Free entry in a Cournot market with overlapping ownership*

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Abstract

We examine the effects of overlapping ownership among existing firms deciding whether to enter a new product market. We show that in most cases—and especially when overlapping ownership is already widespread, an increase in the extent of overlapping ownership will harm welfare by (i) softening product market competition, (ii) reducing entry, thereby (in contrast to standard results) inducing insufficient entry, and (iii) magnifying the negative impact of an increase of entry costs on entry. Overlapping ownership can mostly be beneficial only under substantial increasing returns to scale, in which case industry consolidation (induced by overlapping ownership) leads to sizable cost efficiencies.

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1 Introduction

Overlapping ownership, be it in the form of common or cross-ownership, has generated concern for its potential anti-competitive impact (Elhauge, 2016; Posner et al., 2017), especially due to the rising shares of large investment funds in multiple competitors in several industries; for example airlines (Azar et al., 2018), banks, and supermarkets (Schmalz, 2018). Antitrust authorities take seriously, indeed, the impact of overlapping ownership (e.g., in the 2023 Draft Merger Guidelines).[1] Azar and Vives (2019, 2021) and Backus et al. (2021b) document the dramatic rise in common ownership among publicly listed U.S. companies in the last decades.[2] Cross-ownership is also far from rare. Although antitrust authorities scrutinize horizontal mergers, non-controlling investments in rival firms have gone largely unregulated and are common in multiple industries and countries (e.g., see Shelegia and Spiegel, 2023).

Common owners do influence managers. For example, Shekita (2022) documents 30 separate cases of intervention by common owners—including funds—aimed to influence firm

[1]The 2023 Draft Merger Guidelines put forth by the U.S. Department of Justice (DOJ) and the Federal Trade Commission (FTC) specify: "When an acquisition involves partial ownership or minority interests, the agencies examine its impact on competition." Particularly, they underline that "the agencies have concerns with both cross-ownership, which refers to holding a non-controlling interest in a competitor, as well as common ownership, which occurs when individual investors hold non-controlling interests in firms that have a competitive relationship that could be affected by those joint holdings." The FTC had already held a hearing on the possible anti-competitive effects of common ownership in 2018 (see FTC hearing) and in 2021 authorized a compulsory process for common ownership (see FTC compulsory process). As an example, in 2016, ValueAct agreed to pay $11 million to settle the lawsuit filed by the DOJ (see DOJ press release) alleging that ValueAct violated the reporting and waiting period requirements of the Hart-Scott-Rodino Antitrust Improvements Act of 1976 (HSR Act). The DOJ argued that ValueAct purchased over $2.5 billion of Halliburton and Baker Hughes voting shares with the intent to influence the firms' decisions relating to a proposed merger between them and, therefore, could not rely on the limited "investment-only" exemption to the HSR Act. The European Commission has raised the issue of the impact of overlapping ownership on competition and innovation in the Dow-Dupont merger case in 2017, arguing that "so-called 'passive' shareholders, have more influence than their formal, minority, equity share" and offering multiple examples whereby institutional shareholders "acknowledge that they exert influence on individual firms with an industry-wide perspective" (see Commission 27/03/2017, Case M.7932 Dow/DuPont). Similarly, in the Bayer-Monsanto merge case (see Commission 21/03/2018, Case M.8084 Bayer/Monsanto), the European Commission suggested "common shareholding (. . .) to be taken as an element of context in the appreciation of any significant impediment to effective competition (. . .)." Tzanaki (2021) reviews the treatment of overlapping ownership by antitrust authorities. In addition to antitrust authorities, the OECD also held "a hearing on common ownership by institutional investors and its impact on competition" in 2017 (see OECD hearing) and has since issued a working paper on the topic (Bas et al., 2023).

[2]Ownership links also exist among private firms, for example through the stakes of private equity firms (e.g., see Eldar and Grennan, 2023), which has generated antitrust concerns (Wilkinson and White, 2006). Particularly, Li et al. (2023) find that venture capital firms (VCs) that fund multiple pharmaceutical startups shut down the startups' lagging R&D projects encouraging them to turn to alternative projects. They provide evidence that VCs do so not only to limit the duplication of R&D costs but also to create market power for successful startups.

conduct.[3] Antón et al. (2023) provide evidence that common owners foster anti-competitive outcomes by tolerating managerial slack and offering top management incentives that are not very sensitive to firm performance. In the context of mergers and acquisitions, Antón et al. (2022) note that diversified shareholders of the acquiring firm can profit from value-destroying acquisitions through their stakes in non-merging competing firms. Indeed, they find that ownership by acquirer shareholders in non-merging competitors is associated with a higher likelihood of deal completion and lower acquirer CAR (*i.e.*, cumulative abnormal return). Particularly, funds with more rival ownership are more likely to vote in favor of the acquisition. Azar et al. (2021) find that the Big Three successfully engage with firms in which they hold significant stakes to make them reduce their CO_2 emissions.[4]

At the same time, firm entry patterns have been argued to pose a significant impact on the aggregate economy. Using a panel of U.S. states over the period 1982–2014, Gourio et al. (2016) find that (positive) shocks to the number of new firms have sizable and lasting (positive) effects on a state's real GDP, productivity, and population. Gutiérrez and Philippon (2019) document a decline in entry of firms in the U.S. economy and estimate the elasticity of entry with respect to Tobin's Q to have dropped to zero since the late 1990s, up to which point it was positive and significant.[5] Gutiérrez et al. (2021) argue that increases in entry costs have had a considerable impact on the U.S. economy over the past 20 years, leading to higher concentration, as well as lower entry, investment, and labor income. Figure 1 shows the increase in regulatory restrictions that has accompanied the decrease in entry.

The literature above documents the decline in firm entry rates (accompanied by a milder decrease in firm exit rates) and a concurrent increase in overlapping ownership over

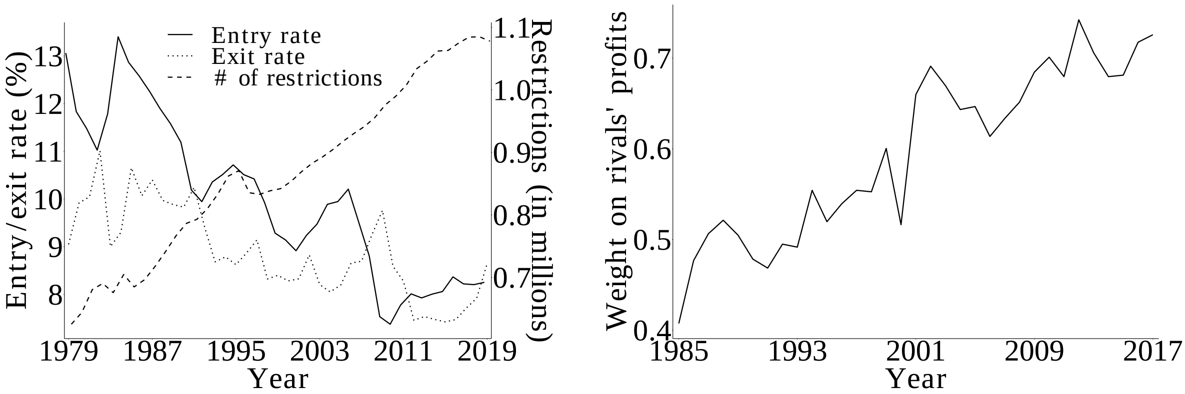
[3]In 2013, Trian Fund Management, a shareholder of DuPont, proposed to split up the company citing DuPont's (i) low performance relative to competitors, (ii) lack of R&D and other measures that could help it gain market share, and (iii) agreement to pay $750 million more than required in a settlement with Monsanto over a patent dispute. DuPont resisted and won the proxy vote with the support of three of its largest shareholders, the "Big Three" (*i.e.*, BlackRock, Vanguard, and State Street Global Advisors; see DuPont proxy vote). While other explanations have also been proposed, in an opinion piece, Schmalz (2015; DuPont opinion piece) offered the index funds' diversified interests in the industry as the reason behind them not enforcing relative performance evaluation, stronger DuPont CEO incentives, and more R&D to boost its market share.

[4]Schmalz (2021) provides a compelling survey of the available evidence on how common owners influence firm decisions. See also Elhauge (2021).

[5]Apart from a generalized decline in entry, Decker et al. (2016) document a particular decline in high-growth young firms in the U.S. since 2000, when such firms could have had a major contribution to job creation.

Figure 1: Firm entry, regulatory restrictions, and overlapping ownership trends in the U.S.

(a) Firm entry/exit rates and regulatory restrictions **(b)** Average weight on competing firms' profits



Note: Firm count and death data are from U.S. Census Bureau Business Dynamics Statistics. The firm entry (resp. exit) rate in year t is calculated as the count of age-zero firms (resp. firm deaths) in year t divided by the average count of firms in year t and $t-1$. The total number of regulatory restrictions data are from McLaughlin et al. (2021). Panel (b) shows the average intra-sector Edgeworth sympathy coefficient for the largest 1500 firms by market capitalization (*i.e.*, the average weight placed by a firm on the profits of another firm in the same sector relative to a weight of 1 placed on its own profit), as calculated in Azar and Vives (2021) based on Thomson-Reuters 13F filings data on institutional ownership.

close to 40 years in the U.S. economy (see Figure 1). There are several explanations for the decreased entry dynamism, an increase in entry costs (for technological or regulatory reasons) being a prominent one. It is possible also that apart from softening competition in pricing, overlapping ownership also contributes to diminishing entry dynamism. Some recent empirical work points in this direction. Newham et al. (2022) find that in the U.S. pharmaceutical industry, higher common ownership between the brand firm and potential generic entrants leads to fewer generic entrants. Relatedly, Xie and Gerakos (2020) and Xie (2021) show that common institutional ownership between a brand-name drug manufacturer and a potential generic entrant increases the likelihood that the two enter into a settlement agreement whereby the brand firm pays the generic to delay entry. Ruiz-Pérez (2019) estimates a structural model of market entry and price competition under common ownership in the U.S. airline industry to find that the higher the common ownership between the incumbents (*i.e.*, airlines that serve a certain route) and a potential entrant, the lower the likelihood that the latter will enter (to serve the same route).

