Free entry in a Cournot market with overlapping ownership*

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Abstract

We examine the effects of overlapping ownership in a Cournot oligopoly where existing firms with overlapping ownership decide whether to enter a new market. We show that entry is either monotonically decreasing in the degree of overlapping ownership or has an inverted-U relationship with it. Although entry is excessive under non-decreasing returns to scale, with decreasing returns to scale entry (in contrast to standard results) is insufficient under high levels of overlapping ownership. Under standard assumptions, we find that overlapping ownership magnifies the negative impact of an increase of entry costs on entry providing a rationale for empirical evidence.

Keywords: common ownership, cross ownership, institutional ownership, minority shareholdings, oligopoly, entry, competition policy
JEL classification codes: D43, E11, L11, L13, L21, L41

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1 Introduction

Overlapping ownership, be it in the form of common or cross ownership, has generated concern for its potential anti-competitive impact (Elhauge, 2016; Posner et al., 2017), especially due to the rising shares of large investment funds in multiple competitors in several industries; for example airlines (Azar et al., 2018), banks and supermarkets (Schmalz, 2018). Azar and Vives (2019, 2021) and Backus et al. (2021b) document the dramatic rise in common ownership in the S&P 500 firms in the last decades.

At the same time, firm entry patterns have been argued to pose a significant impact on the aggregate economy. Using a panel of U.S. states over the period 1982–2014, Gourio et al. (2016) find that (positive) shocks to the number of new firms have sizable and lasting (positive) effects on a state's real GDP, productivity, and population. Gutiérrez and Philippon (2019) document a decline in entry of firms in the U.S. economy and estimate the elasticity of entry with respect to Tobin's Q to have dropped to zero since the late 1990s, up to which point it was positive and significant.[1] Gutiérrez et al. (2021) argue that increases in entry costs have had a considerable impact on the U.S. economy over the past 20 years, leading to higher concentration, as well as lower entry, investment and labor income. Figure 1 documents the increase in regulatory restrictions that accompanies the decrease in entry.

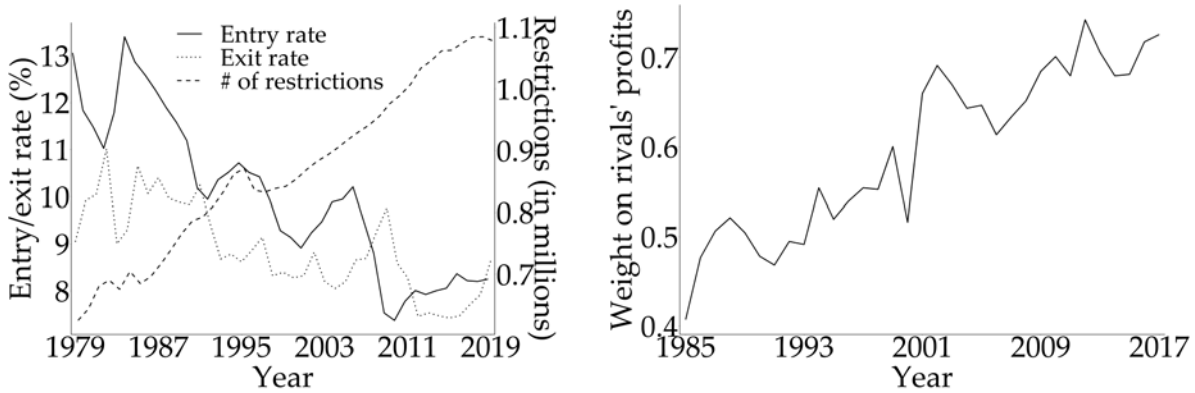
The literature above documents the decline in firm entry rates (accompanied by a milder decrease in firm exit rates) and a concurrent increase in common ownership over close to 40 years in the U.S economy (see Figure 1). There are several explanations for the decreased entry dynamism, an increase in entry costs (for technological or regulatory reasons) being a prominent one. It is possible also that apart from softening competition in pricing, overlapping ownership also contributes to diminishing entry dynamism. Some recent empirical work points in this direction. Newham et al. (2019) find that in the U.S. pharmaceutical industry higher common ownership between the brand firm and potential generic entrants leads to fewer generic entrants. Relatedly, Xie and Gerakos (2020) analyze patent infringement lawsuits filed by brand-name drug manufacturers against generic manufacturers to find that common institutional ownership of the brand and generic

[1] Apart from a generalized decline in entry, Decker et al. (2016) document a particular decline in high-growth young firms in the U.S. since 2000, when such firms could have had a major contribution to job creation.

firms increases the likelihood that the two litigants enter into a settlement whereby the brand firm often pays the generic to delay entry. Ruiz-Pérez (2019) estimates a structural model of market entry and price competition under common ownership in the U.S. airline industry to find that the higher the common ownership between the incumbents and a potential entrant, the lower the likelihood of entry.

Figure 1: Firm entry, regulatory restrictions and overlapping ownership trends in the U.S.

(a) Firm entry/exit rates and regulatory restrictions **(b)** Average weight on competing firms' profits



Note: firm count and death data are from U.S. Census Bureau Business Dynamics Statistics. The firm entry (resp. exit) rate in year t is calculated as the count of age zero firms (resp. firm deaths) in year t divided by the average count of firms in year t and $t-1$. The total number of regulatory restrictions data are from McLaughlin et al. (2021). Panel (b) shows the average intra-sector Edgeworth sympathy coefficient for the largest 1500 firms by market capitalization (*i.e.*, the average weight placed by a firm on the profits of another firm in the same sector relative to a weight of 1 placed on its own profit), as calculated in Azar and Vives (2021) based on Thomson-Reuters 13F filings data on institutional ownership.

In this paper we provide a framework to study the effects of overlapping ownership in a Cournot oligopoly with free entry. We study an industry or product market which established firms with existing ownership ties consider whether to enter; that is, there is pre-entry overlapping ownership. This is common in today's markets with extensive common ownership links among public firms.[2] In Appendix B we consider the case of post-entry overlapping ownership.

We are interested in several questions. How does overlapping affect entry, prices and welfare? Will overlapping ownership suppress entry or will entry still tend to be excessive as in the case without overlapping ownership? What are the forces at play? How does overlapping ownership mediate the (negative) effect of entry costs on entry? What level of overlapping ownership is socially optimal?

[2]For example, one can think of pharmaceuticals considering whether to incur R&D costs to enter a new drug market.

We address the main concern about how overlapping ownership can affect competitive outcomes through entry: by inducing firms to internalize the negative externality that their entry would pose on other firms, thus reducing entry. Modulo the integer constraint, entry is lower than what it would be without the internalization of the entry externality, so that in equilibrium the net profit is positive.

Overlapping ownership differs from collusion in terms of both the mechanism through which it affects competitive outcomes and the actual competitive effects. Indeed, the mechanism in overlapping ownership is through incentives of owners and managers (Schmalz, 2018, 2021; Anton et al., 2022b), while collusion works with dynamic threats, rewards and punishements. Both pre-entry overlapping ownership and collusion induce firms to internalize the effects of their actions on other firms' profits, but the former gives rise to different trade-offs and forces which are not present under collusion. For example, collusion and post-entry overlapping ownership (that is, when ownership links develop after entry) tend to spur entry, since a firm—which decides whether to enter only seeking to maximize its own profit—expects higher profits when there is going to be collusion or softer competition in pricing compared to when there is not (e.g., see Fershtman and Pakes, 2000). Pre-entry overlapping ownership induces a novel trade-off that we describe below.

Our findings follow. First, we distinguish the three channels through which an increase in the level of pre-entry overlapping ownership affects entry; these channels are not specific to our assumption of Cournot competition. Overlapping ownership tends to limit entry by increasing the degree of internalization of the negative externality of entry on other firms' profits but also tends to increase equilibrium profits in the product market competition stage, which tends to increase entry. There also is a channel with an ambiguous effect on entry: overlapping ownership changes the magnitude of the entry externality. The effect of overlapping ownership on entry will depend on the size of the different channels and the direction of the ambiguous channel's effect. We find that an increase in the degree of overlapping ownership can limit or, counter-intuitively, spur entry. For low levels of overlapping ownership, the rise in own profit due to increases in overlapping ownership can dominate. However, for high levels, competition in the product market is already soft enough, so that further increases in overlapping ownership suppress entry. Common ownership among U.S.-listed firms is indeed already high enough, so that if private firms

are treated as a competitive fringe, then further increases in common ownership are likely to limit entry by public firms into product markets where other public firms already operate. In the extreme case of complete indexation of the industry, where firms maximize aggregate industry profits, only one firm enters so that fixed entry costs are minimized unless there are substantial decreasing returns to scale (DRS).

Second, we show that under common assumptions, overlapping ownership exacerbates the negative impact of an increase in the entry cost on entry. Therefore, overlapping ownership could magnify the negative macroeconomic implications documented in Gutiérrez et al. (2021).

Third, we find that whether entry is excessive or insufficient will depend on (i) the level of overlapping ownership, (ii) the magnitude of the entry externality, (iii) whether returns to scale are increasing (IRS) or decreasing and to what degree, and (iv) whether competition is business-stealing or business-enhancing (*i.e.*, whether individual quantity decreases or increases, respectively, with the number of firms in the Cournot game), and to what degree. Under business-stealing competition, increases in the level of overlapping ownership or the magnitude of the entry externality, tend to make entry insufficient with the two forces being complements in inducing insufficient entry. Also, DRS tend to make entry insufficient, since the planner takes advantage of variable-cost savings due to entry to a greater extent than firms do. Increases in the magnitude of the business-stealing effect and IRS tend to make entry excessive. Under business-enhancing competition, entry is always insufficient.

In the standard case of business-stealing competition, we find that with non-DRS entry is indeed excessive under general conditions. However, if returns to scale are decreasing enough, then entry is insufficient under high levels of overlapping ownership. On the one hand, the planner combats DRS by having many firms enter. On the other hand, while with positive overlapping ownership a firm also prefers lower average costs both for itself and for the other firms, the internalization of the price effect that its entry will have suppresses entry relative to the planner's solution.

Fourth, a welfare-maximizing planner that can only regulate overlapping ownership and then allow firms to freely enter may optimally choose (a) no overlapping ownership, (b) intermediate levels of it, or even (c) complete indexation of the industry. For instance, consider constant returns to scale (CRS); then the average variable cost of production is

not affected by entry. The planner has to balance the effect of overlapping ownership—in the Cournot game and in the entry stage—on the total quantity and price, and on the total entry costs. In numerical simulations we find that for low entry cost, the planner chooses no overlapping ownership, while for higher entry cost intermediate levels of overlapping ownership or complete indexation can be optimal. With IRS, complete indexation of the industry (leading to a monopoly) can be optimal under both a total surplus and a consumer surplus standard.

Last, we examine the effects of overlapping ownership in the case where apart from the commonly-owned firms, there are also maverick firms (price-taking and without ownership ties), which may enter the market. The presence of maverick firms essentially changes the demand faced by the commonly-owned firms by depressing it and making it more elastic. This suppresses entry by commonly-owned firms and makes it less sensitive to the level of overlapping ownership. Our results on the effects of overlapping ownership on the price, entry by commonly-owned firms, as well as our comparison of equilibrium and socially optimal levels of entry extend to this case with the demand appropriately adjusted.

The plan of the paper is as follows. Section 2 discusses related literature and section 3 presents the model and studies the pricing stage. Section 4 studies the entry stage, existence and uniqueness of equilibrium in the complete game with entry. Section 5 studies the effects of overlapping ownership under free entry. Section 6 considers post-entry overlapping ownership and the entry of maverick firms. Last, section 7 concludes. Proofs are gathered in Appendix A, and supplementary material (including the examination of the case of post-entry overlapping ownership) and proofs thereof in the Appendices B and C.

2 Related literature

Research attention to the possible anti-competitive effects of overlapping ownership dates back to at least Rubinstein and Yaari (1983) and Rotemberg (1984). Recently, interest on the topic has revived given the rising shares of large diversified funds. As Banal-Estañol et al. (2020) show, the profit loads firms place on competing firms increase if the holdings of more diversified investors increase relative to those of less diversified investors. Multiple empirical studies have been conducted and there is a debate on whether and how common

ownership affects corporate conduct and softens competition.[3]

Theoretical work has considered models where the effects of overlapping ownership are not only through product market competition: when (i) there are diversification benefits because investors are risk-averse (Shy and Stenbacka, 2020) or (ii) firms choose cost-reducing or quality-enhancing R&D investment possibly with R&D spillovers (Bayona and López, 2018; López and Vives, 2019), product positioning (Li and Zhang, 2021) or qualities (Brito et al., 2020), (iii) firms invest in a preemption race (Zormpas and Ruble, 2021), (iv) firms may choose to transfer their innovation technology to a rival firm (Papadopoulos et al., 2019). Last, other studies have examined the effects of overlapping ownership in a general equilibrium setting (Azar and Vives, 2019, 2021) or under alternative models of corporate control (Vravosinos, 2021).

All of the models above treat the number of firms in the industry as exogenous.[4] Sato and Matsumura (SM; 2020) provide a circular-market model with CRS and free entry under pre-entry symmetric common ownership. In their model the welfare effects of common ownership are directly implied by its effects on entry.[5] They show that entry always decreases with common ownership. Thus, given that in their setting for low levels of common ownership entry is excessive while for high it is insufficient, welfare has an inverted-U shaped relationship with the degree of common ownership, which implies a strictly positive optimal degree of common ownership.

Our model differs from theirs in several ways. First, we consider quantity instead of price competition. Second we derive our results under general demand and cost functions and consider examples of parametric assumptions for ease of interpretation. In our setting total surplus depends on equilibrium objects not only through the number of firms. This means, for example, that higher overlapping ownership can induce a social planner that regulates entry (but not overlapping ownership) to allow fewer firms to enter,

[3]While He and Huang (2017), Azar et al. (2018), Park and Seo (2019), Boller and Morton (2020), Banal-Estañol et al. (2020) and Anton et al. (2022a,b) find evidence in favor of this hypothesis, others have found little to no effect (e.g., see Koch et al., 2021; Lewellen and Lowry, 2021; Backus et al., 2021a). Backus et al. (2021c) outline the limitations of the empirical approaches used so far and argue that these make it difficult to draw clear conclusions. Schmalz (2021) provides a compelling survey of the available evidence on how common owners influence firm decisions. See also Elhauge (2021) and Shekita (2022).

[4]Li et al. (2015) show that in a Cournot duopoly the incumbent firm can strategically develop cross ownership to deter the other firm from entering.

[5]Welfare only depends on the number of firms, the cost of transportation and the entry cost. Consumers have a unit demand and pay transportation costs proportional to their distance from the firm that they choose to buy from. The planner's problem is equivalent to minimizing the total transportation and entry costs; the former decrease with the number of firms, while the latter increase with it.

since overlapping ownership decreases the effectiveness of entry in reducing the price. Our modelling allows us to delineate three channels through which pre-entry overlapping ownership affects entry and test the robustness of the results obtained in SM.[6]

Our work can be seen as an extension of the literature on free entry in Cournot markets. Mankiw and Whinston (1986) show that in a symmetric Cournot market with free entry and non-IRS where in the pricing stage (i) the total quantity increases with the number of firms, and (ii) the business-stealing effect is present, entry is never insufficient by more than one firm. Amir et al. (2014) extend these results to the case of limited IRS, showing that still under business-stealing competition entry is never insufficient by more than one firm. We extend the result of Amir et al. (2014) to the case of competition under overlapping ownership, showing that under business-enhancing competition, entry is always insufficient. However, we show that under business-stealing competition, overlapping ownership can lead to insufficient entry (by more than one firm) when returns to scale are decreasing.

The setting of symmetric firms with a symmetric overlapping ownership structure that we consider preserves the properties of the Cournot game being symmetric, which allows for extensions of existing oligopoly results (e.g., see Vives, 1999) to the case of competition under overlapping ownership. Namely, we extend the results of Amir and Lambson (2000), who use lattice-theoretic methods to study equilibrium existence and comparative statics with respect to the (exogenous) number of firms in a symmetric Cournot market, and of Amir et al. (2014), who build on the latter to study free entry.

3 The Cournot-Edgeworth λ-oligopoly model with free entry

There is a (large enough) finite set $\mathcal{F} := \{1,2,\ldots,N\}$ of N symmetric firms that can potentially enter a market. The game has two stages, the entry stage and the pricing stage. In the first stage, each firm chooses whether to enter by paying a fixed cost $f > 0$.[7] In the pricing stage, entrants compete à la Cournot. Namely, each firm i chooses its production quantity, $q_i \in \mathbb{R}_+$, simultaneously with the other firms. We denote by $s_i := q_i/Q$ firm i's

[6]For example, SM find that entry always decreases with overlapping ownership, while in our case overlapping ownership sometimes spurs entry. In addition, in our model equilibrium total surplus can behave in multiple different ways as the extent of overlapping ownership changes—contrary to the inverted-U relationship found in SM. Last, we study how overlapping ownership mediates the effect of the entry cost on entry, which is not examined in SM.

[7]We study pure strategy equilibria. If firms decide whether to enter sequentially, this is indeed without loss of generality. However, if they decide simultaneously, then although the pure equilibrium is still an equilibrium, there can also be equilibria where firms mix in their entry decisions (e.g., see Cabral, 2004).

share of the total quantity $Q \coloneqq \sum_{i=1}^{n} q_i$. We also write $\boldsymbol{q}$ and $\boldsymbol{q}_{-i}$ to denote the production profile of all firms, and all firms expect i, respectively; also, $Q_{-i} \coloneqq \sum_{j \neq i} q_j$.

3.1 The pricing stage

Each firm i's production cost is given by the function $C : \mathbb{R}_+ \to \mathbb{R}_+$ with $C(q_i) \geq 0$ and $C'(q_i) > 0$ for every q_i. Denote by $E_C(q) \coloneqq C'(q)q/C(q)$ the elasticity of the cost function. When $C(q_i) = cq_i^{\kappa}/\kappa$ for some $c,\kappa > 0$, firms have constant elasticity costs and $E_C(q) \equiv \kappa$. (i) For $\kappa = 1$ we have CRS, (ii) for $\kappa \in (0,1)$ we have IRS, (iii) for $\kappa > 1$ DRS (for $\kappa = 2$ costs are quadratic). $AC(q) \coloneqq C(q)/q$ is the average cost.

The inverse demand function $P : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $P'(Q) < 0$ for every $Q \in \left[0,\overline{Q}\right)$, where $\overline{Q} \in (0,+\infty]$ is such that $P(Q) > 0$ if and only if $Q \in \left[0,\overline{Q}\right)$.[8] We assume that there exists $\overline{q} > 0$ such that $P(q) < AC(q)$ for every $q > \overline{q}$, and that P and C are two-times differentiable.[9] For $Q < \overline{Q}$ we denote by $\eta(Q) \coloneqq -P(Q)/(QP'(Q))$ the elasticity of demand, and by $E_{P'}(Q) \coloneqq -P''(Q)Q/P'(Q)$ the elasticity of the slope of inverse demand. An inverse demand function with constant elasticity of slope (CESL), $E_{P'}(Q) \equiv E$, allows for log-concave and log-convex demand encompassing linear and constant elasticity specifications. When we refer to linear demand, we mean $P(Q) = \max\{a - bQ, 0\}$. Every result applies to generic cost function and inverse demand function unless otherwise stated. We assume that the optimal (gross) monopoly profit is higher than the entry cost, that is, $\max_{Q \geq 0}\{P(Q)\,Q - C(Q)\} > f$.

Suppose n firms enter. A quantity profile $\boldsymbol{q}^*$ is an equilibrium of the pricing stage if for each firm $i \in \{1,\ldots,n\}$, $q_i^* \in \arg\max_{q_i \geq 0}\left\{\pi_i\left(q_i,\boldsymbol{q}_{-i}^*\right) + \lambda \sum_{j \neq i} \pi_j\left(q_i,\boldsymbol{q}_{-i}^*\right)\right\}$, where $\pi_i(\boldsymbol{q}) \coloneqq P(Q)\,q_i - C(q_i)$ and $\lambda \in [0,1]$ is the (exogenous) Edgeworth (1881) coefficient of effective sympathy among firms.[10] This coefficient can for example arise from a symmetric overlapping ownership structure (be it common or cross ownership) as in López and Vives (2019) or Azar and Vives (2021).

Our model with symmetric overlapping ownership is rich enough to capture the main forces and allows us to study a number of issues.[11] Although the assumption of a unique

[8] More precisely, $P(Q)$ and its derivatives may be undefined for $Q = 0$ (e.g., with $\lim_{Q \downarrow 0} P(Q) = +\infty$ and $\lim_{Q \downarrow 0} P'(Q) = -\infty$).

[9] P is required to be differentiable for $Q < \overline{Q}$.

[10] Section B.1 in Appendix B presents models that give rise to this objective function.

[11] The assumption of a symmetric overlapping ownership structure greatly facilitates the analysis. First, with an asymmetric overlapping ownership structure, the Cournot game is neither symmetric, nor even aggregative and the analysis would require very strong assumptions. Second, there would be extensive

Edgeworth coefficient of effective sympathy λ across all firm pairs is a simplification, an increase in λ in comparative statics exercises captures a particularly relevant phenomenon. It can for instance represent the expansion of an investment fund's holdings across all firms in an industry, as has recently been the trend that has spurred the antitrust interest in overlapping ownership.[12] This is also the reason why proposed policies have emphasized the industry-wide holdings of each investor rather than only individual stock trades. For instance, Posner et al. (2017) and Posner (2021) propose that an investor holding shares of more than one firm in an oligopoly be not allowed to own more than 1% of the market shares unless they commit to being purely passive. Further, in section 6.2 we discuss the case where there is also a competitive fringe of maverick firms.

Given a quantity profile $\boldsymbol{q}$ where the number of firms that have entered is $n \equiv \dim(\mathbf{q})$, total surplus is given by $\mathrm{TS}(\boldsymbol{q}) \coloneqq \int_0^Q P(X)dX - \sum_{i=1}^n C(q_i) - nf$, while the Herfindahl–Hirschman index (HHI) and modified HHI (MHHI) are given by $\mathrm{HHI}(\boldsymbol{q}) \coloneqq \sum_{i=1}^n s_i^2$ and $\mathrm{MHHI}(\boldsymbol{q}) \equiv (1-\lambda)\,\mathrm{HHI}(\boldsymbol{q}) + \lambda$. We denote the MHHI at a symmetric equilibrium by $H_n \coloneqq (1+\lambda(n-1))/n$.

3.2 Equilibrium in the pricing stage

3.2.1 Existence and uniqueness of a pricing stage equilibrium

Having described the environment we first derive conditions for equilibrium existence and uniqueness in the pricing stage using lattice-theoretic methods as in Amir and Lambson (AL; 2000). Let $\Delta(Q,Q_{-i}) \coloneqq 1 - \lambda - C''(Q-Q_{-i})/P'(Q)$ be defined on the lattice $L \coloneqq \left\{(Q,Q_{-i}) \in \mathbb{R}^2_{++} : \overline{Q} > Q \geq Q_{-i}\right\}$. $\Delta > 0$ allows for decreasing, constant and mildly IRS, while $\Delta < 0$ allows for more significant IRS.

Proposition 1. The following statements hold:

(i) Assume $\Delta(Q,Q_{-i}) > 0$ on L. Then, in the pricing stage

(a) there exists a symmetric equilibrium and no asymmetric equilibria,

multiplicity of equilibria in the entry stage. Even when there are two groups of firms, one with and one without overlapping ownership, there would often be equilibria where (i) firms from only the first group enter, (ii) firms from only the second group enter, and (iii) firms from both groups enter. Third, there would be no single measure of overlapping ownership, as is λ in our model.

[12] In a market with more than two firms, a trade that only involves two firms (e.g., a firm buying shares of another firm) will generally not shift the coefficient of effective sympathy uniformly for all firm pairs.

(b) if also $E_{P'}(Q) < (1 + \lambda + \Delta(Q,Q_{-i})/n)/H_n$ on L, then there exists a unique and symmetric equilibrium.

(ii) Assume that $\Delta(Q,Q_{-i}) < 0$ and $E_{P'}(Q) < \frac{1+\lambda+\Delta(Q,Q_{-i})}{1-(1-\lambda)(1-s_i)}$ on L. Then, in the pricing stage

(a) for every $m \in \{1,2\ldots,n\}$ there exists a unique quantity q_m such that any quantity profile where each of m firms produces quantity q_m and the remaining $n-m$ firms produce 0 is an equilibrium,

(b) no other equilibria exist.

Remark 3.1. The second order of differentiability of $P(Q)$ is inessential. However, it simplifies the arguments and interpretation and emphasizes the tension between the assumption $\Delta < 0$ and the one on $E_{P'}(Q)$. The latter guarantees that π_i is strictly concave in q_i whenever $P(Q) > 0$. IRS are needed for $\Delta < 0$ but at the same time tend to violate profit concavity.[13]

Corollary 1.1 studies existence and uniqueness of the pricing stage equilibrium under linear demand and linear-quadratic cost. The linear-quadratic cost function is of the form $C(q) = c_1 q + c_2 q^2/2$, where $c_1 \geq 0$, for (i) $q \in [0, +\infty)$ if $c_2 \geq 0$, (ii) $q \in [0, -c_1/c_2]$ if $c_2 < 0$.[14]

Corollary 1.1. Let demand be linear, $P(Q) = \max\{a - bQ, 0\}$, and cost be linear-quadratic with $a > c_1 \geq 0$ and $c_2 > -2bc_1/a$. Then,

(i) if $c_2 > -b(1-\lambda)$, then $\Delta > 0$ on L and a unique and symmetric equilibrium exists,

(ii) if $c_2 < -b(1-\lambda)$, then $\Delta < 0$ on L and a unique (in the class of symmetric equilibria), symmetric equilibrium exists.

[13] In the $\Delta > 0$ case, for $\lambda = 0$ we recover the condition $C'' - P' > 0$, under which AL show that a symmetric equilibrium exists and there are no asymmetric equilibria (Theorem 2.1). In the $\Delta < 0$ case, the assumption on $E_{P'}$ guarantees that the firm's objective is quasiconcave in its own quantity, under which condition AL show the same result. For $\lambda = 1$, DRS are necessary for uniqueness of the (symmetric) equilibrium. For example, with CRS, there are infinitely many equilibria (the symmetric one included), all with the same fixed total quantity arbitrarily distributed across firms, since each firm maximizes aggregate industry profits. Analogously, with $C'' < 0$ it is an equilibrium for firms to concentrate all production in one firm to take advantage of the IRS, as indicated in part (ii-a) of the proposition.

[14] Cost is indeed increasing over $q \leq -c_1/c_2$ when $c_2 < 0$. The value of $C(q)$ for higher q will not matter in applications, as parameter values will be such that firms do not produce more than $-c_1/c_2$.

In light of Proposition 1 we maintain from now on the following assumption unless otherwise stated in a specific result. The assumption should be understood to hold at the relevant values of (n,λ) for each result.[15]

Maintained Assumption. The conditions in part (i-a,b) or part (ii) of Proposition 1 hold.

Remark 3.2. When in a result we assume $\Delta > 0$ (resp. $\Delta < 0$) it is thus understood that the additional assumption of part (i) (resp. part (ii)) of Proposition 1 also holds. In section B.10 of the Appendix we discuss what happens when the condition in part (i-b) need not hold.

The maintained assumption guarantees that firms will play a symmetric equilibrium in the pricing stage subgame of any SPE. Given that monopoly profit is positive, that equilibrium will be interior.[16] When $\Delta < 0$, the pricing subgame also has asymmetric equilibria; however, these cannot be part of an SPE of the complete game, since the entering firms that do not produce would prefer to avoid the entry cost by not entering.

We denote by $\boldsymbol{q}_n$ the symmetric Cournot equilibrium when n firms are in the market (which is unique under out maintained assumption), and with some abuse of notation by q_n the quantity each firm produces in that profile, where the subscript n now does *not* refer to the identity of the n-th firm; we also write $Q_n := nq_n$, $\mathrm{TS}_n := \mathrm{TS}\left(\boldsymbol{q}_n\right)$. To simplify notation, for any $n > 0$ we also denote by $\Pi(n,\lambda) := P\left(Q_n\right) q_n - C(q_n)$ the individual (gross) profit in the symmetric equilibrium of the Cournot game with n firms and Edgeworth coefficient λ. When we ignore the integer constraint on n, we allow all equilibrium objects, such as $\Pi(n,\lambda)$, to be defined for $n \in \mathbb{R}_{++}$. We refer to $\Pi(n,\lambda) - f$ as net profit. The Cournot equilibrium pricing formula is

$$\frac{P(Q_n) - C'(q_n)}{P(Q_n)} = \frac{H_n}{\eta(Q_n)}. \tag{1}$$

3.2.2 Comparative statics of the pricing stage equilibrium

Proposition 2 describes some comparative statics for the pricing stage (*i.e.*, under a fixed number of firms).

[15]For example, for global comparative statics of the Cournot game as λ changes, the assumption is assumed to hold for fixed n and every $\lambda \in [0,1]$. For existence of a free entry equilibrium for a fixed λ, it is sufficient that the assumption hold for every $n \in \mathbb{R}_{++}$ and that fixed λ.

[16]Proposition 7 in Appendix B.3 studies the stability of the pricing stage equilibrium.

Proposition 2. The following statements hold:

(i) total and individual quantity, and total surplus (resp. individual profit) are decreasing (resp. increasing) in λ,

(ii) individual profit is deceasing in n,

(iii) if $E_{P'}(Q) < 2$ (resp. $E_{P'}(Q) > (1+\lambda)/\lambda$) for every $Q < \overline{Q}$, then individual quantity is decreasing (resp. increasing) in n over $n \geq 2$,[17]

(iv) if $\Delta > 0$ (resp. $\Delta < 0$), then total quantity is increasing (resp. decreasing) in n.

