q_n))}{n-\Lambda}\right)(n+\Lambda-\Lambda E_{P'}(Q_n))\end{array}\right\} \\
&= \operatorname{sgn}\left\{\begin{array}{r}\left(\lambda(n-1)+2\lambda-\frac{\Lambda(2n+(n-1)(n+\Lambda)-n\Lambda E_{P'}(Q_n))}{n-\Lambda}\right)\times \\ (n+\Lambda-\Lambda E_{P'}(Q_n))+\lambda(2n-\Lambda)(2-E_{P'}(Q_n))-\frac{\lambda\Lambda(n-\Lambda)Q_n E'_{P'}(Q_n)}{n+\Lambda-\Lambda E_{P'}(Q_n)}\end{array}\right\} \\
&\leq \operatorname{sgn}\left\{2+\lambda(n-1)+2\lambda-\frac{\Lambda(2n+(n-1)(n+\Lambda)-n\Lambda E_{P'}(Q_n))}{n-\Lambda}\right\} \\
&< \operatorname{sgn}\left\{2+\lambda(n-1)+2\lambda-\frac{\Lambda(2n+(n-1)(n+\Lambda)-2n\Lambda)}{n-\Lambda}\right\} \\
&= \operatorname{sgn}\left\{2+\lambda(n-1)+2\lambda-\frac{\Lambda(n-\Lambda)(n+1)}{n-\Lambda}\right\} = \operatorname{sgn}\{1+2\lambda-\Lambda n\},
\end{aligned}
$$

which is non-positive given $n \geq 2$. **Q.E.D.**

Proof of Claim 4 Under constant marginal costs and linear demand

$$
\begin{aligned}
Q_n &= \frac{n(a-c)}{b(n+\Lambda)}, \quad \Pi(n,\lambda) = \left(a-\frac{n(a-c)}{n+\Lambda}-c\right)\frac{a-c}{b(n+\Lambda)} = \frac{\Lambda(a-c)^2}{b(n+\Lambda)^2}, \\
\frac{\partial\Pi(n,\lambda)}{\partial n} &= \frac{(a-c)^2(\lambda(n+\Lambda)^2-2(n+\Lambda)(1+\lambda)\Lambda)}{b(n+\Lambda)^4} = -\frac{(a-c)^2(2\Lambda-\lambda(n-\Lambda))}{b(n+\Lambda)^3}, \\
\frac{\partial^2\Pi(n,\lambda)}{(\partial n)^2} &= \frac{(a-c)^2\{-\lambda(1+\lambda)(n+\Lambda)^3-3(1+\lambda)(n+\Lambda)^2[\lambda(n-\Lambda)-2\Lambda]\}}{b(n+\Lambda)^6} \\
&= \frac{2(a-c)^2(1+\lambda)[3\Lambda-\lambda(2n-\Lambda)]}{b(n+\Lambda)^4} = \frac{2(a-c)^2(1+\lambda)[2\Lambda-\lambda(n-\Lambda)+1-\lambda]}{b(n+\Lambda)^4},
\end{aligned}
$$

$$
\begin{aligned}
E_{\partial\Pi/\partial n,n}(n,\lambda) &= \frac{2[n+\Lambda-(1-\lambda)]\,[2\Lambda-\lambda(n-\Lambda)+1-\lambda]}{(n+\Lambda)\,[2\Lambda-\lambda(n-\Lambda)]} \\
&= 2\left(1-\frac{1-\lambda}{n+\Lambda}\right)\left(1+\frac{1-\lambda}{2\Lambda-\lambda(n-\Lambda)}\right),
\end{aligned}
$$

Thus, under linear demand and constant marginal costs, $E_{\partial\Pi/\partial n,n}(n,\lambda)$ is decreasing in n, and thus bounded from above by

$$
2\left(1-\frac{1-\lambda}{2}\right)\left(1+\frac{1-\lambda}{2-\lambda(1-1)}\right) = 2\left[1-\left(\frac{1-\lambda}{2}\right)^2\right] \leq 2.
$$

Q.E.D.

Proof of Proposition 15 See the proof of Claim 1. **Q.E.D.**

Proof of Proposition 16 We have seen that the derivative of equilibrium total surplus (in the Cournot game with a fixed number of firms) with respect to n is given by

$$
\frac{d\,\mathrm{TS}(q_n)}{dn} = \Pi(n,\lambda) - f - \Lambda_n Q_n P'(Q_n)\frac{\partial q_n}{\partial n}.
$$

Given (8) we then have that

$$
\left.\frac{d\,\mathrm{TS}(\boldsymbol{q}_n)}{dn}\right|_{n=n^*(\lambda)} = -\,\lambda n^*(\lambda)\Pi_n\left(n^*(\lambda),\lambda\right) - \Lambda_{n^*(\lambda)}Q_{n^*(\lambda)}P'\left(Q_{n^*(\lambda)}\right)\left.\frac{\partial q_n}{\partial n}\right|_{n=n^*(\lambda)}
$$

and the result follows as in the proof of Proposition 11. **Q.E.D.**