However, these empirical results do *not* imply that entry decreases with overlapping ownership in any market or under any conditions. More importantly, they do not speak to the welfare effects of overlapping ownership (through its effect on both product market competition and entry). Therefore, a theoretical analysis is needed that will (i) examine in more detail the relationship between entry and overlapping ownership and (ii) gauge the welfare effects of overlapping ownership in markets without barriers to entry.

To this end, in this paper, we provide a framework to study the effects of overlapping ownership in a Cournot oligopoly with free entry. We study an industry or product market which established firms with existing ownership ties consider whether to enter; that is, there is pre-entry overlapping ownership. This is common in today's markets with extensive common ownership links among public firms.[6] In Appendix B we consider the case of post-entry overlapping ownership.

We are interested in several questions. How does overlapping affect entry and welfare? Will overlapping ownership suppress entry (by inducing firms to internalize the negative externality that their entry would pose on other firms) making it insufficient (from a welfare standpoint) or will entry still tend to be excessive as in the case without overlapping ownership? What are the forces at play? How does overlapping ownership mediate the (negative) effect of entry costs on entry?

The main takeaway of our analysis is that in most relevant cases—and especially when overlapping ownership is already high, an increase in the extent of overlapping ownership will harm welfare not only by (i) softening product market competition but also by (ii) reducing entry, thereby (in contrast to standard results) inducing or exacerbating insufficient entry, and (iii) magnifying the negative impact of an increase in entry costs on entry. Overlapping ownership can mostly be beneficial only under increasing returns to scale, in which case industry consolidation (induced by overlapping ownership) leads to significant cost efficiencies.

Overlapping ownership differs from collusion in terms of both the mechanism through which it affects competitive outcomes and the actual competitive effects. The mechanism in common ownership is through incentives of owners and managers as stated above (Schmalz, 2018, 2021; Antón et al., 2023). Collusion works with dynamic threats, rewards, and punishments.[7] Both pre-entry overlapping ownership and collusion induce firms to internalize the effects of their actions on other firms' profits, but we show that the former

[6]Firms continuously decide whether to enter new product markets or expand their presence to new locations. For example, chain stores decide whether to pay a fixed cost to open a new branch, pharmaceuticals whether to incur R&D and regulatory costs to enter a new drug market, tech firms whether to perform R&D to enter new product markets, and airlines which routes to serve.

[7]The effect of overlapping ownership on incentives could in turn have coordinated effects facilitating or impeding collusion (e.g., see Malueg, 1992; Gilo et al., 2006).

gives rise to trade-offs and forces that are not present under collusion.

In infinitely repeated oligopoly games, (the prospect of) entry acts as a constraint on incumbents. It limits the attainable collusive outcomes since by colluding incumbents increase prices thereby enhancing the incentives for entry by new firms (e.g., see McAfee and McMillan, 1992; Asker and Nocke, 2021). Similarly, post-entry overlapping ownership (*i.e.*, when ownership links develop after entry) tends to spur entry by softening pricing competition (thereby increasing profits), since a firm decides whether to enter only seeking to maximize its own profit.

On the other hand, pre-entry overlapping ownership induces a novel trade-off (absent in the context of collusion or post-entry overlapping ownership) in terms of its effect on entry. We distinguish the three channels (not specific to our assumption of Cournot competition) through which an increase in the level of pre-entry overlapping ownership affects entry. Overlapping ownership tends to limit entry by increasing the degree of internalization of the negative externality of entry on other firms' profits but also tends to increase equilibrium profits in the product market competition stage, which tends to increase entry. Overlapping ownership also changes the magnitude of the entry externality, and this channel has an ambiguous effect on entry.

The effect of overlapping ownership on entry will depend on the size of the different channels and the direction of the ambiguous channel's effect. We find that an increase in the degree of overlapping ownership can limit but also spur entry. For markets with many active firms and low levels of overlapping ownership, the rise in own profit due to an increase in overlapping ownership can dominate.[8] However, in markets with only a few firms or already high levels of overlapping ownership, competition in the product market is already soft enough, so further increases in overlapping ownership suppress entry. Common ownership among U.S.-listed firms is indeed already widespread and increasing (see Figure 1) and several U.S. markets are dominated by a few large public companies (e.g., see Shapiro, 2019). Thus, further increases in common ownership are likely to limit entry by public firms into product markets where other public firms already operate (or also consider operating).

However, it is not immediately obvious that this suppression of entry is welfare-damaging. Particularly, if entry is excessive (compared to the total surplus-maximizing

[8]The result also applies to markets that do not exist yet but which many firms will enter if there is low overlapping ownership.

level of entry)—as is generally the case in homogeneous product markets without overlapping ownership (see Mankiw and Whinston, 1986; Amir et al., 2014), then the suppression of entry caused by overlapping ownership will tend to improve welfare. This tendency toward excessive entry is due to business-stealing competition (*i.e.*, the fact that the individual quantity decreases with the number of firms). Namely, if by entering a firm causes incumbent firms to reduce output, entry is more attractive to the entrant than it is socially desirable.[9]

But—given that ownership links cause firms to partly internalize the effect that their entry would have on other firms' profits—does this general excessive entry result apply to markets *with* overlapping ownership? We show that it does not. In the standard case of business-stealing competition, we find that under decreasing returns to scale (DRS) and high levels of overlapping ownership, entry is insufficient.[10] Then, any decrease in entry (e.g., due to an expansion of overlapping ownership) will harm welfare. On the other hand, entry is (weakly) excessive under substantial increasing returns to scale (IRS). In that case, the socially optimal level of entry under both a total surplus and a consumer surplus standard is a monopoly, which can be achieved through extreme (high) levels of overlapping ownership.

Also, we show that under common assumptions, overlapping ownership exacerbates the negative impact of an increase in entry costs on entry. Therefore, overlapping ownership could magnify the negative macroeconomic implications documented in Gutiérrez et al. (2021).

Last, we examine the effects of overlapping ownership in the case where apart from the commonly-owned firms, there are also maverick firms (price-taking and without ownership ties), which may enter the market. The presence of maverick firms essentially changes the demand faced by the commonly-owned firms by depressing it and making it more elastic. This suppresses entry by commonly-owned firms and makes it less sensitive to the level of

[9]Remember that the (free-entry) equilibrium markup is positive. That is, the price—which is equal to the value to consumers of a marginal increase in a firm's quantity—is higher than the marginal cost of production. In other words, the marginal social benefit of production is higher than its marginal cost. Therefore, the reduction in each firm's output caused by entry leads to a reduction in total surplus not internalized by the entrant.

[10]That is because the planner takes advantage of cost savings due to entry to a greater extent than firms do. Although—with overlapping ownership—a firm also prefers lower average costs both for itself and for the other firms, the internalization of the price effect (that its entry would have) suppresses entry relative to the planner's solution. Under business-enhancing competition (*i.e.*, when the individual quantity increases with the number of firms in the Cournot game), entry is again insufficient.

overlapping ownership. Our results on the effects of overlapping ownership on consumer surplus, entry by commonly-owned firms, as well as our comparison of equilibrium and socially optimal levels of entry extend to this case with the demand appropriately adjusted.

The plan of the paper is as follows. Section 2 discusses related literature and section 3 presents the model and studies the quantity-setting stage. Section 4 studies the entry stage, existence, and uniqueness of equilibrium in the complete game with entry. Section 5 studies the effects of overlapping ownership under free entry. Section 6 discusses the robustness of our results and considers post-entry overlapping ownership and the entry of maverick firms. Last, section 7 concludes. Proofs are gathered in Appendix A. Supplementary material (including the examination of the case of post-entry overlapping ownership) and proofs thereof are in Appendices B and C.

2 Related literature

Research attention to the possible anti-competitive effects of overlapping ownership dates back to at least Rubinstein and Yaari (1983) and Rotemberg (1984). Recently, interest in the topic has revived given the rising shares of large diversified funds. As Banal-Estañol et al. (2020) show, the profit loads firms place on competing firms increase if the holdings of more diversified investors increase relative to those of less diversified investors. Multiple empirical studies have been conducted and there is a debate on whether and how common ownership affects corporate conduct and softens competition.[11]

Theoretical work has considered models where the effects of overlapping ownership are not only through product market competition: when (i) there are diversification benefits because investors are risk-averse (Shy and Stenbacka, 2020) or (ii) firms choose cost-reducing or quality-enhancing R&D investment possibly with R&D spillovers (Bayona and López, 2018; López and Vives, 2019), product positioning (Li and Zhang, 2021) or qualities (Brito et al., 2020), (iii) firms invest in a preemption race (Zormpas and Ruble, 2021), (iv) firms may choose to transfer their innovation technology to a rival firm (Papadopoulos et al., 2019). Last, other studies have examined the effects of overlapping ownership in a general equilibrium setting (Azar and Vives, 2019, 2021) or under alternative models of corporate control (Vravosinos, 2023).

[11]While He and Huang (2017), Azar et al. (2018), Park and Seo (2019), Boller and Morton (2020), Banal-Estañol et al. (2020), and Antón et al. (2023) find evidence in favor of this hypothesis, others have found little to no effect (e.g., see Koch et al., 2021; Lewellen and Lowry, 2021; Backus et al., 2021a).