Competition is business-stealing (*i.e.*, q_n is decreasing in n) under standard assumptions. As in AL, for $\Delta > 0$ the Cournot market is quasi-competitive (*i.e.*, Q_n is increasing in n) while for $\Delta < 0$ it is quasi-anticompetitive (*i.e.*, Q_n is decreasing in n). Also, increases in overlapping ownership cause the price to increase and total surplus to fall.[18]

4 The entry stage

Assume that potential entrants have overlapping ownership with a coefficient of effective sympathy $\lambda \in [0,1]$. Given that $n-1$ firms enter, it is optimal for an n-th firm to enter if and only if $(1+\lambda(n-1))\left(\Pi\left(n,\lambda\right)-f\right) \geq \lambda(n-1)\left(\Pi\left(n-1,\lambda\right)-f\right)$. This can equivalently be written as

$$\Psi(n,\lambda) := \overbrace{\Pi\left(n,\lambda\right)}^{\text{own profit from entry}} - \lambda \overbrace{(n-1)\left(\Pi\left(n-1,\lambda\right)-\Pi\left(n,\lambda\right)\right)}^{=:\Xi(n,\lambda)>0\text{, entry externality on other firms}} \geq \overbrace{f}^{\text{cost of entry}}, \qquad (2)$$

where $\Xi(n,\lambda)$ denotes the externality that the entry of the n-th firm poses on the other firms, that is, the absolute value of the reduction in the aggregate profits of all other firms

[17] Of course, the condition $E_{P'}(Q) > (1+\lambda)/\lambda$ is very strong, especially given the assumption $E_{P'}(Q) < (1+\lambda+\Delta/n)/H_n$ on L. Also, it pushes against profit concavity in own quantity, which can make even make the monopolist's problem ill-behaved. For example, with CESL demand, when $E > 2$, $\lim_{Q\downarrow 0}(P(Q)Q - C(Q)) = +\infty$.

[18] Aggregate industry profits depend on the number of firms in the following way (see Appendix B for details). Under IRS, monopoly maximizes aggregate industry profits. As we will see, this combined with the fact that the Cournot market is quasi-anticompetitive for $\Delta < 0$ will imply that monopoly maximizes total surplus when $\Delta < 0$. Similarly, under constant elasticity costs and non-DRS ($E_C \leq 1$), aggregate industry profits decrease with the number of firms. However, under DRS the effect of entry on aggregate industry profits depends on two main factors: entry causes price to fall, but as more firms enter, production is distributed across more firms, which induces savings in variable costs. For returns to scale decreasing strongly enough the latter effect dominates and aggregate industry profits increase with the number of firms. This is why multiple firms may enter in that case even when $\lambda = 1$.

caused by the entry of the n-th firm.[19] $\Psi(n,\lambda)$ is a firm's own profit from entry minus the part of the entry externality that is internalized by the firm (*i.e.*, the entry externality multiplied by λ). We call $\Psi(n,\lambda)$ a firm's "internalized profit" from entry. We assume that when indifferent, firms enter. Then, $\boldsymbol{q}_n$ is a free entry equilibrium if and only if

$$\Psi(n,\lambda) \geq f > \Psi(n+1,\lambda), \tag{3}$$

which for $\lambda = 0$ reduces to the standard free entry condition $\Pi(n,0) \geq f > \Pi(n+1,0)$.[20] We assume that $\Psi(N,\lambda) < f$ for every λ.

In deciding whether to enter a firm compares the profit it will make to the cost of entry and the negative externality its entry will pose to the other firms.[21] Pre-existing overlapping ownership directly alters the incentives of firms to enter in a way additional to its effect on individual profit in the Cournot game.

The planner's problem We will consider the problem of a total surplus-maximizing planner who takes λ as given and chooses the number of firms that will compete à la Cournot. Denote by $n^o(\lambda) := \arg\max_{n\in\mathbb{N}} \mathrm{TS}_n$ the number of firms that given λ maximizes total surplus.[22] Clearly, if the planner could choose both n and λ, she would set $\lambda = 0$, since total surplus is decreasing in λ. Define also $\widehat{n}^o(\lambda) := \arg\max_{n\in\mathbb{R}_+} \mathrm{TS}_n$, the number of firms that given λ maximizes total surplus if we ignore the integer constraint on n.

We will also look at comparative statics with respect to λ—including how free entry

[19]This externality can be further decomposed into two effects:

$$\Xi(n,\lambda) = \overbrace{(n-1)\frac{\Pi(n-1,\lambda)}{n}}^{\text{profit-stealing}} + \frac{n-1}{n}\overbrace{[(n-1)\Pi(n-1,\lambda) - n\Pi(n,\lambda)]}^{\text{change in aggregate industry profits}}$$

Even if the entry of the n-th firm did not affect aggregate industry profits, the firm still steals $1/n$-th of the profit of each of the other $n-1$ firms; this corresponds to the profit-stealing effect. At the same time, the n-th firm's entry affects aggregate industry profits—share $(n-1)/n$ of which is earned by the other $n-1$ firms—as shown in Proposition 9 in Appendix B.

[20]For $\lambda = 1$, (3) reduces to $n\Pi(n,1) - (n-1)\Pi(n-1,1) \geq f > (n+1)\Pi(n,1) - n\Pi(n,1)$. Each firm seeks to maximize aggregate profits, so firms enter as long as entry increases aggregate gross profits by enough to cover entry costs. Hence, provided that the savings in variable costs are not large enough to compensate for additional entry costs, only one firm enters. In the simulation of Figure 7(c) in the appendix savings in variable costs are large enough compared to the fixed cost to make five firms enter when $\lambda = 1$.

[21]If we compare this with the post-entry overlapping ownership case, where—modulo the integer constraint—net profit is zero, we see that investors would prefer to become common owners before rather than after entry.

[22]Since monopoly net profit is positive, it follows that $n^o(\lambda) \geq 1$. Also, the planner can give subsidies in case the net profit in the symmetric Cournot equilibrium is negative.

equilibrium total surplus varies with λ. This way we will deduce the optimal choice of a planner that only controls overlapping ownership and allows firms to freely enter.

Existence and uniqueness of equilibrium Define $\Delta\Pi(n,\lambda) \coloneqq \Pi(n,\lambda) - \Pi(n-1,\lambda) < 0$, the decrease in individual profit caused by the entry of an extra firm. Proposition 3 identifies a condition under which a unique equilibrium exists. We treat n as a continuous variable and differentiate with respect to it.

Proposition 3. Assume that for every $n \in [1, +\infty)$

$$E_{\Delta\Pi,n}(n,\lambda) \coloneqq -\frac{\frac{\partial \Delta\Pi(n,\lambda)}{\partial n}}{\Delta\Pi(n,\lambda)}(n-1) < \frac{(n-1)\left(1+\lambda+\varepsilon(n,\lambda)\right)}{1+\lambda(n-1)}$$

where $E_{\Delta\Pi,n}$ is (a measure of) the elasticity with respect to n of the slope of individual profit with respect to n, and $\varepsilon(n,\lambda) \coloneqq \partial\Pi(\nu,\lambda)/\partial\nu|_{\nu=n-1}/\Delta\Pi(n,\lambda) - 1$. Then, $\Psi(n,\lambda)$ is decreasing in n, and thus a unique equilibrium with free entry exists.

Remark 4.1. $\varepsilon(n,\lambda)$ will be close to 0 since by the mean value theorem $\Delta\Pi(n,\lambda) = \partial\Pi(\nu,\lambda)/\partial\nu|_{\nu=\nu^*}$ for some real number $\nu^* \in [n-1,n]$.

Remark 4.2. For example, it can be checked that for $\lambda < 1$, $\Psi(n,\lambda)$ is indeed decreasing in n under linear demand and linear-quadratic cost with $a > c_1 \geq 0$ and $c_2 \geq 0$.

The condition in Proposition 3 requires that equilibrium profit in the pricing stage be not too convex in n; that is, the rate at which individual profit decreases with n should not decrease (in absolute value) too fast with n. With regard to internalized profit $\Psi(n,\lambda)$, see (2), an increase in n (i) decreases the first term $\Pi(n,\lambda)$, (ii) tends to increase the entry externality $\Xi(n,\lambda)$ through the increase in $(n-1)$ (as entry affects the profits of more firms), which tends to decrease $\Psi(n,\lambda)$, and (iii) affects $\Xi(n,\lambda)$ through its effect on the magnitude of the entry externality $\Pi(n-1,\lambda) - \Pi(n,\lambda)$ on a single firm. As long as the per-firm entry externality does not decrease with n too fast, $\Psi(n,\lambda)$ decreases with n.[23] We maintain the assumption that $\Psi(n,\lambda)$ is decreasing in n. Then, for a given λ, the

[23]For example, if $\Pi(n,\lambda)$ is concave in n, then the condition is satisfied given $\varepsilon(n,\lambda) \approx 0$. For $\lambda = 0$ the condition reduces to $\Pi(n,\lambda)$ being decreasing in n, which has been shown in Proposition 2. For $\lambda = 1$, the proposition requires that as the number of firms increases, aggregate industry profits increase (e.g., due to variable cost-savings) by less and less. The case $\lambda = 1$ can be special. Under CRS, (gross) aggregate industry profits are independent of the number of firms, so $\Psi(n,\lambda) = 0$ for every $n \geq 2$, and thus only one firm enters in equilibrium.

number $\widehat{n}^*(\lambda)$ of firms that enter in equilibrium if we ignore the integer constraint on n is pinned down by $\Psi\left(\widehat{n}^*(\lambda),\lambda\right) = f$, and $n^*(\lambda) = \lfloor \widehat{n}^*(\lambda) \rfloor$ is the number of firms that enter if we respect the integer constraint.

5 The effects of overlapping ownership under free entry

This section studies the following concerns about the anti-competitive effects that overlapping ownership can have: suppress entry by inducing firms to internalize the effect their entry would have on other firms' profits (subsections 5.1 and 5.2), magnify the impact of entry costs on entry (subsection 5.3), and induce entry which is not welfare optimal (subsection 5.4).

5.1 Overlapping ownership effects on entry

The effect of changes in λ on entry will be determined by the sign of the (partial) derivative of $\Psi(n,\lambda)$ with respect to λ. If $\partial\Psi(n,\lambda)/\partial\lambda$ is positive (resp. negative), then increases in λ should be met with increases (resp. decreases) in n for (3) to continue to hold. Proposition 4 studies the effects of overlapping ownership on entry.

Proposition 4. Equilibrium entry (locally) changes with λ in direction given by[24]

$$\operatorname{sgn}\left\{\frac{d\widehat{n}^*(\lambda)}{d\lambda}\right\} = \operatorname{sgn}\left\{\overbrace{\frac{1}{\lambda}\frac{E_{\Pi,\lambda}(\widehat{n}^*(\lambda),\lambda)}{\widehat{E}_{\Pi,n}(\widehat{n}^*(\lambda),\lambda)}}^{\substack{\text{increase in own}\\ \text{profit from entry}}} - \overbrace{E_{\Xi,\lambda}\left(\widehat{n}^*(\lambda),\lambda\right)}^{\substack{\text{change in magnitude}\\ \text{of entry externality}}} - \overbrace{1}^{\substack{\text{increase in internalization}\\ \text{of entry externality}}}\right\}$$

where $\widehat{E}_{\Pi,n}(n,\lambda) := -\left(\Pi\left(n,\lambda\right) - \Pi\left(n-1,\lambda\right)\right)(n-1)/\Pi(n,\lambda) > 0$ is a measure of the elasticity of individual profit with respect to n, $E_{\Pi,\lambda}(n,\lambda) := \lambda\partial\Pi(n,\lambda)/\partial\lambda/\Pi(n,\lambda) > 0$ is the elasticity of individual profit with respect to λ, and $E_{\Xi,\lambda}(n,\lambda) := \lambda\partial\Xi(n,\lambda)/\partial\lambda/\Xi(n,\lambda)$ is the elasticity of the entry externality with respect to λ.[25]

Remark 5.1. With the integer constraint $n^*(\lambda)$ does not change with an infinitesimal change $d\lambda$ in λ unless we are the knife-edge case where $\Psi(n^*(\lambda),\lambda) = f$. Thus, as λ increases everything will behave as in the case with a fixed number of firms, until λ reaches knife-edge cases causing a jump in $n^*(\lambda)$ as implied by Proposition 4.

[24] For $\lambda = 0$, cancel the λ in $1/\lambda$ with the one in $E_{\Pi,\lambda}(n,\lambda)$.

[25] $E_{\Xi,\lambda}$ which can also be seen as a measure of the elasticity with respect to λ of the slope of individual profit with respect to n.

An increase in overlapping ownership affects entry through three separate channels. On the one hand, it increases the degree of internalization of the negative externality of entry on other firms' profits; this increased internalization tends to limit entry. On the other hand, it tends to increase equilibrium profits in the Cournot game, which tends to increase entry.[26] Last, there is a channel with an ambiguous effect on entry: overlapping ownership changes the magnitude of the entry externality; that is, it affects how strongly equilibrium profits in the pricing stage decrease with the number of firms. A high (and positive) elasticity $E_{\Xi,\lambda}$ of the entry externality Ξ with respect to λ tends to make entry decreasing in λ, while $E_{\Xi,\lambda}$ being negative tends to make entry increasing in λ. Indeed, the magnitude of the entry externality $\Xi(n^*(\lambda),\lambda)$ can increase or decrease with λ.[27]

These three channels are not specific to our assumption of Cournot competition in the pricing stage. Nevertheless, the direction of the change in the entry externality $\Xi(n,\lambda)$ and the magnitudes of the different channels depend on the market structure. For example, the three channels are also present in the circular-market model with common ownership of Sato and Matsumura (SM; 2020)—although the authors discuss only the first two channels. The direction of the third channel's effect is not ambiguous in their model, where the magnitude of the entry externality monotonically increases with the extent of overlapping ownership.

Remark 5.2. Evaluating the expressions in Propositions 3 and 4 requires evaluation of profits and derivatives thereof in different equilbria of the pricing stage (with n and $n-1$ firms). This is possible under parametric assumptions while the problem remains intractable in general. In what follows, we present numerical results.[28]

Numerical Result 1. Under CESL demand, CRS, $\lambda < 1$ and $\widehat{n}^*(\lambda) \geq 2$, it holds that

(i) entry is decreasing in λ if (a) $E \in (1,2)$ and $\lambda \geq 1/2$, or (b) $E < 1$ and $\lambda \geq 2/5$,

(ii) entry is increasing in λ if $\widehat{n}^*(\lambda) \geq 7$ and (a) $E \in (1,2)$ and $\lambda \leq 1/4$, or (b) $E \in [0,1)$ and $\lambda \leq 1/5$,

[26] The model of Stenbacka and Van Moer (2022), where two firms choose how much to invest in product innovation and can only produce if they successfully innovate, has two similar forces.

[27] Under the parametrizations of Figures 7(a) and 7(b) in the appendix, where entry is low, Ξ is decreasing in λ. Ξ being decreasing in λ is expected under low entry given Proposition 16 on the modified model in section B.11 of Appendix B where firms decide whether to enter by examining a differential version of (2). However, Ξ is increasing in λ under the parametrization of Figure 2(a).

[28] Propositions 14 and 15 in Appendix B are differential versions of Propositions 3 and 4, respectively. Corollary 15.1 shows that in the modified model of Appendix B, total quantity decreases with λ under constant marginal costs and general assumptions on demand.

(iii) the total quantity is decreasing in λ.

For high enough levels of overlapping ownership further increases in these levels suppress entry. In the U.S. for example, common ownership levels among publicly listed firms have indeed been "high enough" during at least the last decade (see Figure 1 and Azar and Vives, 2021; Backus et al., 2021b; Amel-Zadeh et al., 2022).[29] Thus, if private firms are treated as a competitive fringe that only affects the residual demand in the oligopolies of public firms, then further increases in common ownership among the latter are likely to reduce entry by public firms in product markets where other public firms are already present (see section 6.2).

These results can be loosely interpreted as follows. For λ low and entry high, competition is intense, so that there is ample room for an increase in λ to soften it and increase individual profit in the Cournot game. For λ high, pricing competition is already soft enough so that the increase in the internalization of the entry externality (due to an increase in λ) dominates and entry decreases with λ.

Under $\Delta > 0$, when entry is decreasing in λ, the price will be increasing in λ, since both the increase in λ and the resulting decrease in entry tend to increase the price. On the other hand, for low levels of overlapping ownership and not too low entry, overlapping ownership spurs entry (up to the point where λ is too high and then entry decreases with it). However, Numerical Result 1 asserts that with CRS the direct effect of λ on the total quantity dominates, so that the price always increases with λ.

Last, a few words on the interpretation of this comparative statics exercise on a change in λ are in place. Strictly speaking, this exercise amounts to changing the level of overlapping ownership *before* firms make their entry decisions. Therefore, it can be thought of as a counterfactual or a comparison of otherwise similar markets that have different levels of overlapping ownership (before firms enter). When interpreting changes in λ in a market where firms have already entered, one should consider the following. If our model predicts that a change in λ will cause the number of firms to fall, whether incumbent firms will indeed exit can depend on the extent to which the entry cost f is a sunk cost or a fixed operating cost that they can avoid by exiting.

[29] This means that the average value of λ (across pairs of firms) has surpassed $0.4 - 0.5$ in recent years. Clearly, the average λ estimate depends on the particular corporate control assumptions. Also, notice that we compare the average value of λ to the threshold of λ in our model of symmetric firms and overlapping ownership structure.

5.2 Overlapping ownership effects: the linear-quadratic model

We examine in this section the case of linear demand and quadratic costs under DRS, CRS or IRS and the impact of λ on entry, prices, quantities and welfare.

In Figure 2(a), for λ low, the rise in own profit due to increases in λ dominates the other two channels—in this parametrization the magnitude of the entry externality increases with λ.[30] However, for high λ competition in the product market is already soft enough, so that further increases in λ suppress entry.

Within the simple framework of linear demand and CRS or quadratic costs, equilibrium total surplus can behave in multiple different ways as the extent of overlapping ownership changes.[31] (i) It can have a U relationship with λ (Figure 2(a)), so that an intermediate level of overlapping ownership actually minimizes total surplus. (ii) It can have an inverted-U relationship with λ (Figure 7(b) in the appendix) with an intermediate level of overlapping ownership maximizing total surplus. (iii) It can be monotonically increasing (Figure 7(a) in the appendix) in λ with complete indexation of the industry being optimal ($\lambda = 1$). (iv) Last, it can be decreasing (Figure 7(c) in the appendix) in λ with $\lambda = 0$ maximizing total surplus. With CRS, the average variable cost of production is not affected by entry. Thus, the planner has to balance the effect of overlapping ownership—direct (in the Cournot game), and indirect (through its effect on entry)—on the total quantity and price, and on the total entry costs (through its effect of entry). For low entry cost (Figure 2(a)), the planner chooses no overlapping ownership, while for higher entry cost, intermediate values of λ (Figure 7(b) in the appendix) or even $\lambda = 1$ can be optimal (Figure 7(a) in the appendix). With IRS (Figure 2(b)), an increase in overlapping ownership can both decrease the price and increase total surplus. Particularly, choosing λ high enough (e.g., $\lambda = 1$)—inducing a monopoly—is socially optimal under both a total surplus and a consumer surplus standard. Also, a planner that only controls overlapping ownership will still achieve what a planner that can control both would (*i.e.*, a monopoly).

Corollary 4.1 studies how entry, the total quantity and total surplus change with overlapping ownership around $\lambda = 0$. Figure 3 summarizes the results.

Corollary 4.1. Ignore the integer constraint on n (so that entry is given by $\widehat{n}^*(\lambda)$). Let

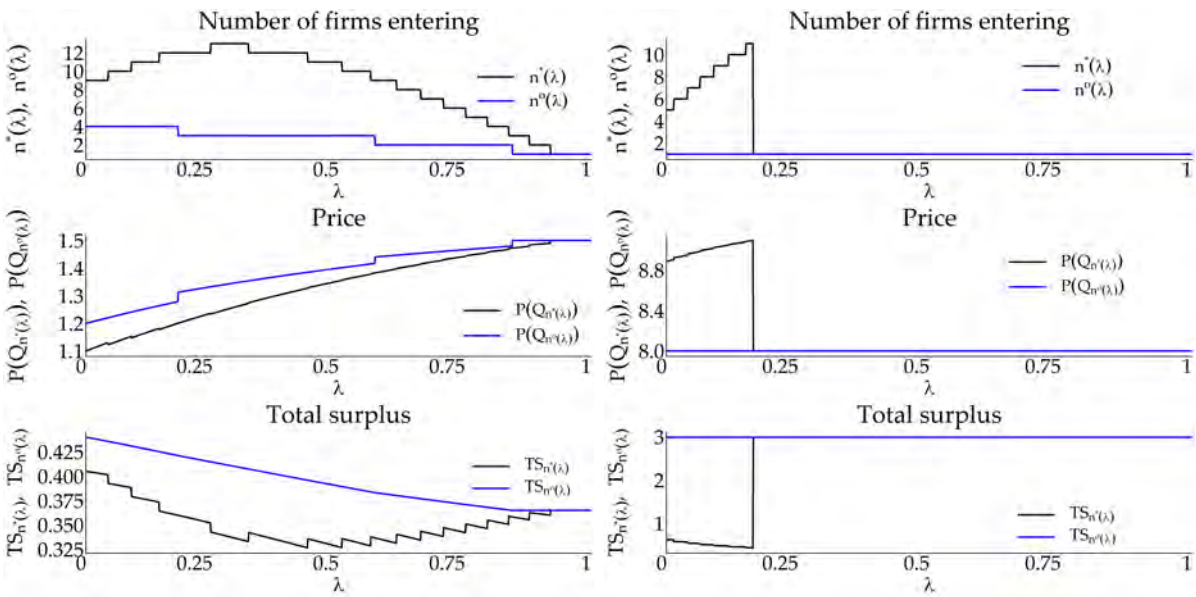
[30] Also, under the IRS parametrization of Figure 2(b), the number of firms increases in λ up to a point where it jumps to 1.

[31] See Figure 2, as well as Figure 7 in the appendix. In describing the relationship of equilibrium total surplus with λ we ignore the integer constraint on n, which is taken into account in the figures.

Figure 2: Equilibrium and planner outcomes for varying λ

(a) linear demand, CRS: $a = 2$, $b = c = 1$, $f = 0.01$

(b) linear demand, linear-quadratic costs (IRS): $a = 10$, $b = 1$, $c_1 = 9$, $c_2 = -3/2$, $f = 0.01$



Note: black lines represent values in equilibrium; blue represent values in the planner's solution.

demand be linear and cost be linear-quadratic with $a > c_1 \geq 0$, $c_2 > -2bc_1/a$, $c_2 \neq -b$, and assume $\widehat{n}^*(0) \geq 2$. Then, there exist thresholds $n(b,c_2) \in \mathbb{R}^3$ (that depend on b and c_2) such that

(i) if $c_2 > -3b/2$, then $n_3(b,c_2) > n_2(b,c_2) > 2$ and starting from $\lambda = 0$: (a) entry is locally increasing (resp. decreasing) in λ if $\widehat{n}^*(0) \overset{\text{(resp. <)}}{>} n_3(b,c_2)$, (b) the total quantity is locally decreasing in λ,

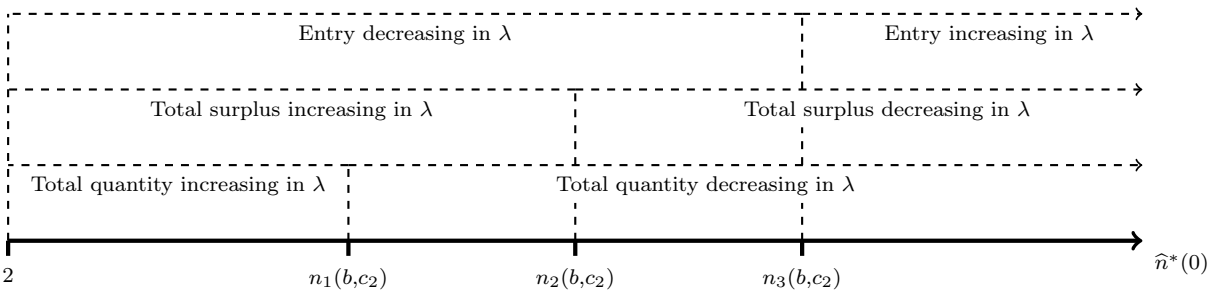
(ii) if $c_2 < -3b/2$, then $n_3(b,c_2) > n_2(b,c_2) > n_1(b,c_2) > 2$ and starting from $\lambda = 0$: (a) entry is locally increasing (resp. decreasing) in λ if $\widehat{n}^*(0) \overset{\text{(resp. <)}}{>} n_3(b,c_2)$, (b) the total quantity is locally increasing (resp. decreasing) in λ if $\widehat{n}^*(0) \overset{\text{(resp. >)}}{<} n_1(b,c_2)$,

(iii) the total surplus is locally increasing (resp. decreasing) in λ if $\widehat{n}^*(0) \overset{\text{(resp. >)}}{<} n_2(b,c_2)$.

Parts (i-a) and (ii-a) of the Corollary extend our finding that if without overlapping ownership many (resp. few) firms enter, then marginally increasing overlapping ownership will increase (resp. decrease) entry.

Parts (i-b) and (ii-b) show that introducing a small amount of overlapping ownership may only increase the total quantity if there are significant IRS (so that the Cournot market is quasi-anticompetitive) and entry is low. In that case, the softening of pricing

Figure 3: Comparative statics around $\lambda = 0$ under linear demand and linear-quadratic cost



Note: see Corollary 4.1 for precise statement. $n_1(b,c_2) > 2$ only under significant IRS.

competition due to the increase in overlapping ownership is dominated by the concurrent decrease in entry—which tends to increase the total quantity since the market is quasi-anticompetitive. This yields a sufficient condition for consumer surplus to be maximized by some $\lambda > 0$. As shown in Figure 2(b), this condition is not necessary, since with IRS a positive level of overlapping ownership can be optimal under a consumer surplus standard even when overlapping ownership decreases the total quantity around $\lambda = 0$.

Part (iii) shows that marginally increasing λ above 0 increases total surplus if and only if entry is low. Particularly, the direct (negative) effect of an increase in λ on total surplus is dominated by the alleviation of excessive entry (since for $\lambda = 0$ entry is excessive) due to the increase in λ. We thus obtain another sufficient condition: if absent overlapping ownership, entry would be low, then a planner that regulates overlapping ownership (but not entry) should choose a positive level of it.

5.3 Entry cost effect on entry

Proposition 5 below studies the effect of the entry cost on entry, as well as how this effect depends on the extent of overlapping ownership (with the level of entry held fixed). Note that λ affects the slope $d\widehat{n}^*(\lambda)/df$ directly but also through its effect on $n^*(\lambda)$. We are interested in the direct effect so we keep $n^*(\lambda)$ fixed as we vary λ.

Proposition 5. Ignore the integer constraint on n (so that entry is given by $\widehat{n}^*(\lambda)$). Then

(i) entry is decreasing in the entry cost,

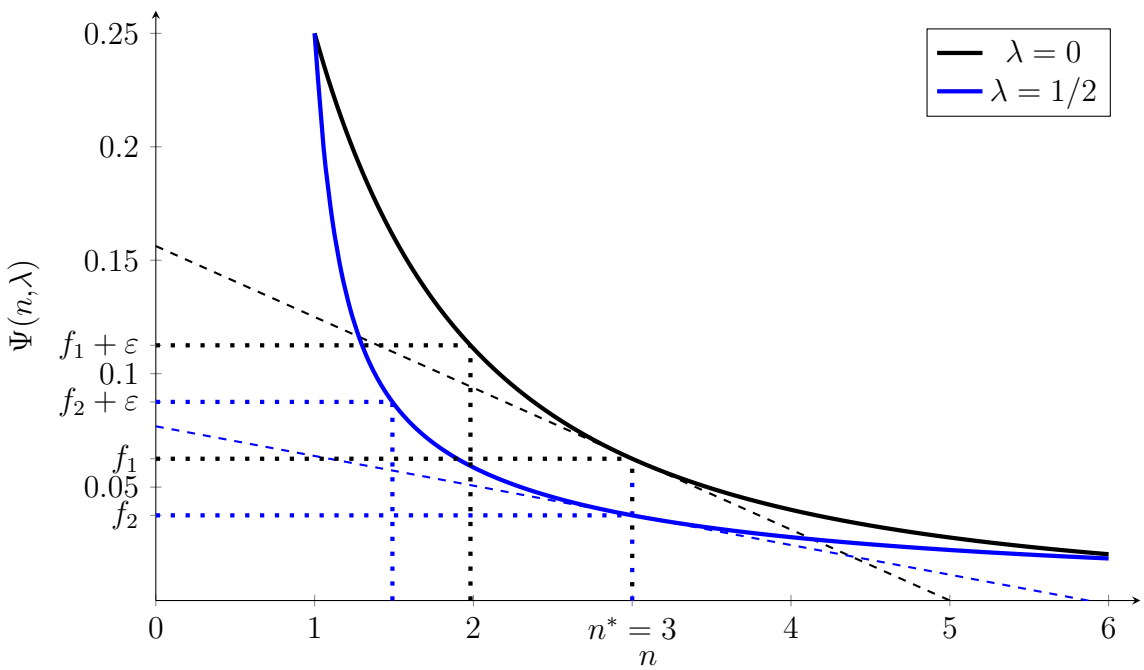
(ii) if λ increases and other parameters x (e.g., demand, cost parameters) change infinitesimally so that $\widehat{n}^*(\lambda)$ stays fixed and $\partial^2\Psi(n,\lambda)/(\partial x \partial n) = 0$ (e.g., (f,λ) infinitesimally

changes in direction $\mathbf{v} := (-(d\widehat{n}^*(\lambda)/d\lambda)/(d\widehat{n}^*(\lambda)/df),1))$, then $|d\widehat{n}^*(\lambda)/df|$ changes in direction given by $\text{sgn}\left\{\partial^2\Psi\left(n,\lambda\right)/(\partial\lambda\partial n)|_{n=\widehat{n}^*(\lambda)}\right\}$.