All of the models above treat the number of firms in the industry as exogenous.[12] Sato and Matsumura (SM; 2020) provide a circular-market model with constant marginal cost (MC) of production and free entry under pre-entry symmetric common ownership. In their model, the welfare effects of common ownership are directly implied by its effects on entry.[13] They show that entry always decreases with common ownership. Thus, given that in their setting for low levels of common ownership entry is excessive while for high it is insufficient, welfare has an inverted-U shaped relationship with the degree of common ownership, which implies a strictly positive optimal degree of common ownership.

Our model differs from theirs in several ways. First, we consider quantity instead of price competition and discuss the main forces behind our results, which do not depend on the mode of competition. Second, we derive our results under general demand (in SM demand is inelastic) and cost functions—allowing for increasing, constant, and decreasing marginal cost—and consider examples of parametric assumptions for ease of interpretation. In our setting, total surplus depends on equilibrium objects not only through the number of firms. Our modeling allows us to delineate three channels through which pre-entry overlapping ownership affects entry, examine how the type of returns to scale mediates the welfare effects of overlapping ownership, and test the robustness of the results obtained in SM.[14]

Our work can be seen as an extension of the literature on free entry in homogeneous product markets. Mankiw and Whinston (1986) show that in a symmetric homogeneous product market with free entry and non-decreasing marginal cost (MC) where in the quantity-setting stage (i) the total quantity increases with the number of firms and (ii) the business-stealing effect is present, entry is never insufficient by more than one firm. Amir et al. (2014) extend these results to the case of mildly decreasing MC, showing that still under business-stealing competition, entry is never insufficient by more than one firm. We extend the result of Amir et al. (2014) to the case of competition under overlapping

[12]Li et al. (2015) show that in a Cournot duopoly, the incumbent firm can strategically develop cross-ownership to deter the other firm from entering.

[13]Welfare only depends on the number of firms, the cost of transportation, and the entry cost. Consumers have a unit demand and pay transportation costs proportional to their distance from the firm that they choose to buy from. The planner's problem is equivalent to minimizing the total transportation and entry costs; the former decrease with the number of firms, while the latter increase with it.

[14]For example, SM find that entry always decreases with overlapping ownership, while in our case overlapping ownership sometimes spurs entry. In addition, in our model equilibrium total surplus can behave in multiple different ways as the extent of overlapping ownership changes—contrary to the inverted-U relationship found in SM. Last, we study how overlapping ownership mediates the effect of the entry cost on entry, which is not examined in SM.

ownership, showing that under business-enhancing competition, entry is always insufficient. However, we show that under business-stealing competition, overlapping ownership can lead to insufficient entry (by more than one firm) when returns to scale are decreasing.

The setting of symmetric firms with a symmetric overlapping ownership structure that we consider preserves the symmetry of the Cournot game, which allows for extensions of existing oligopoly results (e.g., see Vives, 1999) to the case of competition under overlapping ownership. Namely, we extend the results of Amir and Lambson (2000), who use lattice-theoretic methods to study equilibrium existence and comparative statics with respect to the (exogenous) number of firms in a symmetric Cournot market, and of Amir et al. (2014), who build on the latter to study free entry.

3 The Cournot-Edgeworth λ-oligopoly model with free entry

There is a (large enough) finite set $\mathcal{F} \coloneqq \{1,2,\ldots,N\}$ of N symmetric firms that can potentially enter a market. The game has two stages, the entry stage and the quantity-setting stage. In the first stage, each firm chooses whether to enter by paying a fixed cost $f > 0$.[15] In the quantity-setting stage, entrants compete à la Cournot. Namely, each firm i chooses its production quantity, $q_i \in \mathbb{R}_+$, simultaneously with the other firms. We denote by $s_i \coloneqq q_i/Q$ firm i's share of the total quantity $Q \coloneqq \sum_{i=1}^{n} q_i$. We also write $\boldsymbol{q}$ and $\boldsymbol{q}_{-i}$ to denote the production profile of all firms and all firms expect i, respectively; also, $Q_{-i} \coloneqq \sum_{j \neq i} q_j$.

3.1 The quantity-setting stage

Each firm i's production cost is given by the function $C : \mathbb{R}_+ \to \mathbb{R}_+$ with $C'(q) > 0$ for every q. Denote by $E_C(q) \coloneqq C'(q)q/C(q)$ the elasticity of the cost function. We say that there is decreasing/constant/increasing marginal cost (MC) when C' is (globally) strictly decreasing/constant/strictly increasing, respectively. Notice that under decreasing or constant MC, returns to scale (which take into account the fixed entry cost f) are increasing, while increasing MC tends to make returns to scale decreasing.[16] When

[15]We study pure-strategy equilibria. If firms decide whether to enter sequentially, this is without loss of generality. However, if they decide simultaneously, then there can also be equilibria where firms mix in their entry decisions (e.g., see Cabral, 2004).

[16]Because of the fixed entry cost, returns to scale will not be globally decreasing. That is, the average total cost of production $(f + C(q))/q$, will not be globally increasing in the firm's output.

$C(q) = cq^{\kappa}/\kappa$ for some $c,\kappa > 0$, firms have constant-elasticity costs and $E_C(q) \equiv \kappa$. (i) For $\kappa = 1$, we have constant MC, (ii) for $\kappa \in (0,1)$, we have decreasing MC, (iii) for $\kappa > 1$, increasing MC. $AC(q) \coloneqq C(q)/q$ is the average variable cost.

The inverse demand function $P : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $P'(Q) < 0$ for every $Q \in [0,\overline{Q})$, where $\overline{Q} \in (0, +\infty]$ is such that $P(Q) > 0$ if and only if $Q \in [0,\overline{Q})$. We assume that there exists $\overline{q} > 0$ such that $P(q) < AC(q)$ for every $q > \overline{q}$, and that P and C are twice differentiable.[17] For $Q < \overline{Q}$ we denote by $\eta(Q) \coloneqq -P(Q)/(QP'(Q))$ the elasticity of demand, and by $E_{P'}(Q) \coloneqq -P''(Q)Q/P'(Q)$ the elasticity of the slope of inverse demand. An inverse demand function with constant elasticity of slope (CESL), $E_{P'}(Q) \equiv E$, allows for log-concave and log-convex demand encompassing linear and constant elasticity specifications. When we refer to linear demand, we mean $P(Q) = \max\{a - bQ, 0\}$. Every result applies to generic cost and inverse demand functions unless otherwise stated. We assume that the optimal (gross) monopoly profit is higher than the entry cost; that is, $\max_{Q \geq 0}\{P(Q)Q - C(Q)\} > f$.

Suppose n firms enter. A quantity profile $\boldsymbol{q}^*$ is an equilibrium of the quantity-setting stage if for each firm $i \in \{1,\ldots,n\}$, $q_i^* \in \arg\max_{q_i \geq 0}\left\{\pi_i(q_i,\boldsymbol{q}_{-i}^*) + \lambda\sum_{j\neq i}\pi_j(q_i,\boldsymbol{q}_{-i}^*)\right\}$, where $\pi_i(\boldsymbol{q}) \coloneqq P(Q)q_i - C(q_i)$ and $\lambda \in [0,1]$ is the (exogenous) Edgeworth (1881) coefficient of effective sympathy among firms.[18] This coefficient can for example arise from a symmetric overlapping ownership structure (be it common or cross-ownership) as in López and Vives (2019) or Azar and Vives (2021).

Given a quantity profile $\boldsymbol{q}$ where the number of firms that have entered is $n \equiv \dim(\mathbf{q})$, total surplus is given by $\text{TS}(\boldsymbol{q}) \coloneqq \int_0^Q P(X)dX - \sum_{i=1}^n C(q_i) - nf$, while the Herfindahl–Hirschman index (HHI) and modified HHI (MHHI) are given by $\text{HHI}(\boldsymbol{q}) \coloneqq \sum_{i=1}^n s_i^2$ and $\text{MHHI}(\boldsymbol{q}) \equiv (1-\lambda)\,\text{HHI}(\boldsymbol{q}) + \lambda$. We denote the MHHI at a symmetric equilibrium by $H_n \coloneqq (1 + \lambda(n-1))/n$.

3.2 Equilibrium in the quantity-setting stage

3.2.1 Existence and uniqueness of a quantity-setting stage equilibrium

Having described the environment we first derive conditions for equilibrium existence and uniqueness in the quantity-setting stage using lattice-theoretic methods as in Amir

[17] P is required to be differentiable for $Q \in (0,\overline{Q})$. $P(Q)$ and its derivatives may be undefined for $Q = 0$ (e.g., with $\lim_{Q\downarrow 0} P(Q) = +\infty$ and $\lim_{Q\downarrow 0} P'(Q) = -\infty$).

[18] Section B.1 in Appendix B presents models that give rise to this objective function.

and Lambson (AL; 2000). Let $\Delta(Q,Q_{-i}) \coloneqq 1 - \lambda - C''(Q - Q_{-i})/P'(Q)$ be defined on the lattice $L \coloneqq \{(Q,Q_{-i}) \in \mathbb{R}^2_{++} : \overline{Q} > Q \geq Q_{-i}\}$. $\Delta > 0$ allows for increasing, constant, and mildly decreasing MC, while $\Delta < 0$ allows for substantially decreasing MC.[19]

Proposition 1. The following statements hold:

(i) Assume $\Delta(Q,Q_{-i}) > 0$ on L. Then, in the quantity-setting stage

- (a) there exists a symmetric equilibrium and no asymmetric equilibria,
- (b) if also $E_{P'}(Q) < (1 + \lambda + \Delta(Q,Q_{-i})/n)/H_n$ on L, then there exists a unique and symmetric equilibrium.