As long as $\Psi(n,\lambda)$ is decreasing in n, the results of Proposition 5 are not specific to Cournot competition. Part (ii) says that if an increase in λ makes the internalized profit in the pricing stage equilibrium more (resp. less) strongly decreasing in the number of firms, then an increase in the entry cost needs to be met with a smaller (resp. larger) increase (resp. decrease) in the number of firms for the condition $\Psi\left(\widehat{n}^*(\lambda),\lambda\right) = f$ to continue to hold.

Figure 4 explains the reasoning behind this. There are initially $n^* = 3$ firms, which can be a result of $\lambda = 0$ and $f = f_1$, or $\lambda = 1/2$ and $f = f_2$. Also, an increase of λ from 0 to 1/2 makes the internalized profit less strongly decreasing in n (*i.e.*, $\partial^2\Psi\left(n,\lambda\right)/(\partial\lambda\partial n) > 0$).[32] Thus, an increase in the entry cost by ε will decrease entry by more when $\lambda = 1/2$ (and initially $f = f_2$) compared to when $\lambda = 0$ (and initially $f = f_1$).

Figure 4: Entry cost effect on entry mediated by λ under linear demand and CRS



Note: $a = 2$, $b = 1$, $c = 1$. The black and blue solid lines represent $\Psi(n,0)$ and $\Psi(n,1/2)$, respectively. The black and blue dashed lines are tangent to the corresponding solid lines at $n = n^*$.

Numerical result 2 provides conditions under which the cross derivative of $\Psi(n,\lambda)$

[32]For example, at $n = 3$, the slope of $\Psi(n,1/2)$ (see the tangent blue dashed line) is smaller in absolute value than the slope of $\Psi(n,0)$ (see the tangent black dashed line). $\Psi(n,\lambda)$ decreases less strongly with n when $\lambda = 1/2$ than when $\lambda = 0$.

is positive, which by Proposition 5 implies that overlapping ownership exacerbates the negative effect of the entry cost on entry.

Numerical Result 2. Under CESL demand and CRS, $\partial^2\Psi(n,\lambda)/(\partial\lambda\partial n) > 0$ if (i) $E \in (1,1.7]$ and $n \in [2,7]$, or (ii) $E < 1$ and $n \in [2,8]$.

Under CESL demand and CRS, markets with low entry are particularly susceptible to further decreases in entry when there is overlapping ownership. In such markets, apart from the direct effect it has on entry, overlapping ownership also makes entry more strongly decreasing in the entry cost. This means that overlapping ownership could exacerbate the negative macroeconomic implications of rising entry costs documented by Gutiérrez et al. (2021) in the U.S. over the past 20 years. Empirical estimates for various markets place E in the range specified in Numerical Result 2.[33]

5.4 Equilibrium entry versus the socially optimal level of entry

We now study whether equilibrium entry is excessive or insufficient TS_n is single-peaked and ignoring the integer constraint on n.[34] The internalization of the entry externality makes the analysis significantly different from the one without overlapping ownership. We have that

$$\left.\frac{d\,\mathrm{TS}_n}{dn}\right|_{n=\widehat{n}^*(\lambda)} = \Pi\left(\widehat{n}^*(\lambda),\lambda\right) - f - \left(1+\lambda(n-1)\right) Q_n P'(Q_n) \left.\frac{\partial q_n}{\partial n}\right|_{n=\widehat{n}^*(\lambda)}$$

$$\propto \underbrace{\lambda \frac{\Xi\left(\widehat{n}^*(\lambda),\lambda\right)}{\Pi\left(\widehat{n}^*(\lambda),\lambda\right)}}_{\substack{>0 \\ \text{internalized} \\ \text{normalized} \\ \text{entry externality}}} + \underbrace{\left(1 - \frac{E_C\left(q_{\widehat{n}^*(\lambda)}\right) - 1}{\dfrac{P\left(Q_{\widehat{n}^*(\lambda)}\right)}{AC\left(q_{\widehat{n}^*(\lambda)}\right)} - 1}\right)}_{\substack{>0 \\ \text{if DRS, then } < 1 \\ \text{if CRS, then } = 1 \\ \text{if IRS, then } > 1;}} \underbrace{\left.\frac{\partial q_n}{\partial n}\frac{n}{q_n}\right|_{n=\widehat{n}^*(\lambda)}}_{\substack{<0 \text{ if } E_{P'}\left(Q_{\widehat{n}^*(\lambda)}\right)<2 \\ \text{and } \widehat{n}^*(\lambda)\geq 2; \\ \text{if negative, the higher} \\ \text{in absolute value,} \\ \text{the stronger the} \\ \text{business-stealing effect}}}, \tag{4}$$

where we have used the $\Psi\left(\widehat{n}^*(\lambda),\lambda\right) = f$ entry condition and the pricing formula (1). Let us have a closer look at the two terms in the above expression. $\Xi(n,\lambda)/\Pi(n,\lambda) \equiv$

[33] See for example Duso and Szücs (2017), Mrázová and Neary (2017) and Bergquist and Dinerstein (2020).

[34] Lemma 2 in Appendix B provides sufficient conditions for it to be concave. Ignoring the integer constraint is not as important, since as we will see both cases of excessive and insufficient entry are possible (and by more than one firm under the integer constraint as can be seen in Figure 5).

$(n-1)(\Pi(n-1,\lambda)-\Pi(n,\lambda))/\Pi(n,\lambda)$ is the normalized entry externality.[35]

$$1-\frac{E_C(q)-1}{\frac{P(nq)}{AC(q)}-1}=\frac{P(nq)-C'(q)}{P(nq)-AC(q)}>0$$

is a (coarse) measure of the elasticity of the cost function, and thus of the extent to which returns to scale are decreasing or increasing.[36] For example, under constant elasticity costs, $(E_C(q)-1)/(P(nq)/(AC(q))-1)$ is higher than (resp. lower than/equal to) 0 if and only if returns to scale are decreasing (resp. increasing/constant).

We see then that whether entry is excessive or insufficient will depend on (i) the level of overlapping ownership λ, (ii) the magnitude of the normalized entry externality, (iii) whether returns to scale are increasing or decreasing and to what extent, and (iv) whether competition is business-stealing or business-enhancing, and to what degree $\left|\frac{\partial q_n}{\partial n}\frac{n}{q_n}\right|$.

Under business-stealing competition and all else constant, we distinguish the following forces. Increases in the level of overlapping ownership or the magnitude of the entry externality, tend to make entry insufficient; these forces are complements in inducing insufficient entry. Also, DRS tend to make entry insufficient, since the planner takes advantage of variable-cost savings due to entry to a greater extent than firms do. Firms do not fully internalize the variable cost-savings of entry (except in the case of complete indexation) and at the same time more strongly consider the effect of entry on profits through the price. On the other hand, increases in the magnitude of the business-stealing effect and IRS tend to make entry excessive.

Under business-enhancing competition, entry is always insufficient. This can be seen as a generalization of the result of Amir et al. (2014), who prove that entry is insufficient under business-enhancing competition and $\lambda=0$.

We now formally compare equilibrium entry $\widehat{n}^*(\lambda)$ to the socially optimal level of entry $\widehat{n}^o(\lambda)$. First, define

$$\phi(n,\lambda):=\frac{(n-1)(\Pi(n,\lambda)-\Pi(n-1,\lambda))}{n\partial\Pi(n,\lambda)/\partial n}\approx 1,$$

which is positive, since the numerator and denominator are negative. It is close to 1, since (i) by the mean value theorem $\Pi(n,\lambda)-\Pi(n-1,\lambda)=\partial\Pi(\nu,\lambda)/\partial\nu|_{\nu=\nu^*}$ for some real

[35] It is equal to $(n-1)$ times the percentage increase in the profit of each of the $n-1$ other firms when the n-th firm decides *not* to enter compared to the case where it did enter.

[36] It is positive by the pricing formula (1) and since gross profit is positive in equilibrium.

number $\nu^* \in [n-1,n]$, and (ii) $(n-1)/n \approx 1$ for n not too small.[37]

Proposition 6. Assume that TS_n is single-peaked in n, $1-\lambda\phi\left(\widehat{n}^*(\lambda),\lambda\right) > 0$. Then $\widehat{n}^*(\lambda) \overset{\text{(resp. <)}}{>} \widehat{n}^o(\lambda)$ if and only if

$$E_{P'}\left(Q_{\widehat{n}^*(\lambda)}\right) \overset{\text{(resp. >)}}{<} H_n^{-1}\left[1+\lambda\left(1-\frac{\phi\left(n,\lambda\right)}{1-\lambda\phi\left(n,\lambda\right)}\frac{\Delta(Q_n,(n-1)\,q_n)}{1+\lambda(n-1)}\right)\right]\Bigg|_{n=\widehat{n}^*(\lambda)}.$$

Substituting $\lambda = 0$ we recover the standard excessive entry result. Entry is excessive if and only if $E_{P'}\left(Q_{\widehat{n}^*(\lambda)}\right) < \widehat{n}^*(\lambda)$, which is indeed satisfied under standard assumptions on demand.[38] Proposition 6 also asserts that entry is still excessive unless there are DRS (with C'', and thus Δ, high).[39] Indeed, the following remarks show this in more detail.

Remark 5.3. If $\Delta(Q_{\widehat{n}^*(\lambda)},(n-1)q_{\widehat{n}^*(\lambda)}) \leq \frac{[1+\lambda(\widehat{n}^*(\lambda)-1)][1-\lambda\phi(\widehat{n}^*(\lambda),\lambda)]}{\phi(\widehat{n}^*(\lambda),\lambda)}$ and $E_{P'}\left(Q_{\widehat{n}^*(\lambda)}\right) < H^{-1}_{\widehat{n}^*(\lambda)}$, then $\widehat{n}^*(\lambda) > \widehat{n}^o(\lambda)$.

Remark 5.4. If for simplicity we let $\phi\left(\widehat{n}^*(\lambda),\lambda\right) = 1$, then if $C''(q_{\widehat{n}^*(\lambda)}) \leq 0$, $\widehat{n}^*(\lambda) \geq 2$ and $E_{P'}\left(Q_{\widehat{n}^*(\lambda)}\right) < 2-\left(\widehat{n}^*(\lambda)\right)^{-1}$, then $\widehat{n}^*(\lambda) > \widehat{n}^o(\lambda)$.

Remark 5.5. If $\Delta < 0$, then (i) $n^o(\lambda) = 1$ for any $\lambda \in [0,1]$, as $n = 1$ maximizes both Q_n and $n\Pi(n,\lambda)$, and (ii) for $\lambda = 1$, $n^*(1) = n^o(1) = 1$.

On the other hand, under DRS and high levels of overlapping ownership, entry is insufficient. The numerical simulations in Figure 5 verify the result. Particularly, Remark 5.6 shows that all else fixed, an increase in the elasticity E_C of the cost function tends to make entry insufficient.

Remark 5.6. If $C'' > 0$, there exists $\widetilde{q} \in [0,q_{\widehat{n}^*(\lambda)}]$ such that

$$\frac{\Delta(Q_n,(n-1)\,q_n)}{1+\lambda(n-1)} = \frac{1-\lambda}{1+\lambda(n-1)} + \frac{C''(q_n)}{C''(\widetilde{q})}\left(\frac{\frac{P(Q_n)-C'(0)}{AC(q_n)}}{\frac{P(Q_n)}{AC(q_n)}-E_C(q_n)}-1\right),$$

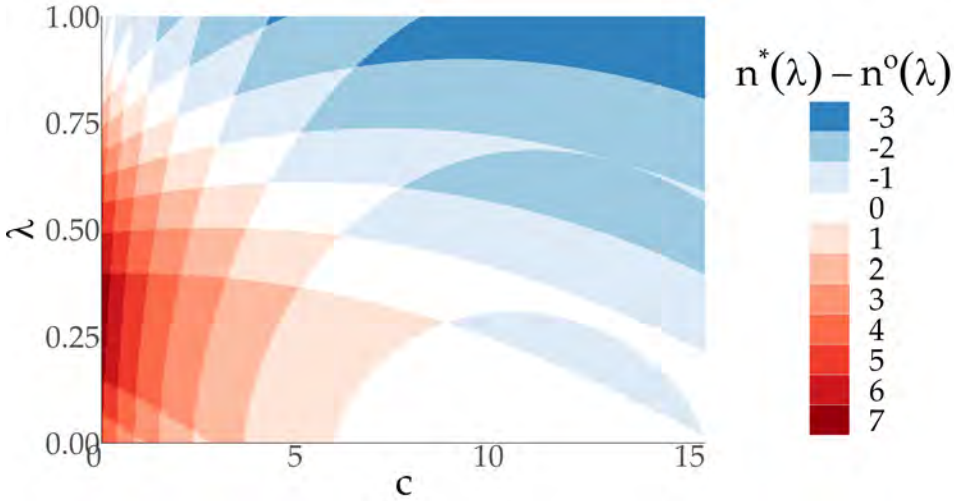
[37] For $\Pi\left(n,\lambda\right)$ strictly convex in n, in which case individual profit decreases with n at a decreasing (in absolute value) rate, $\left(\Pi\left(n,\lambda\right)-\Pi\left(n-1,\lambda\right)\right)/\left(\partial\Pi\left(n,\lambda\right)/\partial n\right) > 1$, which counterbalances $(n-1)/n < 1$. The numerical results of Figure 8 in the Appendix verify that $\phi\left(n,\lambda\right) \approx 1$.

[38] Indeed, given $\widehat{n}^*(\lambda) \geq 2$ and $\Delta > 0$, Proposition 2 asserts that the total quantity in the pricing stage is increasing in n and competition is business-stealing, which are the conditions under which Mankiw and Whinston (1986) show excessive entry. However, we see that $\Delta > 0$ is not necessary, consistent with Amir et al. (2014), who show excessive entry under business-stealing competition and $\Delta > 0$ or $\Delta < 0$.

[39] Notice that the right-hand side in Proposition 6 is decreasing in Δ. The results of Proposition 6 closely resemble those of Proposition 17—which compares $\widehat{n}^*(\lambda)$ and $\widehat{n}^o(\lambda)$ in the model where firms' entry decisions are based on a differential version of (2)—in section B.11 of Appendix B. The difference is that Proposition 6 requires the correction term ϕ, which is replaced with exactly 1 in Proposition 17.

evaluated at $n = \widehat{n}^*(\lambda)$, so (all else constant) the right-hand side in the condition of Proposition 6 is decreasing in $E_C(q_n)$.

Figure 5: Equilibrium versus socially optimal entry under linear demand and quadratic costs



Note: $a = 2$, $b = 1$, $f = 0.05$.

Remark 5.7 shows that if instead of a total surplus, the planner follows a consumer surplus standard, then entry is insufficient (resp. excessive) when returns to scale are at most mildly increasing (resp. sufficiently increasing). In the case of significant IRS (*i.e.*, $\Delta < 0$), as we have already observed in Figure 2(b), an increase in overlapping ownership can stop excessive entry making the market achieve the planner's solution (under both a consumer and a total surplus standard).

Remark 5.7. Under a consumer surplus standard

(i) if $\Delta > 0$, then $n^o(\lambda) = \infty$ (since Q_n is increasing in n), so $n^*(\lambda) < n^o(\lambda)$,

(ii) if $\Delta < 0$, then $n^o(\lambda) = 1$ (since Q_n is decreasing in n), so $n^*(\lambda) \geq n^o(\lambda)$.

Last, Remark 5.8 studies the case of linear demand and linear-quadratic cost.

Remark 5.8. Let $\lambda = 1$, demand be linear and cost be linear-quadratic with $a > c_1 \geq 0$ and $c_2 > -2bc_1/a$. If $c_2 > 0$, then $\Delta > 0$ and there exists $\overline{f} > 0$ such that $\widehat{n}^*(\lambda) \overset{\text{(resp. <)}}{>} \widehat{n}^o(\lambda)$ if and only if $f \overset{\text{(resp. <)}}{>} \overline{f}$, where $\overline{f}$ is increasing (resp. decreasing) in $a - c_1$ (resp. b) and single-peaked in c_2 with the peak at $c_2 = 3b$. If instead $c_2 \leq 0$, then $\Delta \leq 0$ and $n^*(1) = n^o(1) = 1$.

In a fully-indexed industry ($\lambda = 1$) a monopoly arises in the free entry equilibrium under non-DRS, which coincides with the entry-controlling planner's solution. However, if there are DRS, then entry is insufficient when the entry cost f is low, the market is large (*i.e.*, $a - c_1$ high) and/or the demand is highly elastic (*i.e.*, b low).

6 Extensions and robustness

6.1 Post-entry overlapping ownership

Post-entry overlapping ownership applies to the case of a new industry that is to mostly be populated by start-ups without overlapping ownership that will develop ownership links after entry.[40] In this case, firms do not internalize the negative externality their entry has on other firms, as in the standard Cournot model with free entry. Thus, modulo the integer constraint on the number of firms, firms enter until the individual gross profit is equal to the entry cost, so that in equilibrium the net profit is zero. Nevertheless, when deciding whether to enter, they take into account how overlapping ownership will affect product market outcomes. Naturally, an increase in the degree of post-entry overlapping ownership spurs entry, since it tends to increase profits by softening pricing competition (see Proposition 2). However, section B.9 in the Appendix (which studies the model with *post-entry* overlapping ownership) shows that the anti-competitive effect of overlapping ownership prevails causing price to increase and total surplus to fall. The results on the effects of post-entry overlapping ownership are schematically summarized in Figure 6.

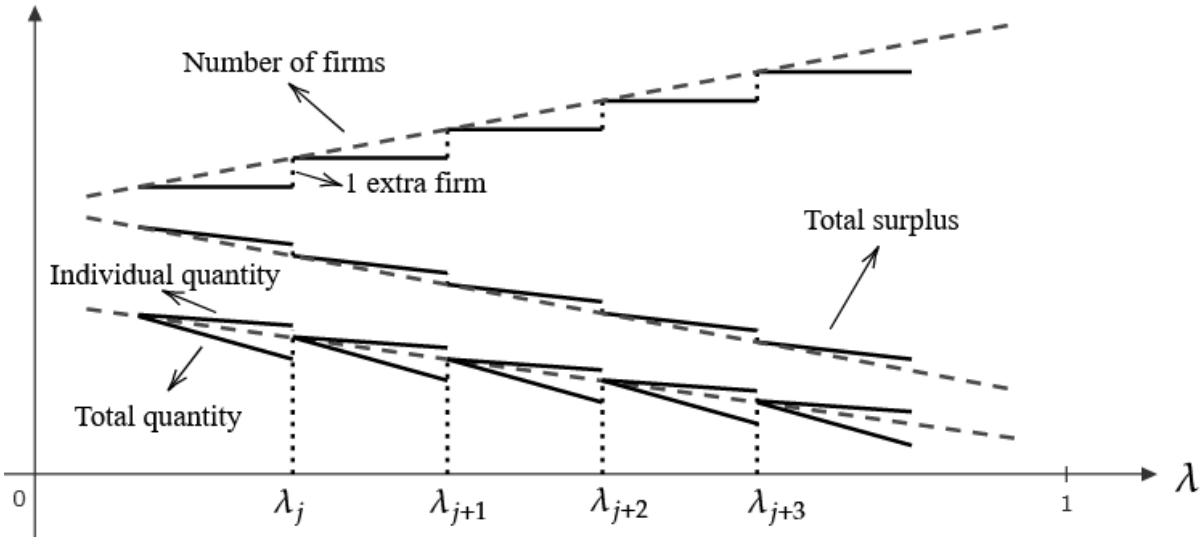
The comparison between the equilibrium and the optimal level of entry also changes since the result on the tendency for excessive entry under business-stealing competition generalizes. Entry is never insufficient by more than one firm (compared to the level of entry chosen by a planner that can regulate entry but not the extent of overlapping ownership) as in the standard Cournot model with free entry (see Mankiw and Whinston, 1986; Amir et al., 2014).

6.2 Entry under the presence of maverick firms

We have examined the effects of overlapping ownership under a symmetric overlapping ownership structure. In that context, overlapping ownership can suppress entry by

[40]The post-entry overlapping ownership case can also be interpreted to address pre-entry overlapping ownership but with overlapping ownership not causing firms to internalize their entry externality.

Figure 6: Equilibrium with post-entry overlapping ownership for varying λ



Note: the solid lines represent equilibrium values under the integer constraint; from bottom to top they represent the behavior of total quantity, individual quantity, total surplus, and the number of firms. The dashed lines represent equilibrium values when we ignore the integer constraint; from bottom to top the first line represents the behavior of both the total and the individual quantity, the second of total surplus, and the third of the number of firms. The solid total quantity line is drawn for the case $\Delta > 0$. The solid individual quantity is drawn above the dashed one given that q_n is decreasing in n (see Proposition 2). To draw the solid total surplus line above the dashed ones we assume that the total surplus is single-peaked in n, and that $n^*(\lambda) \geq n^o(\lambda)$ (see footnote 58 and Proposition 13 in the Appendix). Only the signs of the slopes of the lines and the directions of the jumps are part of the result; the curvatures of lines and spacing of the jumps have been chosen for simplicity in depiction.

inducing firms to internalize the negative externality that their entry would have on other firms. However, if there are also potential entrants without ownership ties—which we call maverick firms, then limited entry by commonly-owned firms may spur entry by maverick ones. This could enhance the incentives of a commonly-owned firm to enter.

In section B.8 of the Appendix we model the maverick firms as a competitive fringe that in the first stage (where oligopolists enter) submit an aggregate supply schedule. We show that the (prospect of) entry by maverick firms essentially changes the demand faced by the commonly-owned firms by depressing it and making it more elastic. With demand adjusted accordingly, the results of the previous sections on the effects of overlapping ownership on entry and the price continue to hold (with the number of firms n not counting maverick firm entry), as does the comparison between the equilibrium and socially optimal levels of entry. Since demand is depressed, we expect lower levels of entry by commonly-owned firms. Also, given that higher (resp. lower) entry by commonly-owned firms leads to lower (resp. higher) entry by maverick firms, we expect entry to be less sensitive to overlapping

ownership due to the presence of the maverick firms.[41] This is indeed verified in section B.8.

7 Conclusion

In this paper we have studied the effects of overlapping ownership in a Cournot oligopoly with free entry. Potential entrants are established firms with overlapping ownership and decide whether to enter a new industry or product market. We derive four main results.

First, an increase in overlapping ownership affects entry through three separate channels. It increases the degree of internalization of the negative externality of entry on other firms' profits, which tends to limit entry. However, it also increases equilibrium profits in the pricing stage, which tends to increase entry. Last, it changes the magnitude of the entry externality on other firms' profits; this channel can affect entry in either direction. With Cournot competition in the pricing stage an increase in the degree of overlapping ownership can limit or—in contrast to prior theoretical and empirical work—spur entry.

The three channels through which overlapping ownership affects entry are not specific to the assumption of Cournot competition. Nevertheless, their direction (especially of the ambiguous channel) and magnitudes can vary with the form of competition. For example, (keeping the number of firms fixed) overlapping ownership may increase profits in a differentiated products market by less than it does in the Cournot game, since with differentiated products competition is already less intense. At the same time, the magnitude of the entry externality may also be diminished. In the extreme case of independent monopolies, overlapping ownership would not affect profits at all.

Second, given the negative macroeconomic implications of rising entry costs documented by Gutiérrez et al. (2021) in the U.S. economy over the past 20 years, we are interested in how overlapping ownership mediates the negative effect of entry costs on entry. We find that overlapping ownership exacerbates the negative impact of an increase of entry costs on entry.

[41] That is because all channels through which λ affects entry will diminish in magnitude when there are maverick firms. First, pricing-stage profit will not increase as strongly with λ, because maverick firms will produce more when oligopolists reduce production as λ increases. In other words, when maverick firms are present, the demand that the oligopolists face is more elastic, so the externality that one oligopolist imposes on the others by producing—thereby pushing down the price—is lower. Thus, as λ increases there is a smaller externality to be internalized in the pricing stage, so the effect of λ on Cournot profit is milder. Second, the entry externality is also lower, since by entering an oligopolist limits the mavericks' entry. Hence, there is a smaller entry externality to be internalized due to an increase in λ.

Third, apart from entry we also study welfare since as Bar-Isaac (2016) notes, the extent of entry is not a sufficient statistic for welfare and the efficiency of a market. We find that entry is excessive under non-DRS. However, with DRS entry will tend to be insufficient in industries with (i) high levels of overlapping ownership, (ii) strong (negative) effects of entry on other firms' profits, (iii) business-enhancing competition.[42] Still, there is a channel through which overlapping ownership can enhance efficiency that is not captured in our model. In a model with asymmetric—especially with regard to production costs—firms, an increase in overlapping ownership can improve production efficiency by causing production to shift towards the more efficient firms.[43]

Out last finding is that in industries with significant barriers to entry—so that the number of firms is fixed—overlapping ownership damages welfare when firms are symmetric (with symmetric firms no efficiency gains due to production shifting to more efficient firms are possible). However, in industries with free entry and symmetric firms a welfare-maximizing planner that can only regulate overlapping ownership and then allow firms to freely enter may choose any level of overlapping ownership, from none at all to complete indexation of the industry. For example, when the entry cost is high, a higher level of overlapping ownership suppresses entry and reduces total entry costs; this effect can be strong enough to make high levels of overlapping ownership welfare-maximizing. Particularly, under increasing returns to scale, complete indexation of the industry will lead to a monopoly, which can be optimal not only under a total welfare standard but also under a consumer surplus standard. Therefore, regulation of overlapping ownership should take into account its effect not only on product market competition but also on entry.

Further, we derive the following *testable implications* for markets that existing firms with overlapping ownership consider entering. First, for low levels of overlapping ownership, an increase in overlapping ownership will (i) increase entry if there are many firms in the market already (low market concentration), but (ii) it will decrease entry if there are only few firms in the industry. Second, for high levels of overlapping ownership, further increases in it will suppress entry. Thus, entry will either depend negatively on overlapping

[42]Particularly, in a completely indexed industry where firms maximize aggregate industry profits, entry is insufficient under DRS when entry costs are low and/or demand is large and elastic.

[43]That is because an increase in production by a firm has—regardless of that firm's identity—the same negative externality on other firms (to be internalized as overlapping ownership increases), while the positive effect on the firm's own profit is larger when the firm has a higher profit margin.

ownership or have an inverted-U relationship with it. Third, unless there are increasing returns to scale, an increase in the extent of overlapping ownership will increase the price. Fourth, increases in the entry cost can suppress entry more in industries with higher levels of overlapping ownership. Fifth, entry by commonly-owned firms is more responsive to the level of overlapping ownership in industries where the prospect of entry by firms without ownership ties to incumbents is less salient.

Finally, given that the extent to which ownership ties affect firm conduct is an open empirical question, our results suggest a test of the common ownership hypothesis. If the common ownership hypothesis fails completely—so that common ownership neither affects pricing decisions nor causes firms to internalize their entry externality, then entry (and other market outcomes) should be independent of common ownership. If the common ownership hypothesis is only *partially* correct in the sense that common ownership influences pricing behavior but does *not* cause the entry externality to be internalized, then entry is expected to increase with the level of common ownership. Finally, if the common ownership hypothesis is correct (*i.e.*, common ownership affects firm conduct in both ways), then entry is expected to either depend negatively on common ownership or have an inverted-U relationship with it.

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A Appendix

A.1 Additional simulation results

Figure 7(a,b) shows that for high enough levels of the entry cost, a planner that regulates overlapping ownership (but not entry) may choose positive levels of it or even complete indexation ($\lambda = 1$). Figure 7(c) shows that under DRS more than one firm may enter in equilibrium even when $\lambda = 1$.

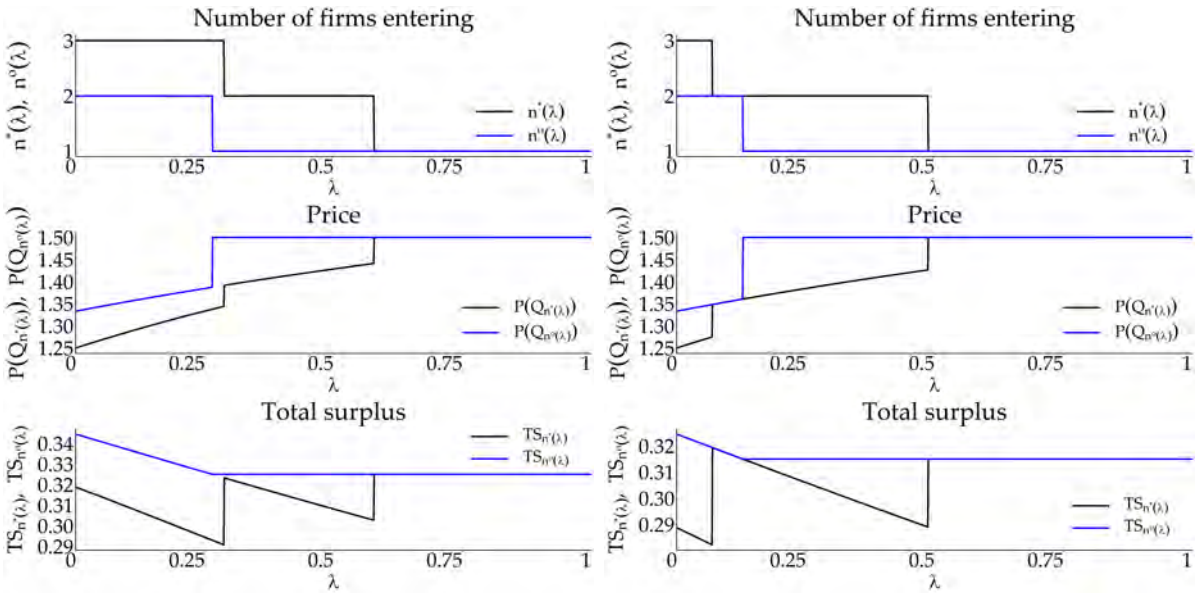
A.2 Some commonly used conditions

Lemma 1 below provides necessary and sufficient conditions for some of our standard assumptions. The proof is elementary and therefore omitted.

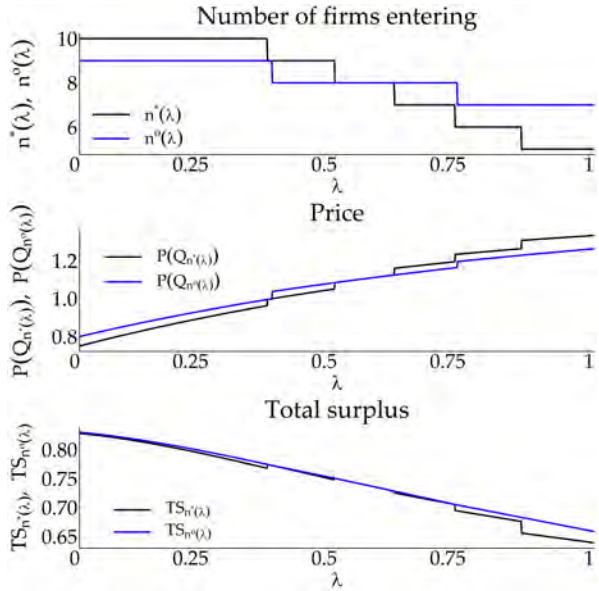
Lemma 1. The following hold:

Figure 7: Equilibrium and planner outcomes for varying λ

(a) linear demand, CRS: $a = 2$, $b = c = 1$, $f = 0.05$ **(b)** linear demand, CRS: $a = 2$, $b = c = 1$, $f = 0.06$



(c) linear demand, quadratic costs (DRS): $a = 2$, $b = 1$, $c = 5$, $f = 0.05$



Note: black lines represent values in equilibrium; blue represent values in the planner's solution.

(i) $\Delta(Q,Q_{-i}) > 0$ on L for every $\lambda \in [0,1)$ if and only if $C''(q) \geq 0$ for every $q < \overline{Q}$.

(ii) $E_{P'}(Q) < (1+\lambda)/H_n$ for every $n \in [2,+\infty)$ (resp. $n \in [1,2]$) and every $\lambda \in [0,1]$ if and only if $E_{P'}(Q) < 2$. (resp. $E_{P'}(Q) < 1$).

In the proofs to come, it will be useful to remember that if $\Delta > 0$ (resp. $\Delta < 0$), then

$$(1+\lambda+\Delta/n)/H_n = 1 + H_n^{-1} - \Lambda_n^{-1} C''(q)/P'(Q)$$

$$\overset{\text{(resp. }\geq)}{\leq} (1+\lambda+\Delta/\Lambda_n)/H_n = 1 + H_n^{-1} + [(1-\lambda)(1-H_n) - C''(q)/P'(Q)]/(\Lambda_n H_n)$$
$$\overset{\text{(resp. }\geq)}{\leq} (1+\lambda+\Delta)/H_n = (2 - C''(q)/P'(Q))/H_n.$$

Also, $E_{P'}(Q) < \frac{1+\lambda+\Delta(Q,Q_{-i})}{1-(1-\lambda)(1-s_i)}$ on L implies that for any $n \in [1,+\infty)$ and any $Q < \overline{Q}$, $E_{P'}(Q) < (1+\lambda+\Delta(Q,(n-1)Q/n))/H_n$. Thus, part (ii) of the maintained assumption implies that when $\Delta < 0$, $E_{P'}(Q)$ is also lower than $(1+\lambda+\Delta/\Lambda_n)/H_n$ and $(1+\lambda+\Delta/n)/H_n$ in the symmetric equilibrium.

A.3 Proofs of section 3

Where clear we may simplify notation (e.g., omitting the subscript n).

Proof of Proposition 1 Wlog we can constrain attention to quantity profiles $\boldsymbol{q} \in \left\{\boldsymbol{x} \in [0,\overline{q}]^n : \sum_{i\in\mathcal{F}} x_i \leq \overline{Q}\right\}$. Also, the best response of firm i depends on $\boldsymbol{q}_{-i}$ only through Q_{-i}. Denote by $r(Q_{-i})$ the best response correspondence of a firm (the same for all firms). If it is a differentiable function, its slope is given by $r_i'(Q_{-i}) = -1 + \Delta(Q,Q_{-i})/[1+\lambda+\Delta(Q,Q_{-i}) - (s_i+\lambda(1-s_i))E_{P'}(Q)]$, for $q_i = r(Q_{-i})$. The proof is then similar to that of Theorem 2.1 in Amir and Lambson (AL; 2000).[44]

Case $\boldsymbol{\Delta > 0}$: We first prove statement (a).

Existence of symmetric equilibrium: Firm i's problem is equivalent to choosing the total quantity to be given by the correspondence $R : [0,\overline{Q}] \to [0,\overline{Q}]$ defined as

$$R(Q_{-i}) := \underset{Q\in[Q_{-i},Q_{-i}+\overline{q}]}{\arg\max} \{P(Q)[Q-(1-\lambda)Q_{-i}] - C(Q-Q_{-i})\} = r(Q_{-i}) + Q_{-i}.$$

taking Q_{-i} as given. The maximand above is strictly supermodular since $\Delta > 0$, so by Theorem A.1 in AL every selection from $R(Q_{-i})$ is non-decreasing in Q_{-i}. Thus, every selection of the correspondence $B : [0,(n-1)\overline{q}] \rightrightarrows [0,(n-1)\overline{q}]$ given by $B(Q_{-i}) := (n-1)R(Q_{-i})/n$ is also non-decreasing in Q_{-i}. By Tarski's fixed point theorem (Theorem A.3 in AL), B has a fixed point, which is a symmetric equilibrium.

Non-existence of asymmetric equilibria: Suppose by contradiction that an asymmetric equilibrium exists, and denote it by $\widetilde{\boldsymbol{q}}$. Then, any permutation of $\widetilde{\boldsymbol{q}}$ should also be an equilibrium, and since $\widetilde{\boldsymbol{q}}$ is asymmetric there exists a permutation $\widehat{\boldsymbol{q}}$ with a firm i such

[44] The proof of uniqueness under $\Delta > 0$ is not considered in AL but is also an extension of standard results.

that $\widehat{q}_i > \widetilde{q}_i$. But $\widetilde{Q} = \widehat{Q}$, so $\widehat{Q}_{-i} < \widetilde{Q}_{-i}$. Thus, $R\left(\widehat{Q}_{-i}\right) = R\left(\widetilde{Q}_{-i}\right) = \widetilde{Q} \geq \widetilde{Q}_{-i} > \widehat{Q}_{-i} \implies R\left(\widehat{Q}_{-i}\right) > \widehat{Q}_{-i}$, so $\widetilde{Q} = R\left(\widehat{Q}_{-i}\right)$ makes the first derivative of the firm's objective non-negative, that is $P\left(\widetilde{Q}\right) + P'\left(\widetilde{Q}\right)\left[\widetilde{Q} - (1-\lambda)\widehat{Q}_{-i}\right] - C'(\widetilde{Q} - \widehat{Q}_{-i}) \geq 0$. Also, since the firm's action space is not bounded from above, it trivially holds that $P\left(\widetilde{Q}\right) + P'\left(\widetilde{Q}\right)\left[\widetilde{Q} - (1-\lambda)\widetilde{Q}_{-i}\right] - C'(\widetilde{Q} - \widetilde{Q}_{-i}) \leq 0$. The last two inequalities imply

$$-(1-\lambda)P'\left(\widetilde{Q}\right) - \frac{C'(\widetilde{Q} - \widetilde{Q}_{-i}) - C'(\widetilde{Q} - \widehat{Q}_{-i})}{\widetilde{Q}_{-i} - \widehat{Q}_{-i}} \leq 0. \tag{5}$$

Last, since every selection from $R(Q_{-i})$ is non-decreasing in Q_{-i}, it follows from $R\left(\widehat{Q}_{-i}\right) = R\left(\widetilde{Q}_{-i}\right) = \widetilde{Q}$ that $R\left(Q_{-i}\right) = \widetilde{Q}$ for all $Q_{-i} \in [\widehat{Q}_{-i}, \widetilde{Q}_{-i}]$. Therefore, in (5) we can let $\widehat{Q}_{-i} \to \widetilde{Q}_{-i}$, which gives $\Delta(\widetilde{Q}, \widetilde{Q}_{-i}) \leq 0$, a contradiction.

For part (b) it remains to show that at most one symmetric equilibrium exists. $E_{P'} < (1+\lambda+\Delta)/H_n$ on L—which holds given that $E_{P'} < (1+\lambda+\Delta/n)/H_n$ and $\Delta > 0$ on L—implies that $\partial^2\left(\pi_i + \lambda\sum_{j\neq i}\pi_j\right)\Big/\left(\partial q_i\right)^2 < 0$, so that $r(Q_{-i})$ is a differentiable function. At a symmetric quantity profile we have $r'(Q_{-i}) = -1 + \Delta(Q, Q_{-i})/(1+\lambda+\Delta(Q, Q_{-i}) - H_n E_{P'}(Q))$. Symmetric equilibria are solutions to $g(q) \equiv r((n-1)q) - q = 0$. Thus, there will be at most one symmetric equilibrium if $g' < 0$, that is, if for any $q \in [0, \overline{Q}/n)$,

$$-\frac{1+\lambda - H_n E_{P'}(nq)}{1+\lambda+\Delta(nq,(n-1)q) - H_n E_{P'}(nq)} < \frac{1}{n-1} \iff E_{P'}(nq) < \frac{1+\lambda+\Delta(nq,(n-1)q)/n}{H_n}$$

which is true, since by assumption it is true on L.

Case $\boldsymbol{\Delta} < \mathbf{0}$: We first prove part (a) for $m = n$. $\Delta < 0$ and $E_{P'}(Q) < \frac{2 - C''(Q-Q_{-i})/P'(Q)}{1-(1-\lambda)(1-s_i)}$ implies that the objective function of each firm is strictly concave in its quantity (in the part where $P(Q) > 0$). Thus, for Q_{-i} such that $r(Q_{-i}) > 0$, $r(Q_{-i})$ is a differentiable function with slope $r_i'(Q_{-i}) = -1 + \Delta/(2 - C''(q_i)/P'(Q) - (s_i + \lambda(1-s_i))E_{P'}(Q)) < -1$ given $\Delta < 0$. Thus, again $g' < 0$ since $r' < -1 < (n-1)^{-1}$ for every $n \geq 2$. Also, $g(0) \geq 0$ and $\lim_{q\to\infty} g(q) = -\infty$, so by continuity of g there exists a unique symmetric equilibrium.

We now prove part (a) for $m < n$. Let q_m be the symmetric equilibrium quantity produced by each firm when m firms are in the market. The m firms are clearly best-responding by producing q_m each. Also, $r'(Q_{-i}) < -1$ (when $r(Q_{-i}) > 0$) implies that $r(mq_m) = r((m-1)q_m + q_m) \leq \max\{r((m-1)q_m) - q_m, 0\} = 0$, since by definition of q_m,

$r((m-1)q_m) = q_m$. Thus, the non-producing firms are also best-responding.

To show part (b) assume by contradiction that there is an equilibrium $\widetilde{\boldsymbol{q}}$ of a different type. Then there exist firms i and j such that $\widetilde{q}_i \neq \widetilde{q}_j$, $\widetilde{q}_i > 0$, $\widetilde{q}_j > 0$ in that equilibrium. Wlog let $\widetilde{q}_i > \widetilde{q}_j$. Given that $R'(Q_{-i}) = r'(Q_{-i}) + 1 < 0$ (when $R(Q_{-i}) > Q_{-i}$) it follows that $R(\widetilde{Q}_{-i}) = R(\widetilde{Q}_{-j}) \implies \widetilde{Q}_{-i} = \widetilde{Q}_{-j} \implies \widetilde{q}_i = \widetilde{q}_j$, a contradiction. **Q.E.D.**

Proof of Corollary 1.1 $\Delta(Q,Q_{-i}) = 1 - \lambda + c_2/b$, constant over L. $E_{P'}(Q) = 0$, also constant. Last, we have that $1 + \lambda + \Delta(Q,Q_{-i}) = 2 + c_2/b$. The result then follows from Proposition 1. Notice also that $Q_n = (a - c_1)/[b(H_n + 1) + c_2/n]$, which is positive since $a > c_1$ and $c_2 > -2bc_1/a > -2b$. $\Pi(n,\lambda) = (a - c_1)^2 \left(bnH_n + c_2/2\right) / \left[bn(H_n + 1) + c_2\right]^2$ is also positive. Last, $C'(q_n) = [bc_1(H_n + 1) + ac_2/n]/[b(H_n + 1) + c_2/n]$ is positive given $c_2 > -2bc_1/a$, so in equilibrium marginal cost is positive. **Q.E.D.**

Proof of Proposition 2 (i) From the pricing formula (1) the Implicit Function Theorem gives $dQ/d\lambda = -(n-1)Q/[n + \Lambda - C''(Q/n)/P'(Q) - \Lambda E_{P'}(Q)] < 0$. For fixed n, total surplus changes with λ in the same direction as total quantity: $d\,\mathrm{TS} = P(Q)dQ - \sum_{i=1}^{n} C'(q)dq = (P(Q) - C'(q))\,dQ$. Differentiating $\Pi\,(n,\lambda)$ with respect to λ we get

$$\frac{\partial \Pi\,(n,\lambda)}{\partial \lambda} = P'(Q_n)\frac{Q_n}{n}\frac{\partial Q_n}{\partial \lambda} + (P(Q_n) - C'(q_n))\,\frac{\partial Q_n}{\partial \lambda}\frac{1}{n} = P'(Q_n)\frac{Q_n}{n}\frac{\partial Q_n}{\partial \lambda}\frac{n - \Lambda_n}{n},$$

which is positive for $\lambda < 1$, where the second equality follows from the pricing formula (1).

(ii) Using the pricing formula (1) we get

$$\begin{aligned}\frac{\partial \Pi\,(n,\lambda)}{\partial n} &= P'(Q_n)\frac{Q_n}{n}\frac{\partial Q_n}{\partial n} - Q_n P'(Q_n) H_n \frac{n\frac{\partial Q_n}{\partial n} - Q_n}{n^2} \\ &\propto -\left[(1-\lambda)\left(H_n^{-1} - 1\right) + n + \Lambda_n - H_n^{-1} C''\left(q_n\right)/P'\left(Q_n\right) - \Lambda_n E_{P'}\left(Q_n\right)\right] < 0,\end{aligned}$$

where the inequality follows from what we have seen in section A.2.

(iii) $dq/dn = d(Q/n)/dn = n^{-1}dQ/dn - Q/n^2 \propto -q\,[n(1+\lambda) - \Lambda E_{P'}(Q)]$.

(iv) From the pricing formula (1) the Implicit Function Theorem gives $dQ/dn = q\Delta(Q,Q_{-i})/(\Lambda_n(1 + \lambda + \Delta/n)/H_n)$. **Q.E.D.**

A.4 Proofs of sections 4 and 5

Proof of Proposition 3 The derivative of $\Psi(n,\lambda)$ with respect to n is equal to

$$\frac{\partial\Psi(n,\lambda)}{\partial n} = \lambda\left(\Pi\left(n,\lambda\right)-\Pi\left(n-1,\lambda\right)\right)+\Lambda_n\frac{\partial\Pi\left(n,\lambda\right)}{\partial n}-\left(\Lambda_n-1\right)\left.\frac{\partial\Pi\left(\nu,\lambda\right)}{\partial\nu}\right|_{\nu=n-1}$$
$$\propto E_{\Delta\Pi,n}-\left(\frac{\Lambda_n-1}{\Lambda_n}+\frac{n-1}{\Lambda_n}\frac{\left.\frac{\partial\Pi(\nu,\lambda)}{\partial\nu}\right|_{\nu=n-1}}{\Pi\left(n,\lambda\right)-\Pi\left(n-1,\lambda\right)}\right)<0,$$

and the result obtains given Proposition 1. **Q.E.D.**

Proof of Proposition 4 The derivative of $\Psi(n,\lambda)$ with respect to λ is given by

$$\frac{\partial\Psi(n,\lambda)}{\partial\lambda} = \left(n-1\right)\left(\Pi\left(n,\lambda\right)-\Pi\left(n-1,\lambda\right)\right)+\Lambda_n\frac{\partial\Pi\left(n,\lambda\right)}{\partial\lambda}-\left(\Lambda_n-1\right)\frac{\partial\Pi\left(n-1,\lambda\right)}{\partial\lambda}$$
$$\propto -\frac{\lambda\left(\frac{\partial\Pi(n,\lambda)}{\partial\lambda}-\frac{\partial\Pi(n-1,\lambda)}{\partial\lambda}\right)}{\Pi\left(n,\lambda\right)-\Pi\left(n-1,\lambda\right)}-\frac{1}{\lambda}\frac{\lambda\frac{\partial\Pi(n,\lambda)}{\partial\lambda}\Big/\Pi(n,\lambda)}{\frac{\Pi(n,\lambda)-\Pi(n-1,\lambda)}{\Pi(n,\lambda)}(n-1)}-1.$$

The result follows by the Implicit Function Theorem given Proposition 3. **Q.E.D.**

Proof of Corollary 4.1 The total derivative of $\widehat{n}^*(\lambda)$ at $\lambda=0$ is

$$\frac{d\widehat{n}^*(\lambda)}{d\lambda} = (n-1)\left.\frac{b\left[(n-1)\left(bn+c_2\right)^2-(n+1+c_2/b)\left(b+c_2/2\right)\left(b(2n+1)+2c_2\right)\right]}{2(b+c_2/2)\left(bn+c_2\right)^2}\right|_{n=\widehat{n}^*(0)},$$

where the denominator is positive and the numerator is a third-degree polynomial in n. In part (i), n_3 is the unique real root of the polynomial, which has a negative discriminant. In part (ii), the discriminant is positive, and the result follows with n_3 the highest of the three real roots of the polynomial equation above. Also,

$$\frac{dQ_{\widehat{n}^*(\lambda)}}{d\lambda}=\frac{\partial Q_n}{\partial\lambda}+\frac{\partial Q_n}{\partial n}\frac{d\widehat{n}^*(\lambda)}{d\lambda}=\frac{Q_{\widehat{n}^*(\lambda)}}{n+1+c_2/b}\left.\left[(1+c_2/b)\frac{d\widehat{n}^*(\lambda)}{d\lambda}\frac{1}{n}-(n-1)\right]\right|_{n=\widehat{n}^*(0)}$$

and for $n_1\equiv(-2b^2-5bc_2-c_2^2)/(2b^2)+\sqrt{(6b^3c_2+11b^2c_2^2+6bc_2^3+c_2^4)/b^4}/2$ the corresponding results follow. For $\lambda=0$, $\Psi(\widehat{n}^*(\lambda),\lambda)=\Pi(\widehat{n}^*(\lambda),\lambda)=f$, we get $d\,\mathrm{TS}_{\widehat{n}^*(\lambda)}/d\lambda\propto dQ_{\widehat{n}^*(\lambda)}/d\lambda-q_{\widehat{n}^*(\lambda)}d\widehat{n}^*(\lambda)/d\lambda$ and for $n_2\equiv\left(2b-c_2+\sqrt{8b^2+6bc_2+c_2^2}\right)/(2b)$ the corresponding result follows. It can be checked that $n_3>n_2>n_1$. **Q.E.D.**

Proof of Proposition 5 We have that $d\widehat{n}^*(\lambda)/df = \left(\partial\Psi\left(n,\lambda\right)/\partial n\right)^{-1}\big|_{n=\widehat{n}^*(\lambda)}$, and part (ii) follows if we take the directional derivative of $d\widehat{n}^*(\lambda)/df$. **Q.E.D.**

Proof of Proposition 6 We have $\partial\,\mathrm{TS}_n/\partial n = \Pi(n,\lambda) - f - \Lambda_n Q_n P'(Q_n)\partial q_n/\partial n$. Given $\Psi\left(\widehat{n}^*(\lambda),\lambda\right) = f$, $d\,\mathrm{TS}_n/dn|_{n=\widehat{n}^*(\lambda)}$ is equal to (denote $\Pi_n(n,\lambda) \equiv \partial\Pi(n,\lambda)/\partial n$)

$$
\begin{aligned}
&-\phi\left(\widehat{n}^*(\lambda),\lambda\right)\lambda\widehat{n}^*(\lambda)\Pi_n\left(\widehat{n}^*(\lambda),\lambda\right) - \Lambda_{\widehat{n}^*(\lambda)}Q_{\widehat{n}^*(\lambda)}P'\left(Q_{\widehat{n}^*(\lambda)}\right)\frac{\partial q_n}{\partial n}\bigg|_{n=\widehat{n}^*(\lambda)} \\
\propto & E_{P'}\left(Q_{\widehat{n}^*(\lambda)}\right) - \frac{\widehat{n}^*(\lambda)}{\Lambda_{\widehat{n}^*(\lambda)}}\left\{1+\lambda\left[1-\frac{\phi}{1-\phi\lambda}\frac{1}{\Lambda_{\widehat{n}^*(\lambda)}}\left(1-\lambda-\frac{C''(q_{\widehat{n}^*(\lambda)})}{P'\left(Q_{\widehat{n}^*(\lambda)}\right)}\right)\right]\right\},
\end{aligned}
$$

and the result follows from single-peakedness of total surplus in n. To see why Remark 5.4 holds notice that for $\phi\left(\widehat{n}^*(\lambda),\lambda\right) = 1$ and CRS

$$
\frac{d\,\mathrm{TS}(\boldsymbol{q}_n)}{dn}\bigg|_{n=\widehat{n}^*(\lambda)} \propto E_{P'}\left(Q_{\widehat{n}^*(\lambda)}\right) - \frac{\widehat{n}^*(\lambda)}{\Lambda_{\widehat{n}^*(\lambda)}}\left[1+\lambda\left(1-\frac{1}{\Lambda_{\widehat{n}^*(\lambda)}}\right)\right].
$$

All else constant (including $\widehat{n}^*(\lambda)$), the expression above is non-increasing in λ, since

$$
\begin{aligned}
\frac{\partial\left(\frac{\widehat{n}^*(\lambda)}{\Lambda_{\widehat{n}^*(\lambda)}}\left[1+\lambda\left(1-\frac{1}{\Lambda_{\widehat{n}^*(\lambda)}}\right)\right]\right)}{\partial\lambda} &\propto -\frac{\widehat{n}^*-1}{\Lambda^2_{\widehat{n}^*}}\left[1+\lambda\left(1-\frac{1}{\Lambda_{\widehat{n}^*}}\right)\right] + \left(1-\frac{1}{\Lambda_{\widehat{n}^*}}+\frac{\lambda(\widehat{n}^*-1)}{\Lambda^2_{\widehat{n}^*}}\right)\frac{1}{\Lambda_{\widehat{n}^*}} \\
&\propto -(\widehat{n}^*-1)\Lambda_{\widehat{n}^*} + \lambda(\widehat{n}^*-1) = \Lambda_{\widehat{n}^*}\left(3-\widehat{n}^*\right) - 2 \leq 0,
\end{aligned}
$$

where the inequality follows given $\widehat{n}^* \geq 2$. Thus, a sufficient condition for excessive entry under constant marginal costs is derived if we set $\lambda = 1$ (but not in $\widehat{n}^*(\lambda)$). The condition for excessive entry is relaxed as C'' decreases, and the remark follows. For Remark 5.5, notice that $\Delta < 0$ on L implies $C''(q) < 0$ for every $q < \overline{Q}$. By Proposition 2 Q_n is decreasing in n, and thus, so is consumer surplus. Also, $n\Pi(n,\lambda) \equiv P(Q_n)Q_n - nC(q_n) < P(Q_n)Q_n - C(Q_n) \leq P(q_1)q_1 - C(q_1) = \Pi(1,\lambda)$, where the first inequality follows from $C'' < 0$. Thus, both consumer surplus and industry profits are maximized for $n = 1$, so $n^o(\lambda) = 1$. Remark 5.6 follows since by the mean value theorem $C''(\widetilde{q}) = (C'(q_n) - C'(0))/q_n$ for some $\widetilde{q} \in [0,q_n]$, and also $P(Q_{\widehat{n}^*(\lambda)}) > C'(q_{\widehat{n}^*(\lambda)}) > C'(0)$.

To see why Remark 5.8 holds, notice that from the proof of Corollary 1.1 we have that for $\lambda = 1$, $\Pi(n,1) = (a-c_1)^2/[2\left(2bn+c_2\right)]$, so that for $n \geq 2$ $\Psi(n,1) = (a-c_1)^2c_2/[2\left(2bn+c_2\right)\left(2b(n-1)+c_2\right)]$. Thus, for $c_2 \leq 0$ we get $n^*(1) = 1$. For

$c_2 > 0$ we get $\widehat{n}^*(1) = \left(b - c_2 + \sqrt{\frac{(a-c_1)^2 c_2}{2f} + b^2}\right)/(2b)$. Pricing stage equilibrium total surplus is $\mathrm{TS}_n = \int_0^{Q_n} (P(X) - P(Q_n))\, dX + n\,(\Pi(n,1) - f) = (a - c_1)^2(3b + c_2/n)/[2\,(2b + c_2/n)^2] - nf$, whose derivative with respect to n is equal to $d\,\mathrm{TS}_n\,/dn = (a - c_1)^2 c_2\,(4bn + c_2)\,/[2\,(2bn + c_2)^3] - f$. Thus, for $c_2 \leq 0$ we get $n^o(1) = 1$. It can be checked that $d^2\,\mathrm{TS}_n\,/(dn)^2 < 0$ for $c_2 > 0$ and $n \geq 1$, so TS_n is concave in n. For $c_2 > 0$, evaluating $d\,\mathrm{TS}_n\,/dn$ at $n = \widehat{n}^*(1)$ gives the first part of the remark with $\overline{f} := (a-c_1)^2(-8b^2 - 2bc_2 - c_2^2 + \sqrt{64b^4 + 64b^3c_2 + 20b^2c_2^2 + 4bc_2^3 + c_2^4})/(32b^3) > 0$. **Q.E.D.**

B Additional material

B.1 Individual firm's objective function under overlapping ownership

Here we briefly describe settings of common and cross ownership which can give rise to the Cournot-Edgeworth λ oligopoly model that we study.

B.1.1 A model of corporate control under common ownership

There is a finite set $\mathcal{J}$ of investors. For each $j \in \mathcal{J}$, β_{ji} denotes investor j's share of firm i, γ_{ji} captures the extent of her control over firm i, and $u_j(\boldsymbol{q}) \coloneqq \sum_{i\in\mathcal{F}} \beta_{ji}\pi_i(\boldsymbol{q})$ is her total portfolio profit, where π_i firm i's profit function. O'Brien and Salop (2000) assume that the manager of firm i maximizes a weighted average of the shareholders' portfolio profits; that is, given $\boldsymbol{q}_{-i}$ she maximizes

$$\sum_{j\in\mathcal{J}} \gamma_{ji} u_j(\boldsymbol{q}) \propto \pi_i(\boldsymbol{q}) + \sum_{k\in\mathcal{F}\setminus\{i\}} \lambda_{ik}\pi_k(\boldsymbol{q}),$$

where $\lambda_{ik} \coloneqq \sum_{j\in J} \gamma_{ji}\beta_{jk} \Big/ \sum_{j\in J} \gamma_{ji}\beta_{ji}$. A common assumption on γ is proportional control, that is $\gamma_{ji} = \beta_{ji}$ for every $j \in \mathcal{J}$ and every $i \in \mathcal{F}$. For appropriate ownership and control structures β and γ it will be that $\lambda_{ik} = \lambda$, fixed for every pair of firms i,k. One such ownership and control structure (β,γ) is described in section B.1.3.

B.1.2 Firm objectives under cross ownership

Firm objectives under cross ownership are also described in Gilo et al. (2006) and López and Vives (2019). Assume that we start with each firm i being held by shareholders who do not hold shares of any of the other firms. Then, each *firm* i buys share $\alpha \in [0, 1/(N-1))$ of every other firm $k \in \mathcal{F} \setminus \{i\}$ *without control rights*. In other words, each firm i acquires a claim to share α of the *total earnings* of every other firm. The total earnings of each firm i now include the profit directly generated by firm i and firm i's earnings from its claims over the other firms' total earnings.

We end up with each firm i being controlled by its initial shareholders, each of whom only hold claims to firm i's total earnings. The controlling shareholders collectively hold a claim to share $(1-(N-1)\alpha)$ of firm i's total earnings. Clearly, all controlling shareholders of firm i agree that firm i should seek to maximize its total earnings.