(ii) Assume that $\Delta(Q,Q_{-i}) < 0$ and $E_{P'}(Q) < \frac{1+\lambda+\Delta(Q,Q_{-i})}{1-(1-\lambda)(1-s_i)}$ on L. Then, in the quantity-setting stage

- (a) for every $m \in \{1,2\ldots,n\}$ there exists a unique quantity q_m such that any quantity profile where each of m firms produces quantity q_m and the remaining $n - m$ firms produce 0 is an equilibrium,
- (b) no other equilibria exist.

Remark 3.1. The second order of differentiability of $P(Q)$ is inessential. However, it simplifies the arguments and interpretation and emphasizes the tension between the assumption $\Delta < 0$ and the one on $E_{P'}(Q)$. The latter guarantees that π_i is strictly concave in q_i whenever $P(Q) > 0$. Decreasing MC is needed for $\Delta < 0$ but at the same time tends to violate profit concavity.[20]

Corollary 1.1 studies the existence and uniqueness of the quantity-setting stage equilibrium under linear demand and linear-quadratic cost. The linear-quadratic cost function is

[19]Studies have indeed found evidence of declining marginal costs ($C'' < 0$), as required by $\Delta < 0$, across several industries (e.g., see Ramey, 1991; Betancourt and Malanoski, 1999; Diewert and Fox, 2008).

[20]In the $\Delta > 0$ case, for $\lambda = 0$ we recover the condition $C'' - P' > 0$, under which AL show that a symmetric equilibrium exists and there are no asymmetric equilibria (Theorem 2.1). In the $\Delta < 0$ case, the assumption on $E_{P'}$ guarantees that the firm's objective is quasiconcave in its quantity, under which condition AL show the same result. For $\lambda = 1$, increasing MC is necessary for the uniqueness of the (symmetric) equilibrium. To see why, notice for example that with constant MC, there are infinitely many equilibria (the symmetric one included), all with the same total quantity arbitrarily distributed across firms, since each firm maximizes aggregate industry profits. Analogously, with $C'' < 0$ it is an equilibrium for firms to concentrate all production in one firm to take advantage of the decreasing MC, as indicated in part (ii-a) of the proposition.

of the form $C(q) = c_1 q + c_2 q^2/2$, where $c_1 \geq 0$, for (i) $q \in [0, +\infty)$ if $c_2 \geq 0$, (ii) $q \in [0, -c_1/c_2]$ if $c_2 < 0$.[21]

Corollary 1.1. Let demand be linear, $P(Q) = \max\{a - bQ, 0\}$, and cost be linear-quadratic with $a > c_1 \geq 0$ and $c_2 > -2bc_1/a$. Then,

(i) if $c_2 > -b(1-\lambda)$, then $\Delta > 0$ on L, and a unique and symmetric equilibrium exists,

(ii) if $c_2 < -b(1-\lambda)$, then $\Delta < 0$ on L, and a unique (in the class of symmetric equilibria) symmetric equilibrium exists.

In light of Proposition 1 we maintain from now on the following assumption unless otherwise stated in a specific result. The assumption should be understood to hold at the relevant values of (n,λ) for each result.[22]

Maintained Assumption. The conditions in part (i-a,b) or part (ii) of Proposition 1 hold.

Remark 3.2. When in a result we assume $\Delta > 0$ (resp. $\Delta < 0$) it is thus understood that the additional assumption of part (i) (resp. part (ii)) of Proposition 1 also holds. In section B.11 of the Appendix we discuss what happens when the condition in part (i-b) need not hold.

The maintained assumption guarantees that firms will play a symmetric equilibrium in the quantity-setting stage subgame of any subgame-perfect equilibrium.[23] Given that monopoly profit is positive, that equilibrium will be interior. When $\Delta < 0$, the quantity-setting subgame also has asymmetric equilibria; however, these cannot be played on the equilibrium path of an SPE of the complete game, since the entering firms that do not produce would prefer to avoid the entry cost by not entering.

We denote by $\boldsymbol{q}_n$ the symmetric Cournot equilibrium when n firms are in the market (which is unique under our maintained assumption) and with some abuse of notation by q_n the quantity each firm produces in that profile, where the subscript n now does

[21] Cost is indeed increasing over $q \leq -c_1/c_2$ when $c_2 < 0$. The value of $C(q)$ for higher q will not matter in applications, as parameter values will be such that firms do not produce more than $-c_1/c_2$.

[22] For example, for global comparative statics of the Cournot game as λ changes, the assumption is assumed to hold for fixed n and every $\lambda \in [0,1]$. For the existence of a free-entry equilibrium for a fixed λ, it is sufficient that the assumption hold for every $n \in \mathbb{R}_{++}$ and that fixed λ.

[23] Proposition 7 in Appendix B.3 studies the stability of the quantity-setting stage equilibrium.

not refer to the identity of the n-th firm; we also write $Q_n \coloneqq nq_n$, $\text{TS}_n \coloneqq \text{TS}(\boldsymbol{q}_n)$.[24] For any $n > 0$ we denote by $\Pi(n,\lambda) \coloneqq P(Q_n)\, q_n - C(q_n)$ the individual (gross) profit in the symmetric equilibrium of the Cournot game with n firms and Edgeworth coefficient λ, and refer to $\Pi(n,\lambda) - f$ as net profit. When we ignore the integer constraint on n, we allow all equilibrium objects, such as $\Pi(n,\lambda)$, to be defined for $n \in \mathbb{R}_{++}$. The Cournot equilibrium pricing formula is

$$\frac{P(Q_n) - C'(q_n)}{P(Q_n)} = \frac{H_n}{\eta(Q_n)}. \tag{1}$$

3.2.2 Comparative statics of the quantity-setting stage equilibrium

As a first step in examining the effects of overlapping ownership, Proposition 2 describes some comparative statics for the quantity-setting stage (*i.e.*, with a fixed number of firms). The total effect of overlapping ownership on consumer and total surplus will then be the sum of two effects: (a) the direct effect (*i.e.*, with the number of firms fixed) studied in part (i) of the proposition and (b) the indirect effect through its effect on entry, which is studied in part (iv) of the proposition and in section 5.

Proposition 2. The following statements hold:

(i) the total and individual quantity, and total surplus (resp. individual profit) are decreasing (resp. increasing) in λ,

(ii) the individual profit is decreasing in n,

(iii) if $E_{P'}(Q) < 2$ (resp. $E_{P'}(Q) > (1+\lambda)/\lambda$) for every $Q < \overline{Q}$, then the individual quantity is decreasing (resp. increasing) in n over $n \geq 2$,[25]

(iv) if $\Delta > 0$ (resp. $\Delta < 0$), then the total quantity is increasing (resp. decreasing) in n.

Part (iii) says that competition is business-stealing (*i.e.*, q_n is decreasing in n) under standard assumptions. Part (iv) says that, as in AL, for $\Delta > 0$ the Cournot market is quasi-competitive (*i.e.*, Q_n is increasing in n) while for $\Delta < 0$ it is quasi-anticompetitive (*i.e.*, Q_n is decreasing in n).

[24] To simplify notation, we suppress the dependence of these objects on λ.

[25] The condition $E_{P'}(Q) > (1+\lambda)/\lambda$ is very strong, especially given the assumption $E_{P'}(Q) < (1 + \lambda + \Delta/n)/H_n$ on L. Also, it pushes against profit concavity in its own quantity, which can even make the monopolist's problem ill-behaved. For example, with CESL demand, when $E > 2$, $\lim_{Q\downarrow 0}(P(Q)Q - C(Q)) = +\infty$.

Part (i) shows the direct effect of an increase in overlapping ownership on consumer surplus to be negative. Combined with part (iv), this implies that if $\Delta > 0$, then an increase in overlapping ownership that suppresses entry will harm consumer welfare, as both the direct and the indirect effect push towards this direction.[26] On the other hand, if $\Delta < 0$, then an increase in overlapping ownership that suppresses entry will indirectly boost consumer welfare. Then, the combined effect on consumer surplus will depend on whether the direct or the indirect effect dominates. The sign of the indirect effect is positive because when $\Delta < 0$, increasing returns to scale are strong enough to make the cost-savings (induced by a reduction in the number of firms) that are passed on to the consumer more than compensate for the harm due to increased market power (that also comes with industry consolidation).

Part (i) shows the direct effect of an increase in overlapping ownership on total surplus to also be negative. Since total surplus is generally not monotone in the level of entry, the sign of the indirect effect will depend on whether the equilibrium level of entry is excessive or insufficient (from a welfare standpoint).[27] Provided that total surplus is single-peaked in n, if entry is insufficient (which section 5.2 shows to be the case under increasing MC and λ high), then an increase in overlapping ownership that suppresses entry will harm total welfare, as both the direct and the indirect effect push towards this direction. On the other hand, if it is excessive (which section 5.2 shows to be the case under decreasing MC or λ low), then the combined effect on total surplus of an entry-suppressing increase in overlapping ownership will depend on whether the direct or the indirect effect dominates.

4 The entry stage

To study the indirect effects, we first need to describe the entry stage. Assume that potential entrants have overlapping ownership with a coefficient of effective sympathy $\lambda \in [0,1]$. Given that $n-1$ firms enter, it is optimal for an n-th firm to enter if and only if $(1+\lambda(n-1))\,(\Pi\,(n,\lambda)-f) \geq \lambda(n-1)\,(\Pi\,(n-1,\lambda)-f)$.[28] This can equivalently

[26]In sections 5.1 and 5.3 we will see that in most relevant cases, an increase in λ suppresses entry.

[27]Appendix B studies how aggregate industry profits depend on the number of firms.