For every $\boldsymbol{q}$, the total earnings $\widetilde{\pi}_i(\boldsymbol{q})$ of each firm i are then given by the solution to the system of equations

$$\widetilde{\pi}_i(\boldsymbol{q}) = \overbrace{\pi_i(\boldsymbol{q})}^{\substack{\text{firm } i\text{'s earnings from} \\ \text{the profit directly} \\ \text{generated by firm } i}} + \overbrace{\alpha \sum_{k \in \mathcal{F} \setminus \{i\}} \widetilde{\pi}_k(\boldsymbol{q})}^{\substack{\text{firm } i\text{'s earnings from its} \\ \text{claims over each firm } k\text{'s,} \\ k \neq i\text{, total earnings}}}, \quad \text{for each } i \in \mathcal{F}.$$

Solving the system of equations we find that each firm i's objective is to maximize

$$\widetilde{\pi}_i(\boldsymbol{q}) \propto \pi_i(\boldsymbol{q}) + \lambda \sum_{k \in \mathcal{F} \setminus \{i\}} \pi_k(\boldsymbol{q}) \qquad \text{where } \lambda := \alpha/[1-(N-2)\alpha] \in [0,1).$$

B.1.3 An example of post-entry overlapping ownership

Post-entry overlapping ownership can for example arise in the form of common ownership as described below. Let all firms be newly-established and the set of investors $\mathcal{J}$ be partitioned into $\{J_0\} \cup \cup_{i \in \mathcal{F}} \{J_i\}$ with $|J_i| = |J_0| = m$ for every $i \in \mathcal{F}$. Before entry each firm i is (exclusively) held by the set J_i of entrepreneurs with $\beta_{ji} = 1/m$ for every $j \in J_i$; there is no common ownership before entry, so when considering entry, the entrepreneurs of each firm unanimously agree to maximize their own firm's profit.[45] After entry, the set J_0 of investors, who previously held no shares of any firm, buy firm shares. Each investor $j \in J_0$ now holds share $\beta'_{ji} = \sigma/m$ of each firm i that has entered, and each entrepreneur $j \in J_i$ holds share $\beta'_{ji} = (1-\sigma)/m$ of her firm for some $\sigma \in [0,1]$. That is, after entry each entrepreneur sells the same amount of shares to the investors, who are now uniformly invested in all firms in the industry. Consider the O'Brien and Salop (2000) model and for every firm i that has entered let $\gamma'_{ji} = \widetilde{\gamma}/m$ be the control each investor $j \in J_0$ has over firm i for some $\widetilde{\gamma} \in [0,1]$, and $\gamma'_{ji} = (1-\widetilde{\gamma})/m$ the control each entrepreneur $j \in J_i$ has over her firm i.[46] After entry, the manager of each firm i maximizes

$$\pi_i(\boldsymbol{q}) + \lambda \sum_{k \neq i} \pi_k(\boldsymbol{q}), \quad \text{where } \lambda = \frac{\widetilde{\gamma}\sigma}{\widetilde{\gamma}\sigma + (1-\widetilde{\gamma})(1-\sigma)} = \frac{1}{1 + (\widetilde{\gamma}^{-1} - 1)(\sigma^{-1} - 1)} \in [0,1].$$

[45] This relies on the fact that a firm's entrepreneurs only hold shares of their own firm both before and after entry. Common ownership develops after entry not through a firm's entrepreneurs investments in other firms but because outside investors invest in multiple firms.

[46] For every other pair of entrepreneur j and firm i, $\beta'_{ji} = \gamma'_{ji} = 0$.

Here λ is increasing in the common owners' level of holdings σ and control $\widetilde{\gamma}$. Under proportional control $\sigma = \widetilde{\gamma}$, and $\lambda = \left[1 + (\sigma^{-1} - 1)^2\right]^{-1}$.

B.2 Pricing-stage equilibria under parametric assumptions

CESL demand is of the form

$$P(Q) = \begin{cases} a + bQ^{1-E} & \text{if } E > 1 \\ \max\{a - b\ln Q, 0\} & \text{if } E = 1 \\ \max\left\{a - bQ^{1-E}, 0\right\} & \text{if } E < 1 \end{cases}$$

for parameters $a \geq 0$ and $b > 0$. For $E = 0$ this reduces to linear demand, while for $a = 0$ and $E > 1$ it reduces to constantly elastic demand with elasticity $\eta = (E-1)^{-1}$.

Claim 1 provides the equilibria under parametric assumptions on the demand and cost functions. The total quantity is decreasing in the level of overlapping ownership, λ.

Claim 1. Under CESL demand and constant returns to scale the total equilibrium quantity in the pricing stage is

$$Q_n = \begin{cases} \left[\frac{b(1-H_n(E-1))}{c-a}\right]^{\frac{1}{E-1}} & \text{if } E \in (1,2) \text{ and } c > a \\ e^{\frac{a-c-bH_n}{b}} & \text{if } E = 1 \\ \left[\frac{a-c}{b(1+H_n(1-E))}\right]^{\frac{1}{1-E}} & \text{if } E < 1 \text{ and } a > c, \end{cases}$$

where $H_n := \Lambda_n/n$, $\Lambda_n := 1 + \lambda(n-1)$. Under linear demand and quadratic costs it is $Q_n = \frac{a}{b(1+H_n)+c/n}$.

B.3 Stability of pricing stage equilibrium

Proposition 7 examines local asymptotic stability of the pricing stage equilibrium in the sense of the myopic continuous adjustment process, as described in al Nowaihi and Levine (1985).

Proposition 7. If $\Delta > 0$, then the pricing stage equilibrium is locally stable.

Proposition 7' studies stability with the maintained assumption relaxed.

Proposition 7'. Assume $\Delta > 0$ but drop the assumption that $E_{P'}(Q) < (1 + \lambda + \Delta(Q,Q_{-i})/n)/H_n$ on L, so that multiple symmetric equilibria may exist. Then, a pricing stage equilibrium is locally stable if and only if $E_{P'}(Q) < (1 + \lambda + \Delta(Q,Q_{-i})/n)/H_n$ in that equilibrium.

Remark B.1. For $\lambda = 0$ we recover the sufficient local (in)stability conditions implied by Theorems 3, 4 and 5 of al Nowaihi and Levine (1985).[47]

Under $\Delta > 0$, when we drop the condition $E_{P'} < (1+\lambda+\Delta/n)/H_n$ on L guaranteeing uniqueness, multiple symmetric equilibria may exist, some of which stable and some unstable. These two sets of equilbria are differentiated by a local version of the dropped condition. An equilibrium is stable if and only if the dropped condition holds *in that equilibrium.*

B.4 Additional comparative statics of pricing stage equilibrium

Let P and C be three-times differentiable. Denote by $E_{P''}(Q) := P'''(Q)Q/P''(Q)$ the elasticity of the curvature of inverse demand.

Proposition 8. The following hold:

(i) If $\Delta > 0$ and also for every $Q < \overline{Q}$, $E_{P'}(Q) < 2$, $E_{P'}(Q)\left[E_{P'}(Q) + E_{P''}(Q)\right] \geq -2$ and for every $q < \overline{q}$, $C''(q)$,$C'''(q) \geq 0$,[48] then $(\partial Q_n)^2/(\partial\lambda\partial n) < 0$.

(ii) $\partial^2 q_n/(\partial\lambda\partial n)$ can be negative or positive (and change sign as λ and/or n changes). For example, for CRS and CESL demand

$$\operatorname{sgn}\left\{\frac{\partial^2 q_n}{\partial\lambda\partial n}\right\} = \operatorname{sgn}\left\{n + \lambda(n-1) - 3 - \left(H_n(n-1) - 1\right)E\right\}.$$

Under the assumptions of part (i), the negative effect of overlapping ownership on the total quantity is strongest in industries with a large number of firms, which would otherwise be the most competitive ones.

[47] al Nowaihi and Levine (1985) deal with a possibly asymmetric equilibrium; they provide analogous conditions where expressions such as Δ vary across firms.

[48] If $P''(Q) = 0$, cancel P'' in $E_{P'}(Q)$ with the one in $E_{P''}(Q)$. Under CESL demand, $E_{P'}(Q)\left[E_{P'}(Q) + E_{P''}(Q)\right] \geq -2$ holds if and only if $E \leq 2$.

Now we study how aggregate industry profits depend on the number of firms.

$$\mu_n := 1 - \frac{\partial Q_n}{\partial n} \frac{n}{Q_n} \frac{1 - H_n}{\eta(Q_n) - H_n}.$$

Proposition 9. Let $\lambda < 1$. Then, the following statements hold:

(i) if $\mu_n \leq 0$, aggregate industry profits are decreasing in n,

(ii) if $\mu_n > 0$, aggregate industry profits are decreasing (resp. increasing) in n if $E_C(q_n) \overset{(\text{ resp. } >)}{<} \mu_n^{-1}$,

(iii) if $C''(q) < 0$ for every $Q \in [0,Q_n]$, then monopoly maximizes aggregate industry profits, $\Pi(1,\lambda) > n\Pi(n,\lambda)$.

Remark B.2. If $\Delta < 0$, then $\partial Q_n/\partial n > 0$, so $\mu_n < 1$, and thus, aggregate industry profits are decreasing in n if $E_C(q_n) \leq 1$. If for example $C'' < 0$ globally (consistent with $\Delta < 0$), then indeed $E_C(q_n) < 1$.

Remark B.3. If $\lambda = 1$ and $C''(q) > 0$ for every $q \in [0,q_n]$, aggregate industry profits are increasing in n.

Consider the extreme case of $\lambda = 1$ and notice the following. Condition $\Delta > 0$ requires decreasing returns to scale, so that aggregate gross profits increase with n (*i.e.*, $n\Pi(n,1) > (n-1)\Pi(n-1,1)$ for any n) due to savings in variable costs as production is distributed across more firms, even though the total quantity increases (see Proposition 2), and thus price decreases with the number of firms. Intuitively, aggregate gross profits being increasing in n for $\lambda = 1$ is tied to uniqueness of the (symmetric) equilibrium in the pricing stage. Since firms jointly maximize aggregate profits, the latter should increase with n for firms to strictly prefer to spread production evenly. On the other hand, under constant returns to scale aggregate profits are constant in n; increasing the number of firms simply increases the ways in which the firms can jointly produce the fixed level of total output that maximizes joint profits.[49] Last, under increasing returns to scale it is an equilibrium for all production to be concentrated in a single firm.

[49] As argued already, in this case the are infinitely many equilibria of the pricing stage, all with the same total quantity.

Claim 2. Under linear demand and quadratic costs

$$\frac{\partial\left[n\Pi\left(n,\lambda\right)\right]}{\partial n}=\frac{c}{2bn}-\frac{b(1-\lambda)+c}{b(n+\Lambda_n)+c}\left(1-H_n\right)\quad\text{with}\quad\frac{\partial^2\left[n\Pi\left(n,\lambda\right)\right]}{\partial\lambda\partial n}>0.$$

(i) for $\lambda=0$, $\operatorname{sgn}\left\{\partial\left[n\Pi\left(n,0\right)\right]/\partial n\right\}=\operatorname{sgn}\left\{c-b(n-1)\right\}$,

(ii) for $\lambda=1$, $\partial\left[n\Pi\left(n,1\right)\right]/\partial n>0$,

(iii) if $c>b(n-1)$, then $\partial\left[n\Pi\left(n,\lambda\right)\right]/\partial n>0$ for every $\lambda\in[0,1]$,

(iv) if $c<b(n-1)$, then there exists $\lambda^*\in(0,1)$ such that $\partial\left[n\Pi\left(n,\lambda\right)\right]/\partial n \overset{(\text{resp. } <)}{>} 0$ if and only if $\lambda \overset{(\text{resp. } <)}{>} \lambda^*$.

In the decreasing returns to scale case of Claim 2 we see that λ and n are complements in increasing aggregate industry profits. Particularly, for λ high enough aggregate industry profits are increasing in the number of firms. This is because with λ high, entry does not reduce the price as much (see point (iii-b) of Proposition 2), so the cost-saving effect of entry under decreasing returns to scale dominates.

B.5 Concavity of total surplus in the number of firms

Lemma 2. TS_n is globally strictly concave in n if for every n

$$\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\left[1-\lambda-H_n\left(\left(\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}-1\right)\left(1-E_{P'}(Q_n)\right)+\frac{\partial^2 Q_n}{(\partial n)^2}\left(\frac{\partial Q_n}{\partial n}\right)^{-1}n-1\right)\right]>\frac{1-\lambda}{n}.$$

Under constant marginal costs and $E_{P'}(Q_n)<2$ for every n, this is true if $E'_{P'}(Q)\equiv\partial E_{P'}(Q)/\partial Q$ is not too high; particularly, $E'_{P'}\leq 0$ is sufficient, and thus so is CESL demand.

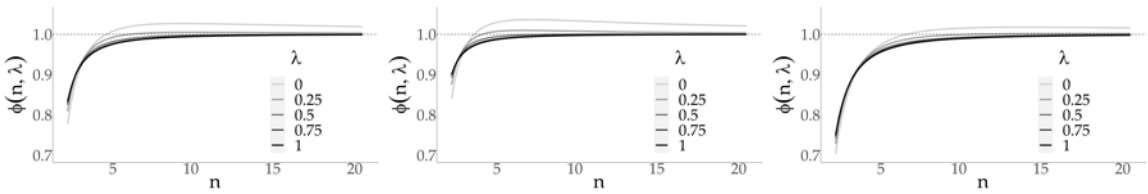
Remark B.4. More generally, all else constant, the condition of Lemma 2 is satisfied if the elasticity of the slope of Q_n with respect to n, $\frac{\partial^2 Q_n}{(\partial n)^2}\left(\frac{\partial Q_n}{\partial n}\right)^{-1}n$, is not too high. Also, remember that $\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\in(0,1)$ under the assumptions of Proposition 2(iii-a), so all else constant, in that case the condition is satisfied if $E_{P'}(Q)$ is not too high.

B.6 Numerical results showing that ϕ is close to 1

The numerical results of Figure 8 verify that $\phi\left(n,\lambda\right)$ is indeed close to 1, especially for $n\geq 3$.

Figure 8: $\phi(n,\lambda)$ under linear demand and quadratic costs

(a) $b = 1$, $c = 1$ **(b)** $b = 2$, $c = 1$ **(c)** $b = 1$, $c = 2$



Note: it can be checked that $\phi(n,\lambda)$ is invariant to the demand parameter a.

B.7 Derivation of Numerical Results

Under CESL demand and constant returns to scale, given Claim 1 we find that

$$\Pi(n,\lambda) = \begin{cases} \frac{1}{n}\left[a + b\left[\frac{b(1-H_n(E-1))}{c-a}\right]^{\frac{1-E}{E-1}} - c\right]\left[\frac{b(1-H_n(E-1))}{c-a}\right]^{\frac{1}{E-1}} & \text{if } E \in (1,2) \text{ and } c > a \\ \frac{1}{n}\left[a - b\ln\left(e^{\frac{a-c-bH_n}{b}}\right) - c\right] e^{\frac{a-c-bH_n}{b}} & \text{if } E = 1 \\ \frac{1}{n}\left[a - b\left[\frac{a-c}{b(1+H_n(1-E))}\right]^{\frac{1-E}{1-E}} - c\right]\left[\frac{a-c}{b(1+H_n(1-E))}\right]^{\frac{1}{1-E}} & \text{if } E < 1 \text{ and } a > c, \end{cases}$$

$$= \begin{cases} \frac{H_n(E-1)b^{\frac{1}{E-1}}}{n}\left[\frac{1-H_n(E-1)}{c-a}\right]^{\frac{2-E}{E-1}} & \text{if } E \in (1,2) \text{ and } c > a \\ \frac{bH_n}{n}e^{\frac{a-c-bH_n}{b}} & \text{if } E = 1 \\ \frac{H_n(1-E)}{nb^{\frac{1}{1-E}}}\left[\frac{a-c}{1+H_n(1-E)}\right]^{\frac{2-E}{1-E}} & \text{if } E < 1 \text{ and } a > c, \end{cases}$$

Derivation of Numerical Result 1 Parameters a, b and c only affect the magnitudes of $d\widehat{n}^*(\lambda)/d\lambda$ and $dQ_{\widehat{n}^*(\lambda)}/d\lambda$, and not their signs. The result then obtains in a way analogous to the one described in the Derivation of Numerical Result 2. ■

Derivation of Numerical Result 2 It is easy to see that the signs of derivatives of $\Psi(n,\lambda)$ are independent of a, b and c. Thus, we can wlog set (i) $a = b = 1$ and $c = 2$ for the case $E \in (1,2)$, and (ii) $a = 2$, $b = c = 1$ for the case $E < 1$.

For $E > 1$ we run the following R code:

```
# load packages #
library(Deriv)
library(optimx)


# define functions #
Lambda = function(n,lambda) {1 + lambda*(n-1)}
```

```
H = function(n,lambda) {(1 + lambda*(n-1))/n}

Pi = function(n,lambda,E,a,b,c) { H(n,lambda)*(E-1)*b^(1/(E-1))*
( (1-H(n,lambda)*(E-1))/(c-a) )^((2-E)/(E-1))/n }
Psi = function(n,lambda,E,a,b,c) {  Pi(n,lambda,E,a,b,c)-lambda*(n-1)*
( Pi(n-1,lambda,E,a,b,c)-Pi(n,lambda,E,a,b,c)) }

# symbolically differentiate Psi #
Deriv_wrt_n_Psi = Deriv(Psi,"n")
Deriv_wrt_nlambda_Psi = Deriv(Deriv_wrt_n_Psi,"lambda")

# define function that creates grid of starting points for optimization #
grid = function(density_n,min_n,max_n,density_l,min_l,max_l,
density_E,min_E,max_E) {
  output = matrix(nrow = (density_n+1)*(density_l+1)*(density_E+1), ncol = 3)
  row_number = 1
  for (i in seq(from = min_n, to = max_n, by = (max_n-min_n)/density_n)) {
    for (j in seq(from = min_l, to = max_l, by = (max_l-min_l)/density_l)) {
      for (k in seq(from = min_E, to = max_E, by = (max_E-min_E)/density_E)) {
        output[row_number,] = c(i,j,k)
        row_number = row_number + 1
      }
    }
  }
  return(output)
}

# minimize cross derivative of Psi from multiple starting points #

minima =  multistart(parmat = grid(15,2,7,15,0,1,30,1.001,1.7),
      fn = function(x) {Deriv_wrt_nlambda_Psi(x[1],x[2],x[3],1,1,2)},
      method = c("L-BFGS-B"), lower = c(2,0,1.001), upper = c(7,1,1.7))
```

The code returns that

$$\min_{(n,\lambda,E)\in[2,7]\times[0,1]\times[1.001,1.7]} \frac{\Psi\,(n,\lambda)}{\partial\lambda\partial n} \approx 2.31\cdot 10^{-6} > 0,$$

which is reached for $n = 7$, $\lambda = 0$ and $E = 1.001$.

In the case of $E < 1$ we similarly find that

$$\min_{(n,\lambda,E)\in[2,8]\times[0,1]\times[-1000,0.999]} \frac{\Psi\,(n,\lambda)}{\partial\lambda\partial n} \approx 1.11\cdot 10^{-7} > 0,$$

which is reached for $n = 8$, $\lambda = 0$ and $E = 0.999$. In additional simulations, allowing E to be even lower than -1000 does not change the result. ■

B.8 Free entry under pre-entry overlapping ownership and the presence of maverick firms

This section presents a model of free entry with pre-entry overlapping ownership under the presence of maverick firms.

For simplicity, model the maverick firms as a competitive fringe that in the first stage (where oligopolists enter) submit an aggregate supply schedule. Namely, there is a set $\mathcal{F}_m$ of infinitesimal firms. Each firm $i \in \mathcal{F}_m$ chooses to either be inactive or produce one (infinitesimal) unit of the good at cost $\chi(i)$.[50] Thus, the aggregate supply function by the maverick firms in the third stage $S : \mathbb{R}_+ \to \mathbb{R}_+$ is given by $S(p) := \int_{i\in\mathcal{F}_m} \mathbb{1}\,(\chi(i) \le p)\,di$. $S : \mathbb{R}_+ \to \mathbb{R}_+$ with $S(p) = 0$ for every $p \in [0,\underline{p}]$ and $S'(p) > 0$ for every $p > \underline{p}$ where $\underline{p} \ge 0$. $S(p)$ gives the total supply of the maverick firms as a function of the market price p. Then, the price $p > 0$ in the competitive equilibrium among the maverick firms will be implicitly given by $P^{-1}(p) = Q + S(p)$, where Q the total quantity produced by the oligopolists.[51] This means that in the second stage the oligopolists are essentially faced with inverse demand $\widetilde{P} : \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$\widetilde{P}(Q) = \begin{cases} P\left(Q + \omega^{-1}(Q)\right) \in \left(\underline{p}, P(Q)\right) & \text{if } P(Q) > \underline{p} \\ P(Q) & \text{if } P(Q) \le \underline{p} \end{cases}$$

[50] This cost can be thought to include any applicable entry costs. Since maverick firms are infinitesimal and each supply an infinitesimal quantity, their entry cost is also infinitesimal.

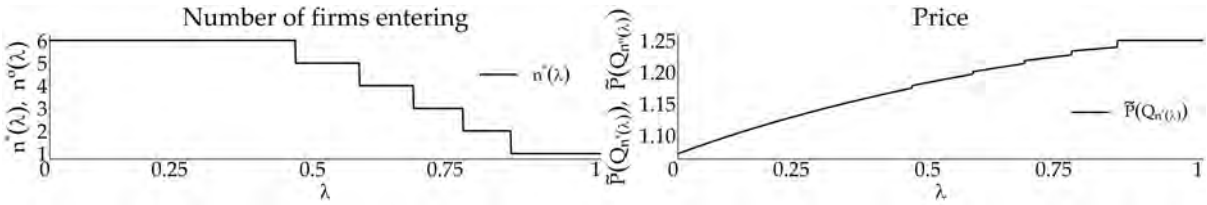
[51] We assume that $S(p) > P^{-1}(p)$ for p large enough.

where $\omega : \mathbb{R}_{++} \to \mathbb{R}_{+}$ is given by $\omega(y) := P^{-1} \circ S^{-1}(y) - y$.[52] $\omega^{-1}(Q)$ gives the quantity supplied in the competitive equilibrium among the maverick firms when the oligopolists produce Q. For example, in the case of (i) linear demand $P(Q) = \max\{a - bQ, 0\}$, (ii) linear maverick aggregate supply schedule $S(p) = \max\{(p - \underline{p})/b_m, 0\}$ with $b_m > 0$ and $\underline{p} \geq 0$, and (iii) CRS (for the oligopolists), $C(q) = cq$, with $a > c \geq \underline{p}$,[53] for any $Q \in [0, (a-c)/b]$, $\widetilde{P}$ is given by[54]

$$\widetilde{P}(Q) = \overbrace{a - \frac{a - \underline{p}}{1 + b_m/b}}^{<a} - \overbrace{\frac{b}{1 + b/b_m}}^{<b} Q.$$

The (prospect of) entry by maverick firms essentially changes the demand faced by the commonly-owned firms by depressing it and making it more elastic. If in the paper wherever P we read $\widetilde{P}$, the results on the effects of overlapping ownership on entry and the price continue to hold (with the number of firms n not counting maverick firm entry). A comparison of Figure 9 with Figure 2(a) in the paper (the two figures use the same parametrization but in the former maverick firms are added) shows entry to be less sensitive to overlapping ownership due to the presence of the maverick firms, as argued in the paper.

Figure 9: Equilibrium with pre-entry overlapping ownership under the presence of maverick firms for varying λ



Note: lines represent values in equilibrium; linear demand, CRS: $a = 2$, $b = c = 1$, $f = 0.01$; linear maverick aggregate supply schedule: $b_m = \underline{p} = 1$.

Last, the total surplus $\widetilde{\mathrm{TS}}(\boldsymbol{q})$ now includes the maverick firms' surplus, where $\boldsymbol{q}$ still is the quantity profile of the oligopolists. Denote by $\widetilde{\mathrm{TS}}_n$ the pricing stage equilibrium

[52] To see this substitute $p = P\left(Q + \omega^{-1}(Q)\right)$ in $P^{-1}(p) = Q + S(p)$, which gives

$$Q + \omega^{-1}(Q) = Q + S \circ P\left(Q + \omega^{-1}(Q)\right) \Longleftrightarrow P^{-1} \circ S^{-1} \circ \omega^{-1}(Q) - \omega^{-1}(Q) = Q,$$

which is true by definition of ω.

[53] For $c = \underline{p}$, the most efficient maverick firms is as efficient as the oligopolists.

[54] The inverse demand $\widetilde{P}$ for higher Q does not play a role since the commonly-owned firms will never produce more than $(a-c)/b$. To derive $\widetilde{P}$, solve for it in $(a - \widetilde{P}(Q))/b = Q + (\widetilde{P}(Q) - \underline{p})/b_m$.

total surplus when n commonly-owned firms enter. Equation (4) also applies in the case with maverick firms but with $\widetilde{\Xi}(n,\lambda) \coloneqq (n-1)\left(\widetilde{\Pi}(n-1,\lambda) - \widetilde{\Pi}(n,\lambda)\right)$, $\widetilde{\Pi}(n,\lambda) \coloneqq \widetilde{P}(Q_n)q_n - C(q_n)$ and $\widetilde{P}$ replacing Ξ, Π and P.[55] Q_n, q_n are still the quantities produced by the commonly-owned firms in the pricing stage equilibrium where n of them enter. $\widehat{n}^*(\lambda)$ is now pinned down by $\widetilde{\Pi}(\widehat{n}^*(\lambda),\lambda) - \lambda\widetilde{\Xi}(\widehat{n}^*(\lambda),\lambda) = f$.

Provided $\widetilde{P}(Q) \geq \underline{p}$ or equivalently $P(Q) \geq \underline{p}$,[56] total surplus now includes the maverick firms' surplus and is thus given by

$$\begin{aligned}\widetilde{\mathrm{TS}}(\boldsymbol{q}) \coloneqq & \overbrace{\int_0^{Q+S\left(\widetilde{P}(Q)\right)} \left(P(X) - \widetilde{P}(Q)\right) dX}^{\text{consumer surplus}} + \overbrace{\int_{\underline{p}}^{\widetilde{P}(Q)} S(p)dp}^{\text{maverick firms' surplus}} + \overbrace{\widetilde{P}(Q)Q - \sum_{i=1}^{n} C(q_i) - nf}^{\text{commonly-owned firms' profits}} \\ = & \,\mathrm{TS}(\boldsymbol{q}) + \underbrace{\int_Q^{Q+S\left(\widetilde{P}(Q)\right)} P(X)dX - S\left(\widetilde{P}(Q)\right)\widetilde{P}(Q)}_{\geq 0;\text{ consumer surplus “due to” maverick firms' production}} + \underbrace{\int_{\underline{p}}^{\widetilde{P}(Q)} S(p)dp}_{\geq 0;\text{ maverick firms' surplus}}\end{aligned}$$

where $\boldsymbol{q}$ still the quantity profile of the oligopolists and $\mathrm{TS}(\boldsymbol{q}) \equiv \int_0^Q P(X)dX - \sum_{i=1}^n C(q_i) - nf$ the total surplus without maverick firms. For any fixed quantity profile of the oligopolists, total surplus is higher when the maverick firms are present (and produce) compared to when they are not. We have then that

$$\begin{aligned}\frac{d\widetilde{\mathrm{TS}}_n}{dn} =& P(Q_n)\left(n\frac{\partial q_n}{\partial n} + q_n\right) - C(q_n) - nC'(q_n)\frac{\partial q_n}{\partial n} - f \\ &+ \begin{bmatrix} \left(1 + S'(\widetilde{P}(Q_n))\widetilde{P}'(Q_n)\right) P\left(Q_n + S(\widetilde{P}(Q_n))\right) - P(Q_n) \\ -S'(\widetilde{P}(Q_n))\widetilde{P}'(Q_n)\widetilde{P}(Q_n) - S(\widetilde{P}(Q_n))\widetilde{P}'(Q_n) + S(\widetilde{P}(Q_n))\widetilde{P}'(Q_n) \end{bmatrix} \frac{\partial Q_n}{\partial n} \\ =& \widetilde{\Pi}(n,\lambda) - f - (1 + \lambda(n-1))\, Q_n\widetilde{P}'(Q_n)\frac{\partial q_n}{\partial n},\end{aligned}$$

where $\widetilde{\mathrm{TS}}_n$ is the pricing stage equilibrium total surplus when n commonly-owned firms enter, $\widetilde{\Pi}(n,\lambda) \coloneqq \widetilde{P}(Q_n)q_n - C(q_n)$, and Q_n, q_n are still the quantities produced by the commonly-owned firms in the pricing stage equilibrium where n of them enter.

Whether there is excessive or insufficient entry by commonly-owned firms will depend on the same forces identified in the previous section but with adjusted magnitude since P is replaced by $\widetilde{P}$. Notice that excessive or insufficient entry is based on a planner that

[55] Refer to Appendix B for a detailed derivation.

[56] Otherwise, wherever $\widetilde{P}(Q)$ substitute $\underline{p}$, and the equation reduces to $\widetilde{\mathrm{TS}}(\boldsymbol{q}) = \mathrm{TS}(\boldsymbol{q})$.

controls the entry of oligopolists and allows them and the maverick firms to produce freely. Importantly, given the production decisions of the oligopolists, the maverick firms' production level maximizes total surplus since the maverick firms are perfect competitors.

B.9 Free entry under post-entry overlapping ownership

In the last section overlapping ownership develops before entry, thus directly affecting the incentives of firms to enter. In this section we study the case where potential entrants have no prior overlapping ownership, but after they enter the market and before they pick quantities in the second stage they develop overlapping ownership, so that they have an Edgeworth coefficient of effective sympathy $\lambda \in [0,1]$. Now, the only channel through which overlapping ownership affects entry is by increasing profits in the post-entry game. Firms expect this and therefore entry increases with overlapping ownership.