[28]When $\Delta < 0$, there are also asymmetric quantity-setting stage equilibria as described in Proposition 1, which are not played on an SPE path. However, these asymmetric quantity-setting stage equilibria can be played on out-of-equilibrium paths, which leads to SPE multiplicity. We focus on SPE where only symmetric quantity-setting stage action profiles are played on out-of-equilibrium paths. This restricts attention to the SPE with the maximum number of entrants.

be written as

$$\Psi(n,\lambda) \coloneqq \overbrace{\Pi(n,\lambda)}^{\substack{\text{own profit}\\ \text{from entry}}} - \lambda \overbrace{(n-1)\left(\Pi(n-1,\lambda) - \Pi(n,\lambda)\right)}^{=:\Xi(n,\lambda)>0,\text{ entry externality on other firms}} \geq \overbrace{f}^{\substack{\text{cost of}\\ \text{entry}}}, \tag{2}$$

where $\Xi(n,\lambda)$ denotes the externality that the entry of the n-th firm poses on the other firms (*i.e.*, the absolute value of the reduction in the aggregate profits of all other firms caused by the entry of the n-th firm). $\Psi(n,\lambda)$ is a firm's own profit from entry minus the part of the entry externality that is internalized by the firm (*i.e.*, the entry externality multiplied by λ). We call $\Psi(n,\lambda)$ a firm's "internalized profit" from entry. We assume that when indifferent, firms enter. Then, $\boldsymbol{q}_n$ is a free-entry equilibrium if and only if

$$\Psi(n,\lambda) \geq f > \Psi(n+1,\lambda), \tag{3}$$

which for $\lambda = 0$ reduces to the standard free entry condition $\Pi(n,0) \geq f > \Pi(n+1,0)$. For $\lambda = 1$, it reduces to $n\Pi(n,1) - (n-1)\Pi(n-1,1) \geq f > (n+1)\Pi(n,1) - n\Pi(n,1)$; firms enter as long as entry increases aggregate gross profits by enough to cover entry costs. We assume that $\Psi(N,\lambda) < f$ for every λ.

In deciding whether to enter a firm compares the profit it will make to the cost of entry and the negative externality its entry will pose to the other firms.[29] Overlapping ownership directly alters the incentives of firms to enter in a way additional to its effect on individual profit in the Cournot game.

The planner's problem We will consider the problem of a total surplus-maximizing planner who takes λ as given and chooses the number of firms that will compete à la Cournot. Denote by $n^o(\lambda) \coloneqq \arg\max_{n\in\mathbb{N}} \mathrm{TS}_n$ the number of firms that given λ maximizes total surplus.[30] If the planner could control both n and λ, she would set $\lambda = 0$, since total surplus is decreasing in λ (this is also true under a consumer surplus standard). Define also $\widehat{n}^o(\lambda) \coloneqq \arg\max_{n\in\mathbb{R}_+} \mathrm{TS}_n$, the number of firms that given λ maximizes total surplus if we ignore the integer constraint on n.

[29]If we compare this with the post-entry overlapping ownership case (see section 6.5), where—modulo the integer constraint—net profit is zero, we see that investors would prefer to become common owners before rather than after entry.

[30]Since monopoly net profit is positive, it follows that $n^o(\lambda) \geq 1$. Also, the planner can give subsidies in case the net profit in the symmetric Cournot equilibrium is negative.

We will also look at comparative statics with respect to λ—including how free-entry equilibrium consumer and total surplus vary with λ. This way we will see when regulation of overlapping ownership is most warranted.

Existence and uniqueness of equilibrium Define $\Delta\Pi(n,\lambda) \coloneqq \Pi(n,\lambda) - \Pi(n-1,\lambda) < 0$, the decrease in individual profit caused by the entry of an extra firm. Proposition 3 identifies a condition under which a unique equilibrium exists. We treat n as a continuous variable and differentiate with respect to it.

Define $E_{\Delta\Pi,n}(n,\lambda) \coloneqq -\frac{\frac{\partial\Delta\Pi(n,\lambda)}{\partial n}}{\Delta\Pi(n,\lambda)}(n-1)$, a measure of the elasticity with respect to n of the slope of individual profit with respect to n, and $\varepsilon(n,\lambda) \coloneqq \partial\Pi(\nu,\lambda)/\partial\nu|_{\nu=n-1}/\Delta\Pi(n,\lambda) - 1$. $\varepsilon(n,\lambda)$ is close to 0, since by the mean value theorem $\Delta\Pi(n,\lambda) = \partial\Pi(\nu,\lambda)/\partial\nu|_{\nu=\nu^*}$ for some real number $\nu^* \in [n-1,n]$.

Proposition 3. Assume that for every $n \in [1,+\infty)$

$$E_{\Delta\Pi,n}(n,\lambda) < \frac{(n-1)(1+\lambda+\varepsilon(n,\lambda))}{1+\lambda(n-1)}.$$

Then, $\Psi(n,\lambda)$ is decreasing in n, so a unique equilibrium with free entry exists.

Remark 4.1. It can be checked that for $\lambda < 1$, $\Psi(n,\lambda)$ is indeed decreasing in n under linear demand and linear-quadratic cost with $a > c_1 \geq 0$ and $c_2 \geq 0$ (and also for $n \geq 5/2$ if $c_2 > -b(1-\lambda)$).

The condition in Proposition 3 requires that equilibrium profit in the quantity-setting stage be not too convex in n; that is, the rate at which individual profit decreases with n should not decrease (in absolute value) too fast with n. Concerning internalized profit $\Psi(n,\lambda)$, see (2), an increase in n (i) decreases the first term $\Pi(n,\lambda)$, (ii) tends to increase the entry externality $\Xi(n,\lambda)$ through the increase in $(n-1)$ (as entry affects the profits of more firms), which tends to decrease $\Psi(n,\lambda)$, and (iii) affects $\Xi(n,\lambda)$ through its effect on the magnitude of the entry externality $\Pi(n-1,\lambda) - \Pi(n,\lambda)$ on a single firm. As long as the per-firm entry externality does not decrease with n too fast, $\Psi(n,\lambda)$ decreases with n.[31] We maintain the assumption that $\Psi(n,\lambda)$ is decreasing in n. Then, for a given λ,

[31] If $\Pi(n,\lambda)$ is concave in n, then the condition is satisfied given $\varepsilon(n,\lambda) \approx 0$. For $\lambda = 0$ the condition reduces to $\Pi(n,\lambda)$ being decreasing in n, which has been shown in Proposition 2. For $\lambda = 1$, provided that entry does not lead to significant savings in variable costs (*i.e.*, MC is not too increasing), only one firm enters. For example, under constant MC, (gross) aggregate industry profits are independent of

the number $\widehat{n}^*(\lambda)$ of firms that enter in equilibrium if we ignore the integer constraint on n is pinned down by $\Psi(\widehat{n}^*(\lambda),\lambda) = f$, and $n^*(\lambda) = \lfloor \widehat{n}^*(\lambda) \rfloor$ is the number of firms that enter if we respect the integer constraint.

5 The effects of overlapping ownership under free entry

This section studies the indirect effects of overlapping ownership (through its impact on entry). Particularly, it addresses the following concerns about the anti-competitive effects that overlapping ownership can have: suppress entry by inducing firms to internalize the effect that their entry would have on other firms' profits (subsection 5.1), induce or exacerbate socially sub-optimal levels of entry (subsection 5.2), and magnify the impact of entry costs on entry (subsection 5.4). Subsection 5.3 summarizes our main findings (except for those of subsection 5.4) using the case of linear demand and linear-quadratic cost.

5.1 Overlapping ownership effects on entry

The effect of changes in λ on entry will be determined by the sign of the (partial) derivative of $\Psi(n,\lambda)$ with respect to λ. If $\partial\Psi(n,\lambda)/\partial\lambda$ is positive (resp. negative), then increases in λ should be met with increases (resp. decreases) in n for (3) to continue to hold. Proposition 4 studies the effects of overlapping ownership on entry.

Proposition 4. Equilibrium entry (locally) changes with λ in direction given by

$$\operatorname{sgn}\left\{\frac{d\widehat{n}^*(\lambda)}{d\lambda}\right\} = \operatorname{sgn}\left\{ \overbrace{\frac{1}{\lambda}\frac{E_{\Pi,\lambda}(\widehat{n}^*(\lambda),\lambda)}{\widehat{E}_{\Pi,n}(\widehat{n}^*(\lambda),\lambda)}}^{\substack{\text{increase in own}\\ \text{profit from entry}}} - \overbrace{E_{\Xi,\lambda}(\widehat{n}^*(\lambda),\lambda)}^{\substack{\text{change in magnitude}\\ \text{of entry externality}}} - \overbrace{1}^{\substack{\text{increase in internalization}\\ \text{of entry externality}}} \right\}$$

where $\widehat{E}_{\Pi,n}(n,\lambda) \coloneqq -(\Pi(n,\lambda) - \Pi(n-1,\lambda))(n-1)/\Pi(n,\lambda) > 0$ is a measure of the elasticity of individual profit with respect to n, $E_{\Pi,\lambda}(n,\lambda) \coloneqq \lambda\partial\Pi(n,\lambda)/\partial\lambda/\Pi(n,\lambda) > 0$ is the elasticity of individual profit with respect to λ, and $E_{\Xi,\lambda}(n,\lambda) \coloneqq \lambda\partial\Xi(n,\lambda)/\partial\lambda/\Xi(n,\lambda)$ is the elasticity of the entry externality with respect to λ.[32]

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the number of firms, so $\Psi(n,1) = 0$ for every $n \geq 2$, and thus $n^*(1) = 1$. It can also be seen that under decreasing MC (see the derivation of Remark 5.3 in the proof of Proposition 5), $\Psi(2,1) < 0$, so $n^*(1) = 1$. On the other hand, in Figure 3c, where there is increasing MC, $n^*(1) = 5$.