This can be interpreted as a long–run equilibrium whereby start-up firms (or already existing firms but without overlapping ownership) enter the industry and then develop overlapping ownership through time. Appendix B.1 describes explicitly how post-entry overlapping ownership can arise. Also, given that the extent to which overlapping ownership affects corporate conduct is an open empirical question, this section can also be interpreted as studying pre-entry overlapping ownership when it affects pricing but does *not* cause firms to internalize their entry externality.

The exogeneity of λ is important with post-entry overlapping ownership, since the incentives of firms to allow for ownership ties after entry are not modeled. For instance, if instead the amount of shares that investors buy from the entrepreneurs depended on the extent of entry—since the latter affects profits, then λ would be a function of n. Although the exogeneity of λ is restrictive, if firms become publicly traded after entry (at least in the long–run), they indeed have limited control over their ownership ties, since for instance investment funds are free to buy shares of all firms.

B.9.1 The entry stage

Each firm only looks at own profit to decide whether to enter as there is no overlapping ownership when it does so.[57] $\boldsymbol{q}_n$ is a free entry equilibrium production profile if and only if

$$\Pi\left(n,\lambda\right) \geq f > \Pi\left(n+1,\lambda\right)$$

as in Mankiw and Whinston (1986). If overlapping ownership develops only after firms enter, it affects the incentives of firms to enter only through its effect on product market outcomes. We assume that there exists n such that $\Pi(n,\lambda) < f$ for any λ.

B.9.2 Existence and uniqueness of equilibrium

Proposition 10 studies existence and uniqueness of a free entry equilibrium.

Proposition 10. $\Pi(n,\lambda)$ is decreasing in n and a unique free entry equilibrium exists.

In equilibrium, firms enter until profits have fallen so much that if an additional firm enters, gross profit will no longer cover the entry cost. $\widehat{n}^*(\lambda)$ is uniquely pinned down by $\Pi\left(\widehat{n}^*(\lambda),\lambda\right) = f$ and $n^*(\lambda) = \max\left\{n \in \mathbb{N} : \Pi\left(n,\lambda\right) \geq f\right\} = \lfloor \widehat{n}^*(\lambda) \rfloor$.

B.9.3 Overlapping ownership effects

Proposition 11 studies the effects of overlapping ownership.

Proposition 11. Ignore the integer constraint on n (so that entry is given by $\widehat{n}^*(\lambda)$). Then

(i) the number of firms entering is increasing in λ,

(ii) individual quantity, total quantity, and total surplus are decreasing in λ,

(iii) if $C'' \geq 0$, then the MHHI is increasing in λ.

Remark B.5. There exists a set of thresholds $\mathcal{L} := \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$, $\lambda_1 < \lambda_2 < \cdots < \lambda_k$, such that

(a) for every $\lambda \in \mathcal{L}$, $\Pi\left(n^*(\lambda),\lambda\right) = f$, and $n^*(\lambda) = \widehat{n}^*(\lambda)$,

[57] Formally, if a firm does not enter, its payoff is 0; if it does, it is $(1+\lambda(n-1))\left(\Pi\left(n,\lambda\right) - f\right)$. Thus, it is optimal for an n-th firm to enter if and only if $\Pi\left(n,\lambda\right) \geq f$.

(b) for λ between two consecutive thresholds $n^*(\lambda)$ remains constant and everything behaves as in the Cournot game with a fixed number of firms.

When we take into account the integer constraint, the number of firms is a step function of in λ, and individual quantity is decreasing with jumps down. Total quantity has a decreasing trend with jumps up (resp. down) for the values of λ at which an extra firm enters under $\Delta > 0$ (resp. $\Delta < 0$). Also, total surplus tends to decrease with λ.[58]

Importantly, even when there is free entry of firms—so that increases in λ lead to the entry of new firms as incumbents suppress their quantities, if the entering firms develop overlapping ownership after entering (up to the level the incumbents have), consumer and total surplus tend to decrease with λ, as in the symmetric case with a fixed number of firms. Also, if one looks at HHI, it will seem as if competition rises as λ increases, which can even be the case with MHHI, although the latter will increase with λ if we slightly strengthen our assumptions. Last, for appropriate levels of λ a small increase in λ can spur the entry of an extra firm causing the total quantity to rise.

The fact that the price increases with λ is to be expected. Remember that an increase in λ is met with an increase in n so that the zero profit condition $\Pi\left(\widehat{n}^*(\lambda),\lambda\right) = f$ is satisfied. When the Cournot market is quasi-anticompetitive ($\Delta < 0$), both the increase in λ and the increase in n cause price to increase. When the Cournot market is quasi-competitive ($\Delta > 0$), the increase in λ tends to increase price, while the increase in n tends to decrease it. The former effect dominates. For example, assume non-DRS and by contradiction that after an increase in λ enough additional firms enter the market to keep the price at its level before the increase in λ (or even make it lower). Then, after the increase in λ (i) each firm has a lower share of the market, (ii) the price has not increased, and (iii) the average (variable) cost of production has not decreased (due to non-DRS and individual quantity having decreased). Thus, individual profit has decreased, violating

[58]To compare total surplus under the integer constraint on n, $\mathrm{TS}_{n^*(\lambda)}$, to its value when we ignore the integer constraint, $\mathrm{TS}_{\widehat{n}^*(\lambda)}$, notice the following. For λ between two consecutive thresholds, $\lambda \in (\lambda_k,\lambda_{k+1})$, it holds that $\widehat{n}^*(\lambda) > n^*(\lambda)$. Thus, given that total surplus is single-peaked in n, if there is (weakly) excessive entry under the integer constraint, ignoring the integer constraint exacerbates excess entry. Therefore, between two λ thresholds $\mathrm{TS}_{\widehat{n}^*(\lambda)} < \mathrm{TS}_{n^*(\lambda)}$, and for λ equal to a thresholds $\mathrm{TS}_{n^*(\lambda)}$ has a jump down. But if under the integer constraint entry is insufficient by 1 firm (which is possible), $n^*(\lambda) = n^o(\lambda) - 1$, then the above does not follow.

the zero profit condition. The result still holds under DRS, since under $\Delta > 0$,

$$\left| \frac{\overbrace{\partial \Pi\left(n,\lambda\right)/\partial\lambda}^{+}}{\underbrace{\partial \Pi\left(n,\lambda\right)/\partial n}_{-}} \right| = \left| \frac{\overbrace{\left(1-H_n\right)\frac{\partial Q_n}{\partial\lambda}}^{-}}{\underbrace{\left(1-H_n\right)\frac{\partial Q_n}{\partial n}}_{+} + \underbrace{H_n\frac{Q_n}{n}}_{+}} \right| < \left| \frac{\partial Q_n/\partial\lambda}{\partial Q_n/\partial n} \right| = \left| \frac{\overbrace{dP\left(Q_n\right)/d\lambda}^{+}}{\underbrace{dP\left(Q_n\right)/dn}_{-}} \right|.$$

This means that for individual profit to stay unchanged after an increase in λ, fewer firms need to enter compared to the number of firms that need to enter for the price to remain unchanged after the increase in λ.

The mechanism behind the effect of λ on entry is akin to the impact of collusion on entry in the dynamic stochastic oligopoly model of Fershtman and Pakes (2000), where firms freely enter, set prices and invest in quality. In their model, for example, a potential entrant only looks at own profit to decide whether to enter foreseeing the possibility of future collusion with an incumbent monopolist. This possibility increases entry incentives (*i.e.* it increases the threshold of quality that the incumbent needs to achieve to deter entry) compared to the equilibrium without collusion. This in turn causes the incumbent monopolist to invest more in quality when future collusion is possible. Overall, the collusive equilibrium features on average higher prices but also more entry and higher qualities and consumer surplus.

B.9.4 Entry cost effect on entry

Proposition 12 studies the effect of the entry cost on entry, as well as how this effect depends on the extent of overlapping ownership. It mirrors Proposition 5 with the role of internalized profit $\Psi(n,\lambda)$ now assumed by profit $\Pi(n,\lambda)$.

Proposition 12. Ignore the integer constraint on n (so that entry is given by $\widehat{n}^*(\lambda)$). Then

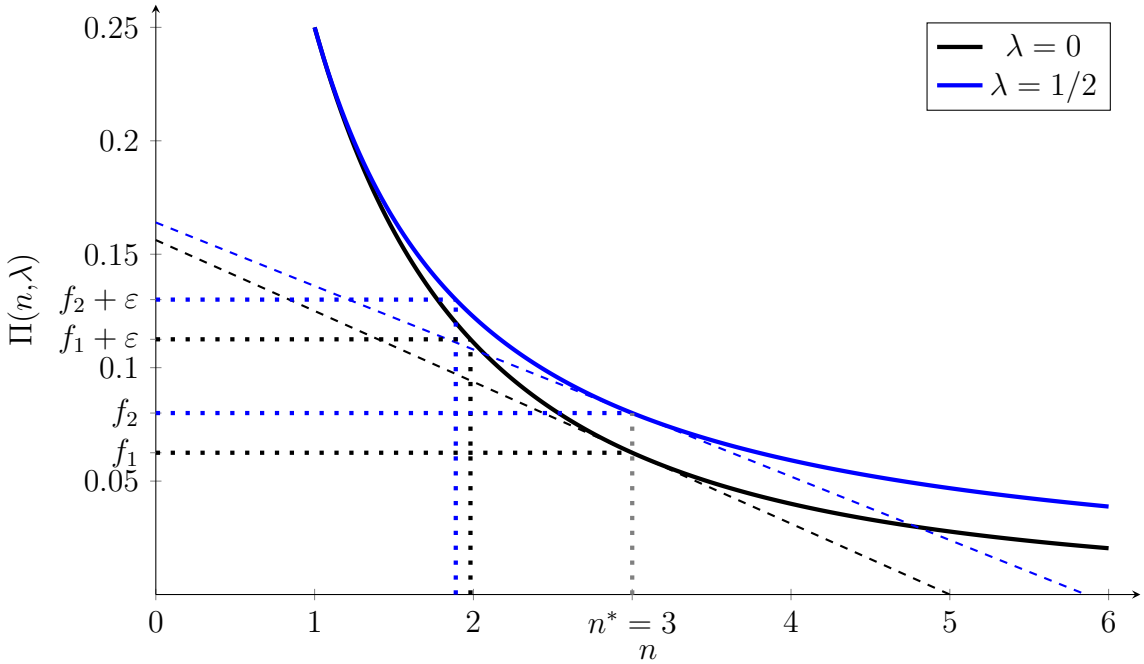
(i) entry is decreasing in the entry cost,

(ii) if λ increases and other parameters x (e.g., demand, cost parameters) change infinitesimally so that $\widehat{n}^*(\lambda)$ stays fixed and $\partial^2\Pi(n,\lambda)/(\partial x\partial n) = 0$ (e.g., (f,λ) infinitesimally changes in direction $\mathbf{v} := (-(d\widehat{n}^*(\lambda)/d\lambda)/(d\widehat{n}^*(\lambda)/df),1))$, then $|d\widehat{n}^*(\lambda)/df|$ changes in direction given by $\operatorname{sgn}\left\{\partial^2\Pi\left(n,\lambda\right)/(\partial\lambda\partial n)|_{n=\widehat{n}^*(\lambda)}\right\}$.

As long as individual profit is decreasing in n, the results of Proposition 12 are not specific to Cournot competition. Part (ii) says that if an increase in λ makes individual profit in the pricing stage equilibrium more (resp. less) strongly decreasing in the number of firms, then an increase in the entry cost needs to be met with a smaller (resp. larger) increase in the number of firms for the zero profit entry condition to continue to hold.

Figure 10 explains the reasoning behind this result. There are initially $n^* = 3$ firms in equilibrium, which can be a result of $\lambda = 0$ and $f = f_1$, or $\lambda = 1/2$ and $f = f_2 > f_1$. Also, for $n \leq 3$, an increase of λ from 0 to 1/2 makes profit less strongly decreasing in n (*i.e.*, $\partial^2\Pi(n,\lambda)/(\partial\lambda\partial n) > 0$). Thus, an increase in the entry cost by ε will decrease entry by more when $\lambda = 1/2$ (and initially $f = f_2$) compared to when $\lambda = 0$ (and initially $f = f_1$).

Figure 10: Entry cost effect on entry mediated by λ under linear demand and CRS



Note: $a = 2$, $b = 1$, $c = 1$. The black and blue solid lines represent $\Pi(n,0)$ and $\Pi(n,1/2)$, respectively. The black and blue dashed lines are tangent to the corresponding solid lines at $n = n^*$.

Claim 3 provides sufficient conditions for the cross derivative of $\Pi(n,\lambda)$ to be negative (resp. positive), which by Proposition 12 implies that overlapping ownership alleviates (resp. exacerbates) the negative effect of the entry cost on entry.

Claim 3. Assume CRS.

(i) If $\partial E_{P'}(Q)/\partial Q \geq 0$, $E_{P'}(Q_n) \in [0,1]$ and $n \geq 5 + E_{P'}(Q_n)$, then $\partial^2\Pi(n,\lambda)/(\partial\lambda\partial n) < 0$ for every $\lambda \in (0,1)$.

(ii) If $\partial E_{P'}(Q)/\partial Q \leq 0$, $E_{P'}(Q_n) \leq 0$ and $n \leq 6/\left(2 - E_{P'}(Q_n)\right)$, then $\partial^2 \Pi\left(n,\lambda\right)/(\partial\lambda\partial n) > 0$ for every $\lambda \in (0,1)$.

Remark B.6. Appendix B provides a more detailed result on $\partial^2 \Pi\left(n,\lambda\right)/(\partial\lambda\partial n)$.

Claim 3 encompasses CESL demand. Therefore, under CESL demand with $E \in [0,1]$ and CRS, in markets with not too low entry ($n \geq 6$ is sufficient), overlapping ownership makes entry less strongly decreasing in the entry cost. This means that as long as it does not induce firms to internalize the entry externality, overlapping ownership could alleviate the negative macroeconomic implications of rising entry costs documented by Gutiérrez et al. (2021) in the U.S. over the past 20 years. The sufficient condition of part (ii) requires $n \leq 3$, as is the case in Figure 10.

The conditions in part (i) of Claim 3 overlap with those of Numerical result 2, which deals with the case of pre-entry overlapping ownership. Thus, under the same parametrization, whether overlapping ownership exacerbates or alleviates the negative effect of the entry cost on entry will depend on the form of overlapping ownership. If overlapping ownership is present prior to entry thus making firms internalize the entry externality, then it exacerbates the effect. If it develops after entry, it alleviates the effect.

B.9.5 Equilibrium entry versus the socially optimal level of entry

The derivative of equilibrium total surplus with respect to n is given by

$$\begin{aligned}\frac{d\,\mathrm{TS}_n}{dn} &= P(Q_n)\left(n\frac{\partial q_n}{\partial n} + q_n\right) - C(q_n) - nC'(q_n)\frac{\partial q_n}{\partial n} - f \\ &= \Pi(n,\lambda) - f + n\left(P(Q_n) - C'(q_n)\right)\frac{\partial q_n}{\partial n},\end{aligned}$$

and therefore

$$\left.\frac{d\,\mathrm{TS}_n}{dn}\right|_{n=\widehat{n}^*(\lambda)} = \overbrace{\Pi\left(\widehat{n}^*(\lambda),\lambda\right) - f}^{=0} + n\left(P(Q_n) - C'(q_n)\right)\left.\frac{\partial q_n}{\partial n}\right|_{n=\widehat{n}^*(\lambda)} \propto \left.\frac{\partial q_n}{\partial n}\right|_{n=\widehat{n}^*(\lambda)},$$

so that with TS_n single-peaked in n, under business-stealing (resp. business-enhancing) competition entry is excessive (resp. insufficient). The results of Mankiw and Whinston (1986) and Amir et al. (2014) generalize to the case of post-entry overlapping ownership. Proposition 13 shows that indeed with business-stealing competition and under the integer constraint, entry is never insufficient by more than one firm.

Proposition 13. The following statements hold:

(i) if $\Delta > 0$ and $E_{P'}(Q) < 2$ on L, then $n^*(\lambda) \geq n^o(\lambda) - 1$,

(ii) if $\Delta < 0$, then $n^*(\lambda) \geq n^o(\lambda) = 1$.

Remark B.7. Under a consumer surplus standard

(i) if $\Delta > 0$, then $n^o(\lambda) = \infty$ (since Q_n is increasing in n), so $n^*(\lambda) < n^o(\lambda)$,

(ii) if $\Delta < 0$, then $n^o(\lambda) = 1$ (since Q_n is decreasing in n), so $n^*(\lambda) \geq n^o(\lambda)$.

Under a consumer surplus standard, entry is insufficient (resp. excessive) when returns to scale are at most mildly increasing (resp. sufficiently increasing).

B.10 Results with (possible) multiplicity of equilibria

This section provides results with the maintained assumption $\Delta > 0$ on L but dropping the assumption that $E'_P < (1 + \lambda + \Delta/n)/H_n$ on L. The second-order condition (SOC) of the firm's problem, that is $E_{P'} < (1 + \lambda + \Delta)/H_n$, will still be assumed to hold strictly in any symmetric pricing stage equilibrium. Then, the Cournot game equilibrium set may consist of multiple symmetric equilibria. Propositions under this relaxed version of the maintained assumption will be marked with an apostrophe (').

B.10.1 Pricing stage equilibrium

Proposition 2' studies the comparative statics of pricing stage equilibria.

Proposition 2'. Let $\Delta > 0$ on L. Then, at extremal equilibria:[59]

(i) total and individual quantity, and total surplus (resp. individual profit) are non-increasing (resp. non-decreasing) in λ,

(ii) individual profit is non-increasing in n,

(iii) total quantity is non-decreasing in n.

[59] By extremal equilibria we mean the equilibrium with minimum quantity among all equilibria and the equilibrium with maximum quantity among all equilibria.

Under $\Delta > 0$, when we drop the condition $E_{P'} < (1+\lambda+\Delta/n)/H_n$ on L guaranteeing uniqueness, the results of Proposition 2 still hold weakly for extremal equilibria. They also hold strictly but only locally around stable equilibria.[60] As observed in AL, a discrete change (e.g., in the integer number n of firms) may even lead to a change in the number of equilibria rendering it hard to make meaningful comparisons between non-extremal equilibra.

B.10.2 Free entry under post-entry overlapping ownership

Proposition 10' studies existence of a free entry equilibrium.

Proposition 10'. Let $\Delta > 0$ on L. Then, at extremal equilibria profit is non-increasing in n and a free entry equilibrium where in the pricing stage firms play an extremal equilibrium exists.

If for example there is multiplicity of pricing stage equilibria for every n, there will exist at least two free entry equilibria: one where the minimum pricing stage equilibrium is played and one where the maximum pricing stage equilibrium is played.[61]

Proposition 13' compares equilibrium entry to the socially optimal level of entry considering also the case of business-enhancing competition. To economize on notation, we are still using q_n, $n^*(\lambda)$ and $n^o(\lambda)$ to denote equilibrium values in a specific extremal equilibrium even though multiple equilibria may exist.

Proposition 13'. Let $\Delta > 0$ on L. Let the same type of extremal equilibrium (*i.e.*, minimum or maximum) be played in the pricing stage of the free entry equilibrium and the planner's solution. Then,

(i) if $q_{n^o(\lambda)-1} \geq q_{n^o(\lambda)}$, then $n^*(\lambda) \geq n^o(\lambda) - 1$.

(ii) if $q_{n^o(\lambda)+1} \geq q_{n^o(\lambda)}$, then $n^*(\lambda) \leq n^o(\lambda)$.

Remark B.8. Proposition 13' and part (ii) of Proposition 13 extend the results of Amir et al. (2014) to the case of post-entry overlapping ownership.

[60] Namely, parts (i)-(iv) of Proposition 2 hold locally in any stable equilibrium (with n treated as a continuous variable in parts (ii)-(iv)).

[61] Observe that extremal equilibria correspond to extremal equilibrium profits. Namely, the minimum (resp. maximum) equilibrium quantity corresponds to the maximum (resp. minimum) equilibrium profit.

Under $\Delta > 0$, when competition is locally business-stealing, equilibrium entry is not insufficient by more than one firm as in the case without overlapping ownership. On the other hand, if competition is locally business-enhancing, entry is not excessive.

B.11 Free entry with pre-entry overlapping ownership: a more tractable framework

In this section we make the model of free entry with pre-entry overlapping ownership more tractable by ignoring the integer constraint on n. The way we do this is *not* just by letting (2) hold with equality. Instead, now each "infinitesimal" firm considers whether to enter or not examining a differential version of (3).[62] Consider firm i of "size" $\varepsilon > 0$ and let $n \in \mathbb{R}_+$ be the number of other firms entering. Firm i's payoff if it enters is $(\varepsilon + \lambda n)\,(\Pi(n+\varepsilon,\lambda) - f)$, while if it does not, it is $\lambda n\,(\Pi(n,\lambda) - f)$. The difference is

$$\varepsilon\Pi(n+\varepsilon,\lambda) + \lambda n\,[\Pi(n+\varepsilon,\lambda) - \Pi(n,\lambda)] - \varepsilon f.$$

Notice that for $\varepsilon = 1$ we recover the case with an integer number of firms. Dividing this expression by ε and letting $\varepsilon \to 0$ gives

$$\Pi(n,\lambda) + \lambda n \frac{\partial \Pi(n,\lambda)}{\partial n} - f.$$

Therefore, $\boldsymbol{q}_n$ is a free entry equilibrium if

$$\overbrace{\Pi(n,\lambda)}^{\substack{\text{own profit}\\ \text{from entry}}} + \lambda \overbrace{n\frac{\partial \Pi(n,\lambda)}{\partial n}}^{\substack{\text{entry externality}\\ \text{on other firms}}} = \overbrace{f}^{\substack{\text{entry}\\ \text{cost}}} \quad \text{and} \tag{6}$$

$$(1+\lambda)\frac{\partial \Pi(n,\lambda)}{\partial n} + \lambda n \frac{\partial^2 \Pi(n,\lambda)}{(\partial n)^2} < 0. \tag{7}$$

Naturally, we only consider the free entry equilibrium and planner's solution with $n \in \mathbb{R}_+$; we denote the number of firms in the two solutions by $n^*(\lambda)$ and $n^o(\lambda)$, respectively. The entry externality is now measured by $n\partial\Pi(n,\lambda)/\partial n$. (6) says that the marginal firm entering is exactly indifferent between entering or not. (7) guarantees that an extra infinitesimal firm does not want to enter, and given that $\partial\Pi(n,\lambda)/\partial n < 0$, can equivalently

[62] Of course, the firm is infinitesimal only for the purpose of the algebra. The firm understands the (marginal) effect of its entry on market outcomes, and in the pricing stage firms still complete á la Cournot but with the symmetric equilibrium solution extended to $n \in \mathbb{R}_{++}$.

be written as

$$1+\lambda-\lambda E_{\partial\Pi/\partial n,n}(n,\lambda)>0, \qquad \text{where} \quad E_{\partial\Pi/\partial n,n}(n,\lambda) := -\frac{\frac{\partial^2\Pi(n,\lambda)}{(\partial n)^2}}{\frac{\partial\Pi(n,\lambda)}{\partial n}}n$$

is the elasticity of the slope of individual profit with respect to n. Also, given that $\partial\Pi(n,\lambda)/\partial n<0$, $\lambda>0$ implies through (6) that the entering firms make positive net profits in equilibrium. For $\lambda=0$, (6) reduces to the standard zero profit condition.

Provided that (7) holds for every n, the (unique) equilibrium level of entry $n^*(\lambda)$ is pinned down by

$$\Pi\left(n^*(\lambda),\lambda\right)+\lambda n^*(\lambda)\left.\frac{\partial\Pi(n,\lambda)}{\partial n}\right|_{n=n^*(\lambda)}=f.$$

Assume that $\Pi(1,\lambda)+\lambda\partial\Pi(n,\lambda)/\partial n|_{n=1}>f$ so that more than 1 firm enters, and

$$\lim_{n\to\infty}\left[\Pi(n,\lambda)+\lambda\partial\Pi(n,\lambda)/\partial n\right]<f.$$

Proposition 14 guarantees that the left-hand side of (6) is decreasing in n, thus ensuring existence of a unique equilibrium.

Proposition 14. If for every n such that $\Pi(n,\lambda)+\lambda\partial\Pi(n,\lambda)/\partial n\geq f$ it holds that $1+\lambda-\lambda E_{\partial\Pi/\partial n,n}(n,\lambda)>0$, then a unique Cournot equilibrium with free entry exists.

Proposition 15. Fix a value for λ and consider the unique symmetric Cournot equilibrium with free entry, where $\partial\Pi(n,\lambda)/\partial n|_{n=n^*(\lambda)}<0$ and $1+\lambda-\lambda E_{\partial\Pi/\partial n,n}(n^*(\lambda),\lambda)>0$.

(i) The number of firms locally changes with λ with direction given by[63]

$$\operatorname{sgn}\left\{\frac{dn^*(\lambda)}{d\lambda}\right\}=\operatorname{sgn}\left\{\overbrace{E_{\partial\Pi/\partial n,\lambda}\left(n^*(\lambda),\lambda\right)}^{\substack{\text{change in magnitude}\\\text{of entry externality}}}+\overbrace{\frac{1}{\lambda}\frac{E_{\Pi,\lambda}\left(n^*(\lambda),\lambda\right)}{E_{\Pi,n}\left(n^*(\lambda),\lambda\right)}}^{\substack{\text{increase in own}\\\text{profit from entry}}}-\overbrace{1}^{\substack{\text{increase in internalization}\\\text{of entry externality}}}\right\}.$$

(ii) The total quantity changes with λ with direction given by

$$\operatorname{sgn}\left\{\frac{dQ_{n^*(\lambda)}}{d\lambda}\right\}=\operatorname{sgn}\left\{\frac{E_{\partial\Pi/\partial n,\lambda}\left(n^*(\lambda),\lambda\right)+\frac{1}{\lambda}\frac{E_{\Pi,\lambda}(n^*(\lambda),\lambda)}{E_{\Pi,n}(n^*(\lambda),\lambda)}-1}{1+\lambda-\lambda E_{\partial\Pi/\partial n,n}\left(n^*(\lambda),\lambda\right)}\frac{\Delta(Q_{n^*(\lambda)},(n-1)\,q_{n^*(\lambda)})}{n^*(\lambda)-1}-1\right\}$$

[63] For $\lambda=0$ cancel the λ in the second term with the one in $E_{\Pi,\lambda}(n,\lambda)$.

where

$$E_{\partial\Pi/\partial n,\lambda}(n,\lambda) := -\frac{\frac{\partial^2\Pi(n,\lambda)}{\partial\lambda\partial n}}{\frac{\partial\Pi(n,\lambda)}{\partial n}}\lambda, \quad E_{\Pi,n}(n,\lambda) := -\frac{\frac{\partial\Pi(n,\lambda)}{\partial n}}{\Pi(n,\lambda)}n > 0, \quad E_{\Pi,\lambda}(n,\lambda) := \frac{\frac{\partial\Pi(n,\lambda)}{\partial\lambda}}{\Pi(n,\lambda)}\lambda > 0$$

are, respectively, the elasticity with respect to λ of the slope of individual profit with respect to n, the elasticity of profit with respect to n, and the elasticity of profit with respect to λ.

Corollary 15.1. In addition to the assumptions of Proposition 15, assume constant returns to scale. Then

$$\operatorname{sgn}\left\{\frac{dn^*(\lambda)}{d\lambda}\right\} = \operatorname{sgn}\left\{\begin{array}{r} \left(n-1+2\lambda-\dfrac{\Lambda_n\left(2n-\Lambda_n E_{P'}\left(Q_n\right)\right)}{n-\Lambda_n}\right)\left(n+\Lambda_n-\Lambda_n E_{P'}\left(Q_n\right)\right) \\ +\lambda(2n-\Lambda_n)\left(2-E_{P'}\left(Q_n\right)\right)-\left.\dfrac{\lambda\Lambda_n(n-\Lambda_n)Q_n E'_{P'}\left(Q_n\right)}{n+\Lambda_n-\Lambda_n E_{P'}\left(Q_n\right)}\right|_{n=n^*(\lambda)} \end{array}\right\},$$

(i) for $\lambda = 0$, given $E_{P'}(Q_{n^*(0)}) < 2$, $dn^*(\lambda)/d\lambda \overset{(\text{ resp. } >)}{<} 0$ if and only if $n^*(0) \overset{(\text{ resp. } >)}{<} 2 + \sqrt{3 - E_{P'}(Q_{n^*(0)})}$.

(ii) If $E'_{P'}(Q_{n^*(\lambda)}) \leq 0$ and $E_{P'}\left(Q_{n^*(\lambda)}\right) > \left[2n - (H_n^{-1} - 1)(n - 1 + 2\lambda)\right]/\Lambda_n$, then $dn^*(\lambda)/d\lambda > 0$.

(iii) If $E_{P'}\left(Q_{n^*(\lambda)}\right) < \max\left\{2, \left[2n - (H_n^{-1} - 1)(n + 1 + 2\lambda)\right]/\Lambda_n\right\}$ and $E'_{P'}(Q_{n^*(\lambda)}) \geq 0$, then $dn^*(\lambda)/d\lambda < 0$.

(iv) If $\lim_{\lambda\to 1^-} E_{P'}(Q_{n^*(\lambda)}) < 2$ (and $E'_{P'}$ bounded), then $dn^*(\lambda)/d\lambda < 0$ for λ close to 1.