[32] $E_{\Xi,\lambda}$ is also a measure of the elasticity with respect to λ of the slope of individual profit with respect to n.

Remark 5.1. With the integer constraint $n^*(\lambda)$ does not change with an infinitesimal change $d\lambda$ in λ unless we are in the knife-edge case where $\Psi(n^*(\lambda),\lambda) = f$. Thus, as λ increases everything will behave as in the case with a fixed number of firms until λ reaches knife-edge cases causing a jump in $n^*(\lambda)$ as implied by Proposition 4.

An increase in overlapping ownership affects entry through three separate channels. On the one hand, it increases the degree of internalization of the negative externality of entry on other firms' profits; this increased internalization tends to limit entry. On the other hand, it tends to increase equilibrium profits in the Cournot game, which tends to increase entry.[33] Last, there is a channel with an ambiguous effect on entry: overlapping ownership changes the magnitude of the entry externality; that is, it affects how strongly equilibrium profits in the quantity-setting stage decrease with the number of firms. A high (and positive) elasticity $E_{\Xi,\lambda}$ of the entry externality Ξ with respect to λ tends to make entry decreasing in λ, while a negative $E_{\Xi,\lambda}$ tends to make entry increasing in λ. The magnitude of the entry externality $\Xi(n^*(\lambda),\lambda)$ can increase or decrease with λ.[34]

Remark 5.2. Evaluating the expressions in Propositions 3 and 4 requires evaluation of profits and derivatives thereof in different equilibria of the quantity-setting stage (with n and $n-1$ firms). This is possible under parametric assumptions while the problem remains intractable in general.[35] In what follows, we present numerical results.

Numerical Result 1. Under CESL demand, constant MC, $\lambda < 1$ and $\widehat{n}^*(\lambda) \geq 2$, it holds that

(i) entry is decreasing in λ if (a) $E \in (1,2)$ and $\lambda \geq 1/2$, or (b) $E < 1$ and $\lambda \geq 2/5$, or (c) $E \in (1,2)$, $\lambda \leq 3/10$, and $\widehat{n}^*(\lambda) \leq 3$, or (d) $E < 1$ and $\widehat{n}^*(\lambda) \leq 3$,

(ii) entry is increasing in λ if $\widehat{n}^*(\lambda) \geq 7$ and (a) $E \in (1,2)$ and $\lambda \leq 1/4$, or (b) $E \in [0,1)$ and $\lambda \leq 1/5$,

(iii) the total quantity is decreasing in λ.

These results can be loosely interpreted as follows. For λ low and entry high, competition is intense, so there is ample room for an increase in λ to soften it and increase

[33]The model of Stenbacka and Van Moer (2022), where two firms choose how much to invest in product innovation and can only produce if they successfully innovate, has two similar forces.

[34]Under the parametrization of Figure 3b, Ξ is decreasing in λ. However, Ξ is increasing in λ under the parametrization of Figure 3a.

[35]See Propositions 14 and 15 in Appendix B for a differential version of Propositions 3 and 4, respectively.

individual profit in the Cournot game. For λ high and/or entry low, competition is already soft enough, so the increase in the internalization of the entry externality (due to an increase in λ) dominates and entry decreases with λ.[36]

Indeed, the case of already high λ seems most relevant. In the U.S. for example, common ownership levels among publicly listed firms have indeed been "high enough" during at least the last decade (see Figure 1).[37] Thus, if private firms are treated as a competitive fringe that only affects the residual demand in the oligopolies of public firms,[38] then further increases in common ownership among the latter are likely to reduce entry by public firms in product markets where other public firms are already present (see section 6.4).

Under $\Delta > 0$, when entry is decreasing in λ, the price is increasing in λ, since both the increase in λ and the resulting decrease in entry tend to increase the price. On the other hand, for low levels of overlapping ownership and high levels of entry, overlapping ownership spurs entry (up to the point where λ is too high and then entry decreases with it). However, even in that case, Numerical Result 1 asserts that with constant MC, the direct effect of λ on the total quantity dominates, so that the price (resp. consumer surplus) always increases (resp. decreases) with λ.[39]

Last, a few words on the interpretation of this comparative statics exercise on a change in λ are in place. Strictly speaking, this exercise amounts to changing the level of overlapping ownership *before* firms make their entry decisions. Therefore, it can be thought of as a counterfactual or a comparison of otherwise similar markets that have different levels of overlapping ownership (before firms enter). When interpreting changes in λ in a market that firms have already entered, one should consider the following. If our model predicts that a change in λ will cause the number of firms to fall, whether incumbent firms will indeed exit can depend on the extent to which the entry cost f is a

[36]Let $\Pi(n,\lambda)$, $n \geq 2$, be (strictly) concave in λ (which is the case, for example, with linear demand and linear-quadratic cost under the conditions $a > c_1 \geq 0$ and $c_2 > -2bc_1/a$ of Corollary 1.1). This means that when λ is already high, the individual profit in the Cournot game increases slowly with λ, and thus, the internalization of the entry externality (due to an increase in λ) dominates and entry decreases with λ.

[37]The average value of λ (across pairs of firms, which depends on the particular corporate control assumptions) has surpassed $0.4 - 0.5$ in recent years. Also, notice that we compare the average value of λ to the threshold of λ in our model of symmetric firms and overlapping ownership structure.

[38]Using data on advanced and emerging economies, Díez et al. (2021) find that in the period 2000-2015, markups of listed firms have been higher and increased faster than markups of private firms, suggesting that private firms have lower market power than listed ones.

[39]Corollary 15.1 shows that in the modified model of Appendix B, total quantity indeed decreases with λ under constant MC and general assumptions on demand.

sunk cost or a fixed operating cost that they can avoid by exiting.

5.2 Equilibrium versus socially optimal levels of entry

In section 5.1 we saw that an increase in the level of overlapping ownership decreases entry unless the extent of overlapping ownership is low to start with and many firms are active. In the latter case, an increase in the level of overlapping ownership will increase entry. Yet, neither of these two effects (especially the decrease in entry) will a priori necessarily reduce welfare. In this section, we study whether equilibrium entry is excessive or insufficient assuming that TS_n is single-peaked.[40] We show that under increasing MC and high overlapping ownership, entry is insufficient. Therefore, the suppression of entry induced by a further increase in the extent of overlapping ownership will be detrimental to welfare in that case. On the other hand, under substantially decreasing MC, entry is excessive, so any entry suppression caused by overlapping ownership is beneficial.

We proceed with the analysis. We have that $d\,\mathrm{TS}_n/dn|_{n=\widehat{n}^*(\lambda)} = \Pi(\widehat{n}^*(\lambda),\lambda) - f + n\left(P(Q_n) - C'(q_n)\right)\partial q_n/\partial n|_{n=\widehat{n}^*(\lambda)}$, which gives

$$\left.\frac{d\,\mathrm{TS}_n}{dn}\right|_{n=\widehat{n}^*(\lambda)} \propto \lambda \underbrace{\overbrace{\frac{\Xi(\widehat{n}^*(\lambda),\lambda)}{\Pi(\widehat{n}^*(\lambda),\lambda)}}^{=1-\overbrace{\frac{f}{\Pi(\widehat{n}^*(\lambda),\lambda)}}^{\substack{\text{share of gross}\\ \text{profit spent on}\\ \text{entry cost}\\ \in(0,1]}}}}_{\substack{\in[0,1)\\ \text{internalized}\\ \text{normalized}\\ \text{entry externality}}} + \underbrace{\left(1-\frac{E_C\left(q_{\widehat{n}^*(\lambda)}\right)-1}{\frac{P\left(Q_{\widehat{n}^*(\lambda)}\right)}{AC\left(q_{\widehat{n}^*(\lambda)}\right)}-1}\right)}_{\substack{>0\\ \text{if increasing MC, then} <1\\ \text{if constant MC, then} =1\\ \text{if decreasing MC, then} >1;}} \underbrace{\left.\frac{\partial q_n}{\partial n}\frac{n}{q_n}\right|_{n=\widehat{n}^*(\lambda)}}_{\substack{<0 \text{ if } E_{P'}\left(Q_{\widehat{n}^*(\lambda)}\right)<2\\ \text{and } \widehat{n}^*(\lambda)\geq 2;\\ \text{if negative, the higher}\\ \text{in absolute value,}\\ \text{the stronger the}\\ \text{business-stealing effect}}}, \tag{4}$$

where we have used the $\Psi(\widehat{n}^*(\lambda),\lambda) = f$ entry condition. Let us have a closer look at the two terms in the above expression. $\Xi(n,\lambda)/\Pi(n,\lambda) \equiv (n-1)(\Pi(n-1,\lambda) - \Pi(n,\lambda))/\Pi(n,\lambda)$

[40] Lemma 2 in Appendix B provides sufficient conditions for it to be concave. For example, it is concave in n under linear demand and linear-quadratic costs with $c_2 \geq 0$.

is the normalized entry externality.[41]

$$1 - \frac{E_C(q) - 1}{\frac{P(nq)}{AC(q)} - 1} = \frac{P(nq) - C'(q)}{P(nq) - AC(q)} > 0$$

is a (coarse) measure of the elasticity of the cost function, and thus of the extent to which marginal cost is decreasing or increasing. For example, $(E_C(q) - 1) / (P(nq)/(AC(q)) - 1)$ is higher than (resp. lower than/equal to) 0 if and only if the marginal cost is increasing (resp. decreasing/constant).

We see then that whether entry is excessive or insufficient will depend on (i) the level of overlapping ownership λ, (ii) the magnitude of the normalized entry externality, (iii) whether marginal cost is decreasing or increasing and to what extent, and (iv) whether competition is business-stealing or business-enhancing, and to what degree $\left|\frac{\partial q_n}{\partial n} \frac{n}{q_n}\right|$. Factors (i) and (ii) can also be measured by the share of gross (*i.e.*, net of variable costs) profit that is spent on entry costs.

Under business-stealing competition and all else fixed, we distinguish the following forces. First, increases in the level of overlapping ownership or the magnitude of the entry externality, tend to make entry insufficient; these forces are complements in inducing insufficient entry. Equivalently, entry tends to be excessive when (due to a low level of overlapping ownership) a large share of revenue goes to the fixed entry cost. Second, increasing MC—which pushes towards DRS—tends to make entry insufficient, since the planner takes advantage of cost savings due to entry to a greater extent than firms do.[42] Conversely, decreasing MC—which pushes towards IRS—tends to make entry excessive. Third, increases in the magnitude of the business-stealing effect make entry excessive. Under business-enhancing competition, entry is always insufficient.

The above analysis is valid *all else fixed*. Thus, it is for example informative when we want to compare two otherwise similar markets (e.g., with similarly strong business-stealing effects) that differ in their type and magnitude of returns to scale. We know then that in the market with increasing MC and high overlapping ownership, entry tends to be insufficient, so a suppression of entry caused by an expansion of overlapping ownership will likely harm welfare. On the other hand, in the market with decreasing MC or low

[41] It is equal to $(n - 1)$ times the percentage increase in the profit of each of the $n - 1$ other firms when the n-th firm decides *not* to enter compared to the case where it did enter.

[42] Firms do not fully internalize the variable cost-savings of entry (except in the case of complete indexation) and at the same time do not internalize the increase in consumer surplus due to higher entry.

overlapping ownership (and thus excessive entry), the indirect effect (*i.e.*, through entry) of an entry-suppressing increase in overlapping ownership on welfare will be positive.

However, if we want to gauge the effects of changes in the extent of overlapping ownership *within* a certain market, then we need to take into account how these changes affect all equilibrium objects. To this end, we now study how overlapping ownership affects the relationship between the equilibrium and socially optimal levels of entry without holding all else fixed. First, define

$$\phi(n,\lambda) \coloneqq \frac{(n-1)(\Pi(n,\lambda)-\Pi(n-1,\lambda))}{n\partial\Pi(n,\lambda)/\partial n} \approx 1,$$

which is positive and close to 1.[43]

Proposition 5. Assume that TS_n is single-peaked in n. Then $\widehat{n}^*(\lambda) \overset{(\text{resp. }<)}{>} \widehat{n}^o(\lambda)$ if and only if, evaluated at $n = \widehat{n}^*(\lambda)$,

$$\lambda\Delta(Q_n,(n-1)q_n) \overset{(\text{resp. }>)}{<} \frac{1-\lambda\phi(n,\lambda)}{\phi(n,\lambda)}[1+\lambda(n-1)](1+\lambda-H_nE_{P'}(Q_n)).$$

Substituting $\lambda = 0$ we recover the standard excessive entry result: entry is excessive if and only if $E_{P'}\left(Q_{\widehat{n}^*(\lambda)}\right) < \widehat{n}^*(\lambda)$, which is indeed satisfied under standard assumptions on demand.[44] Proposition 5 also asserts that (as already discussed), all else (*i.e.*, $\lambda, E_{P'}, P', \widehat{n}^*(\lambda), \phi$) fixed, entry is excessive (resp. insufficient) for C'', and thus also Δ, low (resp. high).[45]

[43] It is positive, because the numerator and denominator are negative. It is close to 1, since (i) by the mean value theorem $\Pi(n,\lambda)-\Pi(n-1,\lambda) = \partial\Pi(\nu,\lambda)/\partial\nu|_{\nu=\nu^*}$ for some real number $\nu^* \in [n-1,n]$ and (ii) $(n-1)/n \approx 1$ for n not too small. For $\Pi(n,\lambda)$ strictly convex in n, in which case individual profit decreases with n at a decreasing (in absolute value) rate, $(\Pi(n,\lambda)-\Pi(n-1,\lambda))/(\partial\Pi(n,\lambda)/\partial n) > 1$, which counterbalances $(n-1)/n < 1$. The numerical results of Figure 8 in Appendix B.6 verify that $\phi(n,\lambda) \approx 1$. Proposition 17 in section B.12 of Appendix B compares $\widehat{n}^*(\lambda)$ and $\widehat{n}^o(\lambda)$ when firms' entry decisions are based on a differential version of (2). The correction term ϕ in Proposition 5 is replaced with exactly 1 in Proposition 17.

[44] Given $\widehat{n}^*(\lambda) \geq 2$ and $\Delta > 0$, Proposition 2 asserts that the total quantity in the quantity-setting stage is increasing in n and competition is business-stealing, which are the conditions under which Mankiw and Whinston (1986) show that entry is excessive. However, we see that $\Delta > 0$ is not necessary, consistent with Amir et al. (2014), who show that entry is excessive under business-stealing competition and $\Delta > 0$ or $\Delta < 0$.

[45] Put differently, all else fixed, an increase in the elasticity E_C of the cost function (which means that returns to scale become more decreasing) tends to make entry insufficient.

5.2.1 Markets with significantly decreasing MC ($\Delta < 0$)

Taking into account the integer constraint on the number of firms, the following remark shows formally that under substantially decreasing MC ($\Delta < 0$, and thus substantial IRS), entry is weakly excessive. Yet, the socially optimal level of entry, one firm, arises in equilibrium when the industry is fully indexed.

Remark 5.3. If $\Delta < 0$, then (i) entry is weakly excessive, $n^*(\lambda) \geq n^o(\lambda)$. Particularly, (ii) $n^o(\lambda) = 1$, as $n = 1$ maximizes both Q_n and $n\Pi(n,\lambda)$, and (iii) for $\lambda = 1$, $n^*(1) = n^o(1) = 1$.[46]

5.2.2 Markets with increasing, constant, or mildly decreasing MC ($\Delta > 0$)

While a result as general and clean as Remark 5.3 cannot be derived for the case of $\Delta > 0$ (partly because cases of both excessive and insufficient entry by more than one firm are possible under $\Delta > 0$), we have already seen that, all else fixed, a high λ and increasing MC tend to make entry insufficient. Numerical Result 2 in section 5.3 indeed shows that this holds without the "all else fixed" qualifier under linear demand and linear-quadratic cost. Namely, under non-decreasing MC, entry is excessive (resp. insufficient) for λ low (resp. high).

Under constant MC, if we let for simplicity $\phi = 1$, then Proposition 5 says that entry is excessive if $\lambda < [1 + \lambda(\widehat{n}^*(\lambda) - 1)]\left(1 + \lambda - H_{\widehat{n}^*(\lambda)} E_{P'}\left(Q_{\widehat{n}^*(\lambda)}\right)\right)$, which always holds if $E_{P'} < 1$ (and thus, also holds under linear demand). Therefore, under constant MC, entry is excessive for every λ (unless $E_{P'}$ is high).[47] Then, an increase in λ that decreases entry tends to indirectly increase total surplus. This indirect effect is particularly strong when the entry cost is high, since in that case, a decrease in entry leads to substantial savings in entry costs. Thus, a high entry cost (which pushes towards IRS) tends to make increases in overlapping ownership more socially desirable.

[46] Notice also that if $1 - C''(Q - Q_{-i})/P'(Q) < 0$ on L, then $\Delta < 0$ on L for every $\lambda \in [0,1]$. In that case, then, $n^o(\lambda) = 1$ for every $\lambda \in [0,1]$, so a planner that controls either overlapping ownership or entry (but not both) will still achieve what a planner that can control both would (*i.e.*, a monopoly). Remember, however, that under $\Delta < 0$, $n^*(\lambda)$ is not the unique equilibrium level of entry, unless $n^*(\lambda) = 1$ (see footnote 28). Particularly, the monopoly solution is always a free-entry equilibrium. Therefore, what $\lambda = 1$ achieves is to break all socially sub-optimal equilibria with higher levels of entry, making a monopoly the unique free-entry equilibrium.

[47] Figure 6b in the appendix shows a case of insufficient entry (by more than 1 firm) for high λ under CESL demand with E high and constant MC.

5.2.3 Results under a consumer surplus standard

Remark 5.4 shows that if instead of a total surplus, the planner follows a consumer surplus standard, then entry is insufficient (resp. excessive) under increasing, constant, or mildly decreasing MC (resp. significantly decreasing MC).

Remark 5.4. Under a consumer surplus standard,

(i) if $\Delta > 0$, then $n^o(\lambda) = \infty$ (since Q_n is increasing in n), so $n^*(\lambda) < n^o(\lambda)$,

(ii) if $\Delta < 0$, then $n^o(\lambda) = 1$ (since Q_n is decreasing in n), so $n^*(\lambda) \geq n^o(\lambda)$.

5.3 Overlapping ownership effects: the linear-quadratic model

In this section, we summarize our main findings so far using the case of linear demand, $P(Q) = \max\{a - bQ, 0\}$, and linear-quadratic cost, $C(q) = c_1 q + c_2 q^2/2$.[48]

First, Figure 2 verifies the results of section 5.2 on the comparison between the equilibrium and socially optimal levels of entry. For increasing MC (i.e., c_2 high), entry is insufficient under high levels of overlapping ownership. Particularly, Numerical Result 2 shows that under non-decreasing MC (and modulo the integer constraint), there exists a threshold $\overline{\lambda}$ such that entry is excessive (resp. insufficient) for λ lower (resp. higher) than $\overline{\lambda}$.[49] Of course, when MC is not substantially increasing (*i.e.*, c_2 low), entry may be excessive for every λ and the result holds for $\overline{\lambda} > 1$.