(v) Under linear demand

$$\operatorname{sgn}\left\{\frac{dn^*(\lambda)}{d\lambda}\right\} = \operatorname{sgn}\left\{\left.\left(n-1+2\lambda-\frac{2n\Lambda_n}{n-\Lambda_n}\right)(n+\Lambda_n)+2\lambda(2n-\Lambda_n)\right|_{n=n^*(\lambda)}\right\}.$$

(vi) If $E_{\partial\Pi/\partial n,n}(n^*(\lambda),\lambda) \leq 2$, $E_{P'}(Q_{n^*(\lambda)}) < 2$, $E'_{P'}(Q_{n^*(\lambda)}) \geq 0$ and $n^*(\lambda) \geq 2$, then the total quantity decreases with λ.

Claim 4. Under linear demand and constant marginal costs $E_{\partial\Pi/\partial n,n}(n,\lambda) \leq 2$ for every $\lambda \in [0,1]$ and $n \geq 1$.

Corollary 15.1 shows that under reasonable assumptions overlapping ownership can spur entry. Proposition 16 shows that the effect of overlapping ownership on the magnitude of the entry externality is ambiguous in our setting. Proposition 17 shows that with pre-entry overlapping ownership both possibilities of excessive and insufficient entry are possible.

Proposition 16. Assume that $\Delta > 0$ and $E_{P'}(Q_n) < 1 + \frac{\Lambda_n}{n-\Lambda_n} \Big/ \left(E_{Q_n,n}(n,\lambda) + \frac{\Lambda_n}{n-\Lambda_n}\right)$ for $n = n^*(\lambda)$. The direction of the change (due to the change in λ) in the magnitude of the entry externality, $\operatorname{sgn}\left\{E_{\partial\Pi/\partial n,\lambda}(n,\lambda)\right\}$, is given by

$$\operatorname{sgn}\left\{ \overbrace{\frac{E_{Q_n,\lambda}(n,\lambda)}{E_{Q_n,n}(n,\lambda)}}^{+} - \frac{\overbrace{E_{\partial Q_n/\partial n,\lambda}(n,\lambda)}^{-\text{ under Prop. 2(ii)}} + \overbrace{\frac{n-1}{n-\Lambda_n}\left[(E_{Q_n,n}(n,\lambda))^{-1} - 1\right]}^{+}}{\left[E_{Q_n,n}(n,\lambda) + \frac{\Lambda_n}{n-\Lambda_n}\right]\left[1 + \frac{\frac{\Lambda_n}{n-\Lambda_n}}{E_{Q_n,n}(n,\lambda)+\frac{\Lambda_n}{n-\Lambda_n}} - E_{P'}(Q_n)\right]} \Bigg|_{n=n^*(\lambda)} \right\}$$

evaluated at $n = n^*(\lambda)$, where

$$E_{\partial Q_n/\partial n,\lambda}(n,\lambda) := \frac{\frac{\partial^2 Q_n}{\partial\lambda\partial n}}{\frac{\partial Q_n}{\partial n}}\lambda, \quad E_{Q_n,n}(n,\lambda) := \frac{\partial Q_n}{\partial n}\frac{n}{Q_n} > 0, \quad E_{Q_n,\lambda}(n,\lambda) := -\frac{\partial Q_n}{\partial\lambda}\frac{\lambda}{Q_n} > 0.$$

Under constant marginal costs

$$\operatorname{sgn}\left\{E_{\partial\Pi/\partial n,\lambda}(n,\lambda)\right\} = \operatorname{sgn}\left\{ \begin{array}{r} 2\Lambda_n^2\left(E_{P'}(Q_n)\right)^2 + \left[n\Lambda_n(n-\Lambda_n-1) - 2n^2 - \Lambda_n^2\right]E_{P'}(Q_n) \\ -n(n-\Lambda_n)(n+\Lambda_n-6) - \left.\dfrac{\Lambda_n(n-\Lambda_n)^2 Q_n E'_{P'}(Q_n)}{n+\Lambda_n-\Lambda_n E_{P'}(Q_n)}\right|_{n=n^*(\lambda)} \end{array} \right\},$$

which can be negative or positive. For linear demand, $\operatorname{sgn}\left\{E_{\partial\Pi/\partial n,\lambda}(n,\lambda)\right\} = \operatorname{sgn}\left\{6 - (n+\Lambda_n)\right\}$.

Proposition 17. Consider the Cournot model with free entry and pre-entry overlapping ownership. Assume that $\mathrm{TS}(\boldsymbol{q}_n)$ is globally concave in n, and $\lambda < 1$. Then in equilibrium there is excessive (insufficient) entry if and only if

$$E_{P'}\left(Q_{n^*(\lambda)}\right) \overset{(\text{ resp. } >)}{<} H_n^{-1}\left\{1 + \lambda\left(1 - \frac{\Delta(Q_n,(n-1)\,q_n)}{(1-\lambda)\,(1+\lambda(n-1))}\right)\right\}\Bigg|_{n=n^*(\lambda)}.$$

The results of this section very closely resemble the ones we obtain under the integer constraint. Therefore, the gain in tractability from dropping the constraint as described above comes at a minimal cost.

C Proofs of additional results

Where clear we may simplify notation, for example omitting the subscript n,λ for equilibrium objects. We may also write for example Q_n instead of $Q_{n^*(\lambda)}$, n instead of $n^*(\lambda)$. Also, we write $\Pi_n(n,\lambda) \equiv \partial\Pi(n,\lambda)/\partial n$, $\Pi_\lambda(n,\lambda) \equiv \partial\Pi(n,\lambda)/\partial\lambda$, $\Pi_{n\lambda}(n,\lambda) \equiv \partial^2\Pi(n,\lambda)/(\partial n\partial\lambda)$, $\Pi_{nn}(n,\lambda) \equiv \partial^2\Pi(n,\lambda)/(\partial n)^2$.

C.1 Proof of section B.2

Proof of Claim 1 Under CESL demand and constant marginal costs the pricing formula $P(Q_n) - C'(q_n) = -H_n Q_n P'(Q_n)$ gives

$$\begin{aligned} a + b(Q_n)^{1-E} - c &= H_n b(E-1)(Q_n)^{1-E} \quad \text{if } E > 1 \\ a - b\ln Q_n - c &= H_n b \quad \text{if } E = 1 \\ a - b(Q_n)^{1-E} - c &= H_n b(1-E)(Q_n)^{1-E} \quad \text{if } E < 1 \end{aligned}$$

and the result follows. In the case $E > 1$, $E < 2$ and $c > a$ guarantee that there is an interior equilibrium. Notice that if $a > c$, then the profit per unit $P(Q) - AC(q) \geq a - c > 0$ is positive and bounded away from zero for every $Q \geq q \geq 0$, and thus there is no equilibrium. In the case $E < 1$, if $a \leq c$, then in the unique equilibrium $Q_n = 0$.

For linear demand and quadratic costs the pricing formula $P(Q_n) - C'(q_n) = -H_n Q_n P'(Q_n)$ gives $a - bQ_n - c(Q_n/n) = H_n b Q_n$, and the result follows. **Q.E.D.**

C.2 Proofs of section B.3

Proof of Propositions 7 and 7' For simplicity, we use the notation Q_n and q_n to refer to values in a specific equilibrium even if that equilibrium is not unique.

Let $\widetilde{\pi}_i(\boldsymbol{q}) := \pi_i(\boldsymbol{q}) + \lambda \sum_{j\neq i} \pi_j(\boldsymbol{q})$. A linearization of the adjustment process around an equilibrium production profile $\boldsymbol{q}_n$ gives

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix} = \overbrace{\begin{bmatrix} k_1 \frac{\partial^2 \widetilde{\pi}_1(\boldsymbol{q})}{(\partial q_1)^2} & k_1 \frac{\partial^2 \widetilde{\pi}_1(\boldsymbol{q})}{\partial q_1 \partial q_2} & \cdots & k_1 \frac{\partial^2 \widetilde{\pi}_1(\boldsymbol{q})}{\partial q_1 \partial q_n} \\ k_2 \frac{\partial^2 \widetilde{\pi}_2(\boldsymbol{q})}{\partial q_2 \partial q_1} & k_2 \frac{\partial^2 \widetilde{\pi}_2(\boldsymbol{q})}{(\partial q_2)^2} & \cdots & k_2 \frac{\partial^2 \widetilde{\pi}_2(\boldsymbol{q})}{\partial q_2 \partial q_n} \\ \vdots & \vdots & \cdots & \vdots \\ k_n \frac{\partial^2 \widetilde{\pi}_n(\boldsymbol{q})}{\partial q_n \partial q_1} & k_n \frac{\partial^2 \widetilde{\pi}_n(\boldsymbol{q})}{\partial q_n \partial q_2} & \cdots & k_n \frac{\partial^2 \widetilde{\pi}_n(\boldsymbol{q})}{(\partial q_n)^2} \end{bmatrix}}^{=:A} \begin{bmatrix} q_1 - q_n \\ q_2 - q_n \\ \vdots \\ q_n - q_n \end{bmatrix},$$

where for $i \neq j$

$$\frac{\partial^2 \widetilde{\pi}_i(\boldsymbol{q})}{\partial q_i \partial q_j} = P'(Q)\left(1+\lambda - E_{P'}(Q)\left((1-\lambda)s_i+\lambda\right)\right),$$
$$\frac{\partial^2 \widetilde{\pi}_i(\boldsymbol{q})}{(\partial q_i)^2} = P'(Q)\left(2 - \frac{C_i''(q_i)}{P'(Q)} - E_{P'}(Q)\left((1-\lambda)s_i+\lambda\right)\right) < 0$$

are evaluated at the equilibrium production profile $\boldsymbol{q}_n$. The second derivative with respect to q_i evaluated at $\boldsymbol{q}_n$ is negative given that $E_{P'}(Q_n) < \left(1+\lambda+\Delta(Q_n,(n-1)q_n)\right)/H_n$.

Notice that $\frac{\partial^2 \widetilde{\pi}_i(\boldsymbol{q})}{\partial q_i \partial q_j}$ does not depend on the identity of firm j as long as $i \neq j$, so that the off-diagonal elements in each row are equal. From Theorem 2(i) in al Nowaihi and Levine (1985)—which also follows from Hosomatsu (1969)—it follows that all eigenvalues of A are real.

Also, we have that $\partial^2 \widetilde{\pi}_i(\boldsymbol{q})/(\partial q_i \partial q_j)\big|_{\boldsymbol{q}=\boldsymbol{q}_n} = P'(Q_n)\left(1+\lambda - H_n E_{P'}(Q_n)\right)$. We distinguish two cases.

Case 1: If $E_{P'}(Q_n) \leq (1+\lambda)/H_n$, then that combined with $\Delta(Q_n,(n-1)q_n) > 0$ imply

$$\left.\frac{\partial^2 \widetilde{\pi}_i(q)}{(\partial q_i)^2}\right|_{\boldsymbol{q}=\boldsymbol{q}_n} < \left.\frac{\partial^2 \widetilde{\pi}_i(q)}{\partial q_i \partial q_j}\right|_{\boldsymbol{q}=\boldsymbol{q}_n} \leq 0,$$

for every $i \neq j$, and it follows from Theorem 2(ii-a) in al Nowaihi and Levine (1985)—also in Hosomatsu (1969)— that all eigenvalues of A are negative. From standard stability theory we then have that the equilibrium is locally stable.

Case 2: For $E_{P'}(Q_n) > (1+\lambda)/H_n$ we get $\partial^2 \widetilde{\pi}_i(q)/(\partial q_i \partial q_j)\big|_{\boldsymbol{q}=\boldsymbol{q}_n} > 0$, and it follows from Theorem 2(ii-b) in al Nowaihi and Levine (1985) that all eigenvalues of A are negative it and only if

$$\sum_{i=1}^{n} \frac{\left.\frac{\partial^2 \widetilde{\pi}_i(q)}{\partial q_i \partial q_j}\right|_{\boldsymbol{q}=\boldsymbol{q}_n}}{\left.\frac{\partial^2 \widetilde{\pi}_i(q)}{\partial q_i \partial q_j}\right|_{\boldsymbol{q}=\boldsymbol{q}_n} - \left.\frac{\partial^2 \widetilde{\pi}_i(q)}{(\partial q_i)^2}\right|_{\boldsymbol{q}=\boldsymbol{q}_n}} < 1, \quad \text{or equivalently}$$
$$-n\frac{1+\lambda - H_n E_{P'}(Q_n)}{1-\lambda - \frac{C''(q_n)}{P'(Q_n)}} < 1 \Longleftrightarrow -\left[1+\lambda - H_n E_{P'}(Q_n)\right] < \Delta(Q_n,(n-1)q_n)/n,$$

Again the result follows from standard stability theory. **Q.E.D.**

C.3 Proofs of section B.4

Proof of Proposition 8 $\operatorname{sgn}\left\{\frac{d^2Q}{d\lambda dn}\right\}$ is equal to

$$= -\operatorname{sgn}\left\{\begin{array}{c}\left[Q+(n-1)\dfrac{dQ}{dn}\right]\left[n+\Lambda-C''(Q/n)/P'(Q)-\Lambda E_{P'}(Q)\right]-(n-1)Q\times \\ \left[1+\lambda-\dfrac{C'''(Q/n)\left(\frac{dQ}{dn}-\frac{Q}{n}\right)\frac{P'(Q)}{n}-C''(Q/n)P''(Q)\frac{dQ}{dn}}{\left(P'(Q)\right)^2}-\lambda E_{P'}(Q)-\Lambda E'_{P'}(Q)\dfrac{dQ}{dn}\right]\end{array}\right\}$$

$$= -\operatorname{sgn}\left\{\begin{array}{c}\dfrac{n}{n-1}\left[2-E_{P'}(Q)\right]+\dfrac{dQ}{dn}\dfrac{n}{Q}\left[n+\Lambda+\Lambda E_{P'}(Q)\left[E_{P'}(Q)+E_{P''}(Q)\right]\right] \\ +\left(\dfrac{dQ}{dn}\dfrac{n}{Q}-1\right)\dfrac{C'''(Q/n)Q/n}{P'(Q)}-\dfrac{C''(Q/n)}{P'(Q)}\left[\dfrac{n}{n-1}+\dfrac{dQ}{dn}\dfrac{n}{Q}\left(1-E_{P'}(Q)\right)\right]\end{array}\right\}.$$

If also $C'', C''' \geq 0$ and $E_{P'}(Q)\left[E_{P'}(Q)+E_{P''}(Q)\right] \geq -2$, then $\operatorname{sgn}\left\{d^2Q/(d\lambda dn)\right\} = -$ given that $n \geq$ and $E_{P'}(Q) < 2$; the latter two imply $\frac{dQ}{dn}\frac{n}{Q} \in (0,1)$.

Also,

$$\begin{aligned}\frac{d^2q}{d\lambda dn} &= \frac{d\left(\frac{dq}{d\lambda}\right)}{dn} = \frac{d\left[\left(\frac{dQ}{d\lambda}\right)\frac{1}{n}\right]}{dn} = \frac{1}{n}\left[\frac{d^2Q}{d\lambda dn}-\frac{1}{n}\frac{dQ}{d\lambda}\right] \\ &\propto \frac{1}{n}\left\{\begin{array}{c}-\dfrac{Q(n-1)}{n}\left[\begin{array}{c}\left(\dfrac{dQ}{dn}\dfrac{n}{Q}-1\right)\dfrac{C'''(q)q}{P'(Q)}-\dfrac{C''(q)}{P'(Q)}\left[\dfrac{n}{n-1}+\dfrac{dQ}{dn}\dfrac{n}{Q}\left(1-E_{P'}(Q)\right)\right] \\ +\dfrac{n}{n-1}\left[2-E_{P'}(Q)\right]+\dfrac{dQ}{dn}\dfrac{n}{Q}\left[n+\Lambda+\Lambda E_{P'}(Q)\left[E_{P'}(Q)+E_{P''}(Q)\right]\right]\end{array}\right] \\ -\dfrac{\left(n+\Lambda-C''(q)/P'(Q)-\Lambda E_{P'}(Q)\right)^2}{n}\dfrac{dQ}{d\lambda}\end{array}\right\} \\ &\propto -\left(\frac{dQ}{dn}\frac{n}{Q}-1\right)\frac{C'''(q)q}{P'(Q)}+\frac{C''(q)}{P'(Q)}\left[\frac{1}{n-1}+\frac{dQ}{dn}\frac{n}{Q}\left(1-E_{P'}(Q)\right)\right]-\frac{n}{n-1}\left[2-E_{P'}(Q)\right] \\ &\quad -\frac{dQ}{dn}\frac{n}{Q}\left[n+\Lambda+\Lambda E_{P'}(Q)\left[E_{P'}(Q)+E_{P''}(Q)\right]\right]+n+\Lambda-\Lambda E_{P'}(Q).\end{aligned}$$

Under CESL demand, for $Q < \overline{Q}$ the elasticity $E_{P''}(Q)$ of the curvature is then given by

$$E_{P''}(Q) \equiv \frac{QP'''(Q)}{P''(Q)} = \begin{cases}\frac{Qb(E+1)E(1-E)Q^{-(E+2)}}{-bE(1-E)Q^{-(E+1)}} = -(E+1) & \text{if } E \neq 1 \\ \frac{-Q2b/Q^3}{b/Q^2} = -2 & \text{if } E = 1\end{cases},$$

so $E_{P''}(Q) = -(E+1)$. Thus, if marginal costs are linear and demand is CESL, we get

$$\begin{aligned}\frac{d^2q}{d\lambda dn} &\propto \frac{n(n-3)}{n-1}+\Lambda-\left(\Lambda-\frac{n}{n-1}\right)E-\frac{1-\lambda}{n+\Lambda-\Lambda E}\left[n+\Lambda-\Lambda E\right] \\ &\propto n-3+\lambda(n-1)-\left(\frac{\Lambda(n-1)}{n}-1\right)E = n+\Lambda-4-\left(\frac{\Lambda(n-1)}{n}-1\right)E.\end{aligned}$$

Q.E.D.

Proof of Proposition 9 (i-ii) Given what we see in the proof of Proposition 10, for aggregate industry profits we have that

$$\begin{aligned}\frac{\partial\left[n\Pi\left(n,\lambda\right)\right]}{\partial n} &= P(Q_n)\frac{Q_n}{n} - C(q_n) + nP'(Q_n)\left(\frac{Q_n}{n}\right)^2\left[\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\left(1-H_n\right)+H_n\right]\\ &\propto -\left[\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\left(1-H_n\right)-\eta(Q_n)\frac{P(Q_n)-C'(q_n)+C'(q_n)\frac{E_C(q_n)-1}{E_C(q_n)}}{P(Q_n)}+H_n\right]\\ &\overset{(1)}{=} -\left[\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\left(1-H_n\right)-\eta(Q_n)\left(1-\frac{H_n}{\eta(Q_n)}\right)\frac{E_C(q_n)-1}{E_C(q_n)}\right]\\ &\propto E_C(q_n)\left(1-\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\frac{1-H_n}{\eta(Q_n)-H_n}\right)-1,\end{aligned}$$

where $H_n < \eta(Q_n)$ from the pricing formula (1).

(iii) We have that

$$n\Pi(n,\lambda) \equiv P(Q_n)Q_n - nC(q_n) \overset{C''<0}{<} P(Q_n)Q_n - C(Q_n) \leq P(q_1)q_1 - C(q_1) = \Pi(1,\lambda),$$

where the last inequality follows by definition of q_1 being the monopolist's optimal quantity.

To see why Remark B.3 holds notice that for $\lambda = 1$

$$\begin{aligned}\frac{\partial\left[n\Pi\left(n,\lambda\right)\right]}{\partial n} &= P(Q_n)\frac{Q_n}{n} - C(q_n) + nP'(Q_n)\left(\frac{Q_n}{n}\right)^2\\ &\overset{C''>0}{>} P(Q_n)\frac{Q_n}{n} - C'(q_n)q_n + P'(Q_n)\frac{Q_n^2}{n} \propto \frac{P(Q_n)-C'(q_n)}{P(Q_n)} - \frac{1}{\eta(Q_n)} \overset{(1)}{=} 0.\end{aligned}$$

Q.E.D.

Proof of Claim 2 From Claim 1 it follows that

$$(Q_n, P(Q_n)) = \left(\frac{a}{b\left(1+H_n\right)+c/n}, a\left(1-\frac{b}{b\left(1+H_n\right)+c/n}\right)\right)$$

and

$$\begin{aligned}\frac{\partial\left[n\Pi\left(n,\lambda\right)\right]}{\partial n} =& P(Q_n)\frac{Q_n}{n} - C(q_n) + nP'(Q_n)\left(\frac{Q_n}{n}\right)^2\left[\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\left(1-H_n\right)+H_n\right]\\ \propto =& \frac{c}{2n} - \frac{b(1-\lambda)+c}{n+\Lambda+c/b}\left(1-H_n\right),\end{aligned}$$

and the rest follow. **Q.E.D.**

C.4 Proof of section B.5

Proof of Lemma 2 We have seen that the first derivative of equilibrium total surplus with respect to n is given by

$$\frac{d\,\mathrm{TS}_n}{dn} = \Pi(n,\lambda) - f - \Lambda_n Q_n P'(Q_n)\frac{\frac{\partial Q_n}{\partial n} - q_n}{n},$$

so if we denote $\Pi_n(n,\lambda) \equiv \partial\Pi(n,\lambda)/\partial n$, the second derivative is given by

$$\begin{aligned}\frac{d^2\,\mathrm{TS}_n}{(dn)^2} =& \Pi_n(n,\lambda) - \lambda Q_n P'(Q_n)\frac{\frac{\partial Q_n}{\partial n} - q_n}{n} - \Lambda_n \frac{\partial Q_n}{\partial n} P'(Q_n)\frac{\frac{\partial Q_n}{\partial n} - q_n}{n} \\ & - \Lambda_n Q_n P''(Q_n)\frac{\partial Q_n}{\partial n}\frac{\frac{\partial Q_n}{\partial n} - q_n}{n} - \Lambda_n Q_n P'(Q_n)\frac{\left(\frac{\partial^2 Q_n}{(\partial n)^2} - \frac{dq_n}{\partial n}\right) n - \frac{\partial Q_n}{\partial n} + q_n}{n^2} \\ \propto & -\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\left[1 - \lambda - H_n\left(\left(\frac{\partial Q_n}{\partial n}\frac{n}{Q_n} - 1\right)(1 - E_{P'}(Q_n)) + \frac{\partial^2 Q_n}{(\partial n)^2}\left(\frac{\partial Q_n}{\partial n}\right)^{-1} n - 1\right)\right] \\ & + \frac{1-\lambda}{n}.\end{aligned}$$

Under constant marginal costs

$$\begin{aligned}\frac{\partial Q_n}{\partial n} &= \frac{1-\lambda}{n + \Lambda - \Lambda E_{P'}(Q_n)}\frac{Q_n}{n} \implies \\ \frac{\partial^2 Q_n}{(\partial n)^2} &= (1-\lambda)\left(-\frac{1 + \lambda - \lambda E_{P'}(Q_n) - \Lambda E'_{P'}(Q_n)\frac{\partial Q_n}{\partial n}}{\left(n + \Lambda - \Lambda E_{P'}(Q_n)\right)^2}\frac{Q_n}{n} + \frac{\frac{\frac{\partial Q_n}{\partial n} n - Q_n}{n^2}}{n + \Lambda - \Lambda E_{P'}(Q_n)}\right), \\ \frac{\partial^2 Q_n}{(\partial n)^2}\left(\frac{\partial Q_n}{\partial n}\right)^{-1} n &= -n\frac{1 + \lambda - \lambda E_{P'}(Q_n) - \Lambda E'_{P'}(Q_n)\frac{\partial Q_n}{\partial n}}{n + \Lambda - \Lambda E_{P'}(Q_n)} + \frac{\partial Q_n}{\partial n}\frac{n}{Q_n} - 1\end{aligned}$$

so that

$$\begin{aligned}\frac{d^2\,\mathrm{TS}_n}{(dn)^2} \propto & -\left\{\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\left[1 - \lambda - H_n\left(\begin{array}{r}\left(\frac{\partial Q_n}{\partial n}\frac{n}{Q_n} - 1\right)(2 - E_{P'}(Q_n)) \\ -n\frac{1 + \lambda - \lambda E_{P'}(Q_n) - \Lambda E'_{P'}(Q_n)\frac{\partial Q_n}{\partial n}}{n + \Lambda - \Lambda E_{P'}(Q_n)} - 1\end{array}\right)\right] - \frac{1-\lambda}{n}\right\} \\ \propto & -\left\{\begin{array}{r} n(1-\lambda)\left(n + \Lambda_n - \Lambda_n E_{P'}(Q_n)\right) - \left(n + \Lambda_n - \Lambda_n E_{P'}(Q_n)\right)^2 \\ -\Lambda_n\left(\begin{array}{r} -\left(n + \Lambda_n - (1-\lambda) - \Lambda_n E_{P'}(Q_n)\right)(2 - E_{P'}(Q_n)) \\ -n\left(1 + \lambda - \lambda E_{P'}(Q_n) - \Lambda_n E'_{P'}(Q_n)\frac{\partial Q_n}{\partial n}\right) - \left(n + \Lambda_n - \Lambda_n E_{P'}(Q_n)\right)\end{array}\right)\end{array}\right\}.\end{aligned}$$

The partial derivative of the expression in the brackets with respect to $E_{P'}(Q_n)$ is given by

$$\begin{aligned}
&-\Lambda n(1-\lambda)+2\Lambda\left(n+\Lambda-\Lambda E_{P'}\left(Q_n\right)\right)\\
&-\Lambda\left(\Lambda\left(2-E_{P'}(Q_n)\right)+n+\Lambda-(1-\lambda)-\Lambda E_{P'}\left(Q_n\right)+\Lambda-(1-\lambda)+\Lambda\right)\\
\propto&\lambda n-(3\Lambda-2(1-\lambda))=-\left(2\Lambda-(1-\lambda)\right)<0,
\end{aligned}$$

so that, given $E_{P'}(Q_n)<2$, for $d^2\,\mathrm{TS}_n/(dn)^2$ to be negative it is sufficient that

$$\begin{aligned}
&n(1-\lambda)\left(n+\Lambda-2\Lambda\right)-\left(n+\Lambda-2\Lambda\right)^2\\
-\Lambda\left(-n\left(1+\lambda-2\lambda-\Lambda E'_{P'}\left(Q_n\right)\frac{\partial Q_n}{\partial n}\right)-\left(n+\Lambda-2\Lambda\right)\right)\geq 0&\Longleftrightarrow\\
1-\lambda-\frac{\Lambda}{n}E'_{P'}(Q_n)Q_n\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}&\geq-\frac{\left(\Lambda+1-\lambda\right)\left(n-\Lambda\right)}{\Lambda n},
\end{aligned}$$

which is true for $E'_{P'}$ not too high. **Q.E.D.**

C.5 Proofs of sections B.9 and B.10

Where clear we may simplify notation, and write for example n instead of $n^*(\lambda)$.

Proof of Proposition 2' (i) Consider $R(Q_{-i})$ as defined in the proof of Proposition 1. For any Q_{-i} the maximand satisfies $\partial^2\left\{P\left(Q\right)\left[Q-(1-\lambda)Q_{-i}\right]-C(Q-Q_{-i})\right\}/(\partial\lambda\partial Q)=P'(Q)Q_{-i}\leq 0$. Thus, by Topkis' Monotonicity Theorem (e.g., see Vives, 1999), for any fixed Q_{-i}, $R(Q_{-i})$ is non-increasing in λ in the strong set order, and thus, so is $B(Q_{-i})$ as defined in the proof of Proposition 1. It follows then (e.g., see Chapter 2, Vives, 1999) that the extreme fixed points of $B(Q_{-i})$ (*i.e.*, the total quantity produced by $n-1$ firms in extremal equilibria) are non-increasing in λ, and the result follows.

(ii) Let q_n denote the individual quantity in an extremal equilibrium with n firms. We have then that $\pi(q_n)=q_nP(q_n+(n-1)q_n)-C(q_n)\geq q_{n+1}P(q_{n+1}+(n-1)q_n)-C(q_{n+1})\geq q_{n+1}P(q_{n+1}+nq_{n+1})-C(q_{n+1})=\pi(q_{n+1})$, where the first inequality follows from q_n being a best response of an individual firm, and the second inequality follows from the fact that $(n-1)q_n\leq nq_{n+1}$ by part (iii) below.

(iii) For any fixed Q_{-i}, $B(Q_{-i})$ is non-decreasing in n, so the total quantity produced by $n-1$ firms in an extremal equilibrium is non-decreasing in n (e.g., see Chapter 2,

Vives, 1999). We have also seen in the proof of Proposition 1 that when $\Delta > 0$, $R(Q_{-i})$ is non-decreasing in Q_{-i} and the result follows. **Q.E.D.**

Proof of Proposition 10 Given that $\Pi(n,\lambda)$ is decreasing in n by Proposition 2, the result follows given that $\Pi(n,\lambda) < f$ for n large. **Q.E.D.**

Proof of Proposition 10' Given that individual profit in extremal equilibria is non-increasing in n by Proposition 2' the result follows since $\Pi(n,\lambda) < f$ for n large. **Q.E.D.**

Proof of Proposition 11 Given $\Pi\left(\widehat{n}^*(\lambda), \lambda\right) = f$, the Implicit Function Theorem gives

$$\frac{d\widehat{n}^*(\lambda)}{d\lambda} = \frac{(n-1)(H_n^{-1}-1)}{1 + H_n + \Lambda_n^{-1}\left[(1-\lambda)(1-H_n) - C''\left(q_n\right)/P'\left(Q_n\right)\right] - H_n E_{P'}\left(Q_n\right)} > 0,$$

where the inequality follows from what we have seen in section A.2.

(ii) The total derivative of the total quantity is then proportional to

$$\begin{aligned}\frac{dQ_{\widehat{n}^*(\lambda)}}{d\lambda} &\propto \frac{\partial Q_n}{\partial \lambda}\frac{\Lambda_n(1+\lambda) + 1 - \lambda - C''\left(q_n\right)/P'\left(Q_n\right) - \Lambda_n^2 E_{P'}\left(Q_n\right)/n}{(n-1)(n-\Lambda_n)} + \frac{\partial Q_n}{\partial n} \\ &= \frac{Q_n\begin{bmatrix} -\left(\Lambda_n(1+\lambda) + \Delta - \Lambda_n^2 E_{P'}\left(Q_n\right)/n\right) \\ +(n-\Lambda_n)\Delta/n \end{bmatrix}}{(n-\Lambda_n)\left(n + \Lambda_n - C''\left(q_n\right)/P'\left(Q_n\right) - \Lambda_n E_{P'}\left(Q_n\right)\right)} = -\frac{\Lambda_n Q_n}{n(n-\Lambda_n)} < 0,\end{aligned}$$

so total quantity decreases with λ, and thus so does individual quantity since the number of firms increases with λ. The total derivative of the total surplus is

$$\begin{aligned}\frac{d\,\mathrm{TS}_{\widehat{n}^*(\lambda)}}{d\lambda} &= P(Q_n)\frac{dQ_{\widehat{n}^*(\lambda)}}{d\lambda} - \frac{d\widehat{n}^*(\lambda)}{d\lambda}C(q_n) - nC'(q_n)\left(\frac{dQ_{\widehat{n}^*(\lambda)}/d\lambda}{n} - \frac{q_n}{n}\frac{d\widehat{n}^*(\lambda)}{d\lambda}\right) - \frac{d\widehat{n}^*(\lambda)}{d\lambda}f \\ &= \frac{dQ_{\widehat{n}^*(\lambda)}}{d\lambda}\left(P(Q_n) - C'(q_n)\right) - \left(P(Q_n) - C'(q_n)\right)q_n\frac{d\widehat{n}^*(\lambda)}{d\lambda} < 0.\end{aligned}$$

where the second equality follows from $\Pi(\widehat{n}^*(\lambda),\lambda) = f$.

(iii) Last, the total derivative of $\mathrm{MHHI}^* = H_{n^*}$ is

$$\begin{aligned}\frac{d\,\mathrm{MHHI}\left(\boldsymbol{q}_{\widehat{n}^*(\lambda)}\right)}{d\lambda} &= \frac{n-1}{n} + \left(\frac{\lambda}{n} - \frac{\Lambda_n}{n^2}\right)\frac{d\widehat{n}^*(\lambda)}{d\lambda} \propto \frac{n-1}{n}\left(\frac{d\widehat{n}^*(\lambda)}{d\lambda}\right)^{-1} + \frac{\lambda n - \Lambda_n}{n^2} \\ &\propto \frac{\left[1 + \lambda + \Delta(Q_n,(n-1)q_n)/n + \left(\frac{1}{n} - \frac{1}{\Lambda_n}\right)\frac{C''(q_n)}{P'(Q_n)}\right]/H_n - E_{P'}\left(Q_n\right)}{n(n-\Lambda_n)} > 0,\end{aligned}$$

where the inequality is implied by $C'' \geq 0$ combined with the maintained assumption (ii) that requires $E_{P'}(Q_n) < (1+\lambda+\Delta(Q_n,(n-1)q_n)/n)/H_n$. **Q.E.D.**

Proof of Proposition 12 We have that $d\widehat{n}^*(\lambda)/df = (\partial\Pi(n,\lambda)/\partial n)^{-1}\big|_{n=\widehat{n}^*(\lambda)}$, and part (ii) follows if we take the directional derivative of $d\widehat{n}^*(\lambda)/df$. **Q.E.D.**

Proof of Claim 3 We have $\partial\Pi(n,\lambda)/\partial n = P'(Q_n)q_n^2\left[\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}(1-H_n)+H_n\right] < 0$, so

$$\frac{\partial^2\Pi(n,\lambda)}{\partial n\partial\lambda} \propto \left\{\begin{array}{r}\left[-(1-E_{P'}(Q_n))\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}-(2-E_{P'}(Q_n))\frac{\Lambda_n}{n-\Lambda_n}\right]\frac{\partial Q_n}{\partial\lambda}\frac{1}{Q_n}\\ -\frac{\partial^2 Q_n}{\partial n\partial\lambda}\frac{n}{Q_n}-\frac{n-1}{n-\Lambda_n}\left(1-\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\right)\end{array}\right\}.$$

Denote $E'_{P'}(Q) \equiv \partial E_{P'}(Q)/\partial Q$. Under constant marginal costs

$$\frac{\partial^2 Q_n}{\partial n\partial\lambda} = \frac{\left\{\begin{array}{c}\left(-Q_n+(1-\lambda)\frac{\partial Q_n}{\partial\lambda}\right)(n+\Lambda-\Lambda E_{P'}(Q_n))\\ -(1-\lambda)Q_n\left[n-1-(n-1)E_{P'}(Q_n)-\Lambda E'_{P'}(Q_n)\frac{\partial Q_n}{\partial\lambda}\right]\end{array}\right\}}{(n+\Lambda-\Lambda E_{P'}(Q_n))^2}\frac{1}{n}, \quad \text{so that}$$

$$\begin{aligned}\frac{\partial^2\Pi(n,\lambda)}{\partial n\partial\lambda} &\propto \left\{\begin{array}{r}2\Lambda^2(E_{P'}(Q_n))^2+\left[n\Lambda(n-\Lambda-1)-2n^2-\Lambda^2\right]E_{P'}(Q_n)\\ -n(n-\Lambda)(n+\Lambda-6)-\frac{\Lambda(n-\Lambda)^2Q_nE'_{P'}(Q_n)}{n+\Lambda-\Lambda E_{P'}(Q_n)}\end{array}\right\}\\ &< E_{P'}(Q_n)\left[2\Lambda^2E_{P'}(Q_n)-n\Lambda-2n^2-\Lambda^2\right] \leq 0,\end{aligned}$$

where the first (resp. second) inequality follows from $n \geq 5+E_{P'}(Q_n)$, $\lambda \in (0,1)$, $E_{P'}(Q_n) \leq 1$, $E'_{P'} \geq 0$ (resp. $0 \leq E_{P'}(Q_n) \leq 1$). Similarly follows part (ii). **Q.E.D.**

Proof of Proposition 13 Part (i): If $n^o(\lambda) \leq 2$, we are done since $n^*(\lambda) \geq 1$ given that monopoly profit is positive. For $n^o(\lambda) \geq 3$ keep in mind that $E_{P'}(Q) < 2$ on L implies that for every $n \in [2,+\infty)$, $E_{P'}(Q) < (1+\lambda)/H_n$ on L. The proof follows the proof of part (a) of Proposition 1 in Amir et al. (ACK; 2014). By definition, $\mathrm{TS}_{n^o(\lambda)} \geq \mathrm{TS}_{n^o(\lambda)-1}$, which implies $\int_{Q_{n^o(\lambda)-1}}^{Q_{n^o(\lambda)}} P(X)dX - n^o(\lambda)C\left(q_{n^o(\lambda)}\right)+(n^o(\lambda)-1)C\left(q_{n^o(\lambda)-1}\right) \geq f$, which then gives $\Pi(n^o(\lambda)-1,\lambda)-f \geq P\left(Q_{n^o(\lambda)-1}\right)q_{n^o(\lambda)-1}-\int_{Q_{n^o(\lambda)-1}}^{Q_{n^o(\lambda)}}P(X)dX + n^o(\lambda)\left(C\left(q_{n^o(\lambda)}\right)-C\left(q_{n^o(\lambda)-1}\right)\right)$, which given $P' < 0$ and that in the Cournot game total

quantity is increasing in n, implies

$$\begin{aligned}\Pi\left(n^o(\lambda)-1,\lambda\right)-f &> P\left(Q_{n^o(\lambda)-1}\right)\left(q_{n^o(\lambda)-1}+Q_{n^o(\lambda)-1}-Q_{n^o(\lambda)}\right)\\ &\quad +n^o(\lambda)\left(C\left(q_{n^o(\lambda)}\right)-C\left(q_{n^o(\lambda)-1}\right)\right) \implies\\ \Pi\left(n^o(\lambda)-1,\lambda\right)-f &> n^o(\lambda)\left(P\left(Q_{n^o(\lambda)-1}\right)-C'\left(\widetilde{q}\right)\right)\left(q_{n^o(\lambda)-1}-q_{n^o(\lambda)}\right),\end{aligned}$$

for some $\widetilde{q}\in\left[q_{n^o(\lambda)-1}-q_{n^o(\lambda)}\right]$, where the implication follows by the mean value theorem. As $R(Q_{-i})$ is non-decreasing in Q_{-i}, it follows as in the proof in ACK that there exists $\widetilde{Q}_{-i}\in\left[(n^o(\lambda)-2)q_{n^o(\lambda)-1},(n^o(\lambda)-1)q_{n^o(\lambda)}\right]$ such that $\widetilde{q}\in r\left(\widetilde{Q}_{-i}\right)$ with $R\left(\widetilde{Q}_{-i}\right)\geq Q_{n^o(\lambda)-1}$ and $P\left(R(\widetilde{Q}_{-i})\right)\geq C'(\widetilde{q})$, so that $P\left(Q_{n^o(\lambda)-1}\right)\geq P\left(R(\widetilde{Q}_{-i})\right)\geq C'(\widetilde{q})$.

Given $E_{P'}<(1+\lambda)/H_n$, Proposition 2 implies that $q_{n^o(\lambda)-1}>q_{n^o(\lambda)}$, which combined with the above gives $\Pi\left(n^o(\lambda)-1,\lambda\right)-f\geq 0$. Also, by Proposition 2 $\Pi(n,\lambda)$ is decreasing in n, so it must be $n^*(\lambda)\geq n^o(\lambda)-1$ for the entry condition to be satisfied.

Part (ii): Since $\Pi(1,\lambda)>f$, $n^*(\lambda)\geq 1$. Also, $\Delta<0$ on L implies that $C''(q)<0$ for every $q<\overline{Q}$. By Proposition 2 Q_n is decreasing in n, and thus, so is consumer surplus. Also, $n\Pi(n,\lambda)\equiv P(Q_n)Q_n-nC(q_n)<P(Q_n)Q_n-C(Q_n)\leq P(q_1)q_1-C(q_1)=\Pi(1,\lambda)$, where the first inequality follows from $C''<0$. Thus, both consumer surplus and industry profits are maximized for $n=1$, so $n^o(\lambda)=1$. **Q.E.D.**

Proof of Proposition 13' For simplicity, we use the notation Q_n, q_n, TS_n and $\Pi(n,\lambda)$ to refer to values in a specific equilibrium even if that equilibrium is not unique.

(i) The proof works like that of part (i) of Proposition 13. The only differences are that (a) in the Cournot game the total quantity in extremal equilibria is non-decreasing in n (instead of increasing in n), (b) $q_{n^o(\lambda)-1}\geq q_{n^o(\lambda)}$ by assumption (instead of $q_{n^o(\lambda)-1}>q_{n^o(\lambda)}$ implied by conditions on the primitives), and (c) $\Pi(n,\lambda)$ is non-increasing in n in extremal equilibria (instead of decreasing). Still, the weak inequality $\Pi\left(n^o(\lambda)-1,\lambda\right)-f\geq 0$ must hold and given that $\Pi(n,\lambda)$ is non-increasing in n, it must be that $n^*(\lambda)\geq n^o(\lambda)-1$.

(ii) The proof follows the one of part (b) of Proposition 1 in ACK. Since P is decreasing,

$$q_{n^o(\lambda)+1}P\left(Q_{n^o(\lambda)+1}\right)<\int_{n^o(\lambda)q_{n^o(\lambda)+1}}^{(n^o(\lambda)+1)q_{n^o(\lambda)+1}}P(Q)dQ. \tag{8}$$

Also, notice that $V_n(q):=\int_0^{nq}P(Q)dQ-nC(q)$ is concave in q (for every fixed n), since

$V_n'(q) = n\left(P(nq) - C'(q)\right)$, so that

$$V_n''(q) = nP'(nq)\left(n - \frac{C''(q)}{P'(nq)}\right) = nP'(nq)\left(\Delta(nq,(n-1)q) + n - 1 + \lambda\right) < 0.$$

Since $V_n(q)$ is concave in n, it follows that for any n and q,q' such that $q' \geq q$ it holds that

$$V_n(q) - V_n(q') \leq V_n'(q)(q - q') = n\left(P(nq') - C'(q')\right)(q - q'). \tag{9}$$

By definition $\mathrm{TS}_{n^o(\lambda)} \geq \mathrm{TS}_{n^o(\lambda)+1}$, which implies that $\Pi\left(n^o(\lambda)+1,\lambda\right) - f$ is less than or equal to

$$\begin{aligned}
&P\left(Q_{n^o(\lambda)+1}\right) q_{n^o(\lambda)+1} - \int_{Q_{n^o(\lambda)}}^{Q_{n^o(\lambda)+1}} P(X)dX + n^o(\lambda)\left(C\left(q_{n^o(\lambda)+1}\right) - C\left(q_{n^o(\lambda)}\right)\right)\\
<&\int_0^{Q_{n^o(\lambda)}} P(Q)dQ - n^o(\lambda)C\left(q_{n^o(\lambda)}\right) - \left[\int_0^{Q_{n^o(\lambda)+1}} P(Q)dQ - n^o(\lambda)C\left(q_{n^o(\lambda)+1}\right)\right]\\
=&V_{n^o(\lambda)}\left(q_{n^o(\lambda)}\right) - V_{n^o(\lambda)}\left(q_{n^o(\lambda)+1}\right)\\
\leq&n^o(\lambda)\left(P\left(n^o(\lambda)q_{n^o(\lambda)+1}\right) - C'\left(q_{n^o(\lambda)+1}\right)\right)\left(q_{n^o(\lambda)} - q_{n^o(\lambda)+1}\right) \leq 0,
\end{aligned}$$

where the first inequality follows from (8), the second inequality follows from (9), $q_{n^o(\lambda)+1} \geq q_{n^o(\lambda)}$, and the last inequality follows from $q_{n^o(\lambda)+1} \geq q_{n^o(\lambda)+1}$ and $P\left(n^o(\lambda)q_{n^o(\lambda)+1}\right) \geq P\left(Q_{n^o(\lambda)+1}\right) \geq C'\left(q_{n^o(\lambda)+1}\right)$ by the pricing formula (1). Thus, $\Pi\left(n^o(\lambda)+1,\lambda\right) < f$, and given that $\Pi(n,\lambda)$ is non-increasing in n, $n^*(\lambda) \leq n^o(\lambda)$. **Q.E.D.**

C.6 Proofs of section B.11

Proof of Proposition 14 The LHS of (6) is globally decreasing, so (6) has a unique solution given that for $n = 0$ the LHS is at least as high as f and for $n \to \infty$ it is lower than f. Also, (7) is immediately satisfied. **Q.E.D.**

Proof of Proposition 15 and Corollary 15.1 Totally differentiating (6) with respect to λ we get

$$\begin{aligned}
&\Pi_n(n^*(\lambda),\lambda)\left(n^*(\lambda) + (1+\lambda)\frac{dn^*(\lambda)}{d\lambda}\right) + \Pi_\lambda(n^*(\lambda),\lambda)\\
&+\lambda n^*(\lambda)\left(\Pi_{nn}(n^*(\lambda),\lambda)\frac{dn^*(\lambda)}{d\lambda} + \Pi_{n\lambda}(n^*(\lambda),\lambda)\right) = 0,
\end{aligned}$$

which gives

$$\frac{dn^*(\lambda)}{d\lambda} = -\frac{n^*(\lambda)\left(\Pi_n(n^*(\lambda),\lambda) + \lambda\Pi_{n\lambda}(n^*(\lambda),\lambda)\right) + \Pi_\lambda(n^*(\lambda),\lambda)}{(1+\lambda)\Pi_n(n^*(\lambda),\lambda) + \lambda n^*(\lambda)\Pi_{nn}(n^*(\lambda),\lambda)}$$
$$= -\frac{n^*(\lambda)\left(1 + \frac{\Pi_\lambda(n^*(\lambda),\lambda)}{\Pi(n^*(\lambda),\lambda)}\left(\frac{\Pi_n(n^*(\lambda),\lambda)}{\Pi(n^*(\lambda),\lambda)}n^*(\lambda)\right)^{-1} - E_{\partial\Pi/\partial n,\lambda}\left(n^*(\lambda),\lambda\right)\right)}{1+\lambda-\lambda E_{\partial\Pi/\partial n,n}\left(n^*(\lambda),\lambda\right)}.$$

Given what we see in the proof of Claim 3, $E_{\partial\Pi/\partial n,\lambda}\left(n,\lambda\right) - \frac{\Pi_\lambda(n,\lambda)}{\Pi_n(n,\lambda)}\frac{1}{n} - 1$ is equal to

$$\frac{-\left[\lambda\left(1-E_{P'}(Q_n)\right)\frac{\partial Q_n}{\partial n}\frac{n}{Q_n} + \lambda\left(2-E_{P'}(Q_n)\right)\frac{\Lambda_n}{n-\Lambda_n} + 1\right]\frac{\partial Q_n}{\partial\lambda}\frac{1}{Q_n} - \frac{\partial^2 Q_n}{\partial n\partial\lambda}\frac{\lambda n}{Q_n} + \frac{2\Lambda_n-n-1}{n-\Lambda_n}\frac{\partial Q_n}{\partial n}\frac{n}{Q_n} - \frac{2\Lambda_n-1}{n-\Lambda_n}}{\frac{\partial Q_n}{\partial n}\frac{n}{Q_n} + \frac{\Lambda_n}{n-\Lambda_n}},$$

where for constant marginal costs

$$\frac{\partial Q_n}{\partial n} = \frac{1-\lambda}{n+\Lambda-\Lambda E_{P'}\left(Q_n\right)}\frac{Q_n}{n} \xRightarrow{C \text{ linear}}$$
$$\frac{\partial^2 Q_n}{\partial n\partial\lambda} = \frac{\left\{\begin{array}{c}\left(-Q_n + (1-\lambda)\frac{\partial Q_n}{\partial\lambda}\right)\left(n+\Lambda-\Lambda E_{P'}\left(Q_n\right)\right)\\ -(1-\lambda)Q_n\left[n-1-(n-1)E_{P'}\left(Q_n\right) - \Lambda E'_{P'}\left(Q_n\right)\frac{\partial Q_n}{\partial\lambda}\right]\end{array}\right\}}{\left(n+\Lambda-\Lambda E_{P'}\left(Q_n\right)\right)^2}\frac{1}{n},$$
$$\frac{\partial^2 Q_n}{(\partial n)^2} = \frac{\left\{(1-\lambda)\frac{Q_n}{n}\left[\begin{array}{c}-\left(1+\lambda-\lambda E_{P'}(Q_n) - \Lambda E'_{P'}(Q_n)\frac{\partial Q_n}{\partial n}\right)\\ +\frac{n+\Lambda-\Lambda E_{P'}\left(Q_n\right)}{n}\left(\frac{\partial Q_n}{\partial n}\frac{n}{Q_n} - 1\right)\end{array}\right]\right\}}{\left(n+\Lambda-\Lambda E_{P'}\left(Q_n\right)\right)^2} \Longrightarrow$$
$$\frac{\partial^2 Q_n}{(\partial n)^2} = \frac{\partial Q_n}{\partial n}\frac{\left[\begin{array}{c}\frac{n+\Lambda-\Lambda E_{P'}\left(Q_n\right)}{n}\left(\frac{\partial Q_n}{\partial n}\frac{n}{Q_n} - 1\right)\\ -\left(1+\lambda-\lambda E_{P'}(Q_n) - \Lambda E'_{P'}(Q_n)\frac{\partial Q_n}{\partial n}\right)\end{array}\right]}{n+\Lambda-\Lambda E_{P'}\left(Q_n\right)}$$

so that $E_{\partial\Pi/\partial n,\lambda}\left(n,\lambda\right) - \frac{\Pi_\lambda(n,\lambda)}{\Pi_n(n,\lambda)}\frac{1}{n} - 1$ has the same sign as

$$\left[\lambda\left(1-E_{P'}(Q_n)\right)\frac{1-\lambda}{n+\Lambda-\Lambda E_{P'}\left(Q_n\right)} + \lambda\left(2-E_{P'}(Q_n)\right)\frac{\Lambda}{n-\Lambda} + 1\right]\frac{n-1}{n+\Lambda-\Lambda E_{P'}\left(Q_n\right)}$$

$$
\begin{aligned}
&+\lambda\frac{\left\{\begin{array}{l}\left(1+(1-\lambda)\dfrac{n-1}{n+\Lambda-\Lambda E_{P'}(Q_n)}\right)(n+\Lambda-\Lambda E_{P'}(Q_n))\\ +(1-\lambda)\left[(n-1)(1-E_{P'}(Q_n))-\dfrac{\Lambda(n-1)Q_nE'_{P'}(Q_n)}{n+\Lambda-\Lambda E_{P'}(Q_n)}\right]\end{array}\right\}}{(n+\Lambda-\Lambda E_{P'}(Q_n))^2}\\
&+\frac{2\Lambda-n-1}{n-\Lambda}\frac{1-\lambda}{n+\Lambda-\Lambda E_{P'}(Q_n)}-\frac{2\Lambda-1}{n-\Lambda}\\
\propto&\left(n-1+2\lambda-\frac{\Lambda(2n-\Lambda E_{P'}(Q_n))}{n-\Lambda}\right)(n+\Lambda-\Lambda E_{P'}(Q_n))+\lambda(2n-\Lambda)(2-E_{P'}(Q_n))\\
&-\frac{\lambda\Lambda(n-\Lambda)Q_nE'_{P'}(Q_n)}{n+\Lambda-\Lambda E_{P'}(Q_n)},
\end{aligned}
$$

which is positive if $E'_{P'}\leq 0$ and $E_{P'}(Q_n)>[2n-(n/\Lambda-1)(n-1+2\lambda)]/\Lambda$. On the other hand, given $E_{P'}(Q_n)<2$, $2(n+\Lambda-\Lambda E_{P'}(Q_n))>\lambda(2n-\Lambda)(2-E_{P'}(Q_n))$, so if $E'_{P'}\geq 0$ and $E_{P'}(Q_n)<[2n-(n/\Lambda-1)(n+1+2\lambda)]/\Lambda$, then the expression is negative.

If $dn^*(\lambda)/d\lambda\leq 0$, then $Q_{n^*(\lambda)}$ clearly decreases with λ. If $dn^*(\lambda)/d\lambda>0$, then in equilibrium

$$
E_{\partial\Pi/\partial n,\lambda}(n,\lambda)-\frac{\Pi_\lambda(n,\lambda)}{\Pi_n(n,\lambda)}\frac{1}{n}-1>0
$$

and the directional derivative of the total quantity when (λ,n) changes in direction $\mathbf{v}:=(1,dn^*(\lambda)/d\lambda)$ is

$$
\begin{aligned}
\nabla_{\mathbf{v}}Q_n=&\frac{\partial Q_n}{\partial\lambda}+\frac{\partial Q_n}{\partial n}\frac{dn^*(\lambda)}{d\lambda}\\
=&\frac{\partial Q_n}{\partial\lambda}-\frac{n^*(\lambda)\left(1+\frac{\Pi_\lambda(n^*(\lambda),\lambda)}{\Pi_n(n^*(\lambda),\lambda)}\frac{1}{n^*(\lambda)}-E_{\partial\Pi/\partial n,\lambda}(n^*(\lambda),\lambda)\right)}{1+\lambda-\lambda E_{\partial\Pi/\partial n,n}(n^*(\lambda),\lambda)}\frac{\partial Q_n}{\partial n}\\
\propto&\frac{E_{\partial\Pi/\partial n,\lambda}(n,\lambda)-\frac{\Pi_\lambda(n,\lambda)}{\Pi_n(n,\lambda)}\frac{1}{n}-1}{1+\lambda-\lambda E_{\partial\Pi/\partial n,n}(n,\lambda)}\frac{1-\lambda-C''(q)/P'(Q)}{n-1}-1,
\end{aligned}
$$

so that under constant marginal costs, $E_{\partial\Pi/\partial n,n}(n,\lambda)\leq 2$, $E_{P'}(Q)'\geq 0$ and $E_{P'}(Q)<2$,

$$
\begin{aligned}
\operatorname{sgn}\{\nabla_{\mathbf{v}}Q_n\}\leq&\operatorname{sgn}\left\{E_{\partial\Pi/\partial n,\lambda}(n,\lambda)-\frac{\Pi_\lambda(n,\lambda)}{\Pi_n(n,\lambda)}\frac{1}{n}-1-(n-1)\right\}\\
=&\operatorname{sgn}\left\{\begin{array}{r}\left(\lambda(n-1)+2\lambda-\dfrac{\Lambda(2n+(n-1)(n+\Lambda)-n\Lambda E_{P'}(Q_n))}{n-\Lambda}\right)\times\\ (n+\Lambda-\Lambda E_{P'}(Q_n))+\lambda(2n-\Lambda)(2-E_{P'}(Q_n))-\dfrac{\lambda\Lambda(n-\Lambda)Q_nE'_{P'}(Q_n)}{n+\Lambda-\Lambda E_{P'}(Q_n)}\end{array}\right\}\\
\leq&\operatorname{sgn}\left\{2+\lambda(n-1)+2\lambda-\frac{\Lambda(2n+(n-1)(n+\Lambda)-n\Lambda E_{P'}(Q_n))}{n-\Lambda}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&< \operatorname{sgn}\left\{2+\lambda(n-1)+2\lambda-\frac{\Lambda\left(2n+(n-1)(n+\Lambda)-2n\Lambda\right)}{n-\Lambda}\right\} \\
&= \operatorname{sgn}\left\{2+\lambda(n-1)+2\lambda-\frac{\Lambda(n-\Lambda)\,(n+1)}{n-\Lambda}\right\} = \operatorname{sgn}\left\{1+2\lambda-\Lambda n\right\},
\end{aligned}
$$

which is non-positive given $n \geq 2$. **Q.E.D.**

Proof of Claim 4 Under constant marginal costs and linear demand

$$
\begin{aligned}
Q_n &= \frac{n(a-c)}{b(n+\Lambda)}, \quad \Pi(n,\lambda) = \left(a-\frac{n(a-c)}{n+\Lambda}-c\right)\frac{a-c}{b(n+\Lambda)} = \frac{\Lambda(a-c)^2}{b(n+\Lambda)^2}, \\
\frac{\partial\Pi(n,\lambda)}{\partial n} &= \frac{(a-c)^2\left(\lambda(n+\Lambda)^2-2(n+\Lambda)(1+\lambda)\Lambda\right)}{b(n+\Lambda)^4} = -\frac{(a-c)^2\left(2\Lambda-\lambda(n-\Lambda)\right)}{b(n+\Lambda)^3}, \\
\frac{\partial^2\Pi(n,\lambda)}{(\partial n)^2} &= \frac{(a-c)^2\left\{-\lambda(1+\lambda)(n+\Lambda)^3-3(1+\lambda)(n+\Lambda)^2\left[\lambda(n-\Lambda)-2\Lambda\right]\right\}}{b(n+\Lambda)^6} \\
&= \frac{2(a-c)^2(1+\lambda)\left[2\Lambda-\lambda(n-\Lambda)+1-\lambda\right]}{b(n+\Lambda)^4}, \\
E_{\partial\Pi/\partial n,n}(n,\lambda) &= \frac{2[n+\Lambda-(1-\lambda)]\left[2\Lambda-\lambda(n-\Lambda)+1-\lambda\right]}{(n+\Lambda)\left[2\Lambda-\lambda(n-\Lambda)\right]} \\
&= 2\left(1-\frac{1-\lambda}{n+\Lambda}\right)\left(1+\frac{1-\lambda}{2\Lambda-\lambda(n-\Lambda)}\right),
\end{aligned}
$$

Thus, under linear demand and constant marginal costs, $E_{\partial\Pi/\partial n,n}(n,\lambda)$ is decreasing in n, and thus bounded from above by

$$
2\left(1-\frac{1-\lambda}{2}\right)\left(1+\frac{1-\lambda}{2-\lambda(1-1)}\right) = 2\left[1-\left(\frac{1-\lambda}{2}\right)^2\right] \leq 2.
$$

Q.E.D.

Proof of Proposition 16 See the proof of Claim 3. **Q.E.D.**

Proof of Proposition 17 We have seen that the derivative of equilibrium total surplus (in the Cournot game with a fixed number of firms) with respect to n is given by

$$
\frac{d\,\mathrm{TS}(q_n)}{dn} = \Pi(n,\lambda) - f - \Lambda_n Q_n P'(Q_n)\frac{\partial q_n}{\partial n}.
$$

Given (6) we then have that

$$\left.\frac{d\,\mathrm{TS}(\boldsymbol{q}_n)}{dn}\right|_{n=n^*(\lambda)} = -\,\lambda n^*(\lambda)\Pi_n\left(n^*(\lambda),\lambda\right) - \Lambda_{n^*(\lambda)}Q_{n^*(\lambda)}P'\left(Q_{n^*(\lambda)}\right)\left.\frac{\partial q_n}{\partial n}\right|_{n=n^*(\lambda)}$$

and the result follows as in the proof of Proposition 6. **Q.E.D.**

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