Numerical Result 2. Let demand be linear and cost be linear-quadratic with $c_2 \geq 0$ (so that $\Delta \geq 0$ for every λ) and assume that $\widehat{n}^*(0) \geq 2$. Then, there exists $\overline{\lambda}$ (which depends on parameters) such that $\widehat{n}^*(\lambda) \overset{(\text{resp. } <)}{>} \widehat{n}^o(\lambda)$ if and only if $\lambda \overset{(\text{resp. } >)}{<} \overline{\lambda}$.

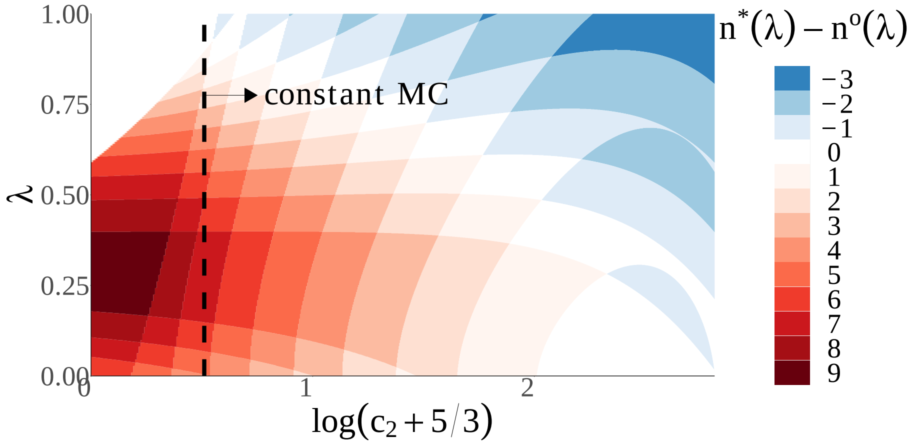
On the other hand, entry is excessive under decreasing MC and/or low λ. Particularly, notice in Figure 2 the area with substantially decreasing MC (*i.e.*, $\Delta < 0$) and high λ in the upper-left corner, where a socially optimal monopoly arises in equilibrium (*i.e.*, $n^*(\lambda) = n^o(\lambda) = 1$), as shown in Remark 5.3.

Second, in section 3 we saw that the direct effect (*i.e.*, if we hold the number of firms fixed) of overlapping ownership on welfare is negative. To gauge its total effect on welfare, we next look at how overlapping ownership affects entry. Observe in Figure 3a that for λ

[48] Claim 3 in Appendix B.8 provides additional results for this case, corroborating our findings.

[49] We ignore the integer constraint on n. This is not important, since cases of both excessive and insufficient entry by more than one firm are possible under $\Delta > 0$ (see Figure 2).

Figure 2: Difference between equilibrium and socially optimal level of entry $n^*(\lambda) - n^o(\lambda)$ (with linear demand and linear-quadratic costs) as a function of the level of λ and the level of c_2 (indicating decreasing, constant, or increasing MC)



Note: $a = 3$, $b = c_1 = 1$, $c_2 \in [-2/3, 15]$, $f = 0.05$. For better readability, an increasing transformation is applied on the x-axis (*i.e.*, c_2). On the dashed line, there is constant MC (*i.e.*, $c_2 = 0$). On the left (resp. right) of the line, marginal cost is decreasing (resp. increasing).

low and entry high, the rise in own profit due to increases in λ dominates the other two channels, and thus, entry increases with λ. However, for high λ or low levels of entry, competition in the product market is already soft enough, and thus further increases in λ suppress entry.

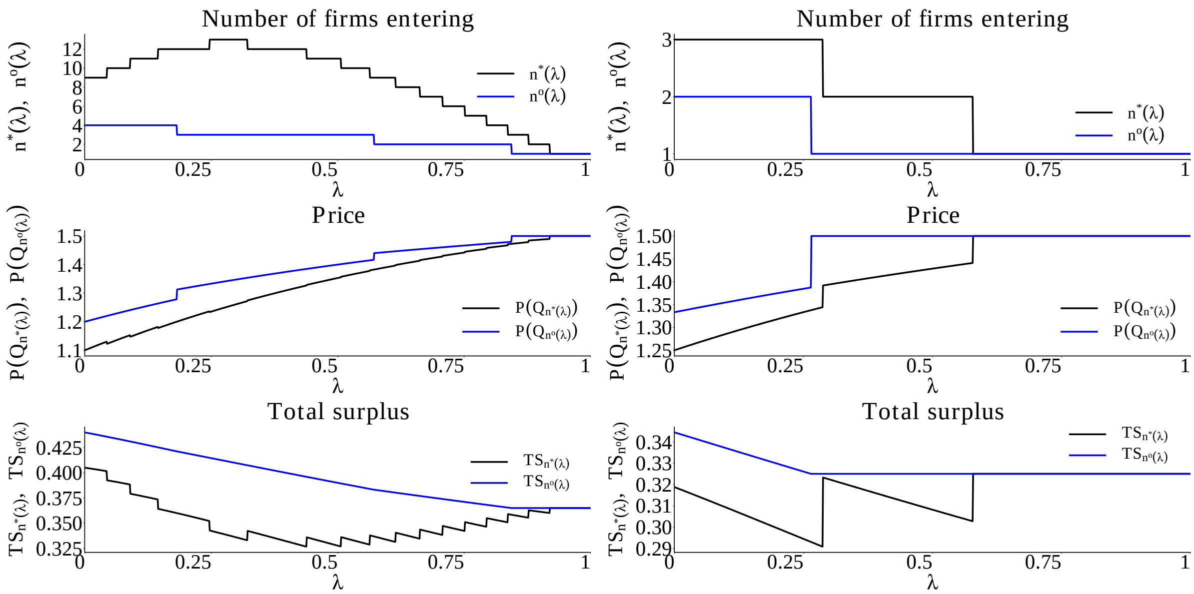
Last, we look at the (total) welfare effects of overlapping ownership. With increasing MC (Figure 3c and Figure 7 in the appendix), overlapping ownership tends to harm consumer and total surplus. On the other hand, with substantially decreasing MC (Figure 3d and Figure 7 in the appendix), a high level of overlapping ownership is optimal both for consumer and total surplus. Particularly, a high value of λ (e.g., $\lambda = 1$) induces entry by only one firm, which is socially optimal. Similarly, when the entry cost f is low (Figure 3a and Figure 7 in the appendix), a low value of λ is socially optimal. On the other hand, when the entry cost f is high (Figure 3b and Figures 6a and 7 in the appendix), a high value of λ (e.g., $\lambda = 1$) maximizes total surplus.

Overall, DRS (*i.e.*, low entry cost f and increasing MC, c_2 high) tend to make overlapping ownership welfare-damaging. Particularly, $\lambda = 0$ or very low tends to maximize total surplus. On the other hand, IRS (*i.e.*, high entry cost f and/or decreasing MC, c_2 low) tend to make high levels of overlapping ownership welfare-optimal. That is because (under IRS) high overlapping ownership leads to entry by only one firm, which implies substantial efficiency gains (compared to the case of low overlapping ownership, where more firms would enter).[50]

Figure 3: Equilibrium and socially optimal outcomes for varying λ

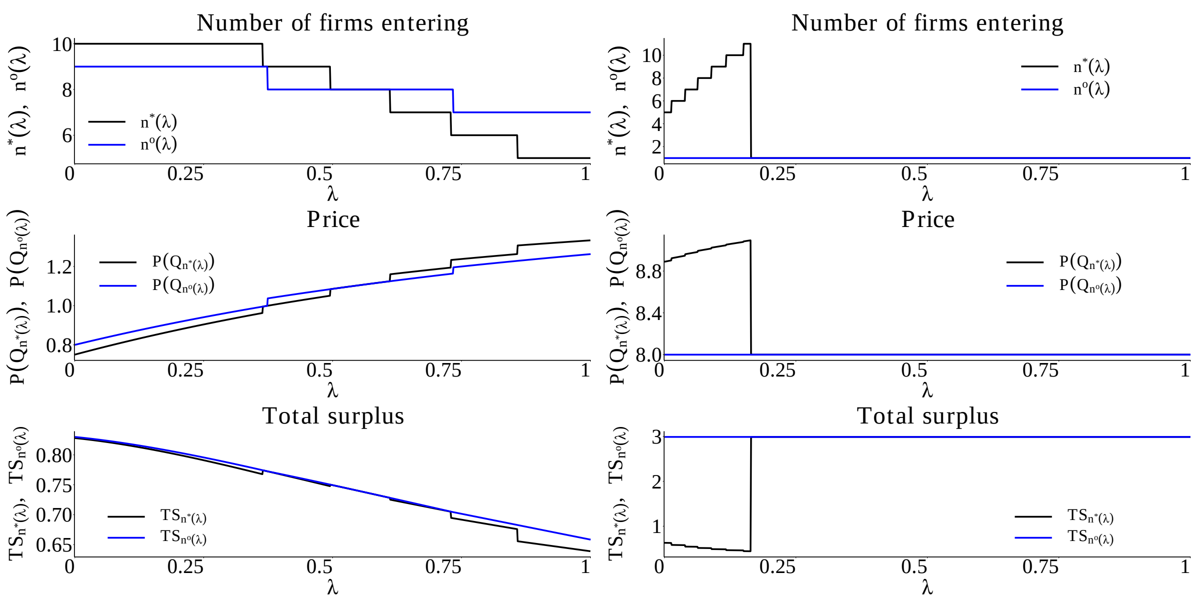
(a) linear demand, constant MC: $a = 2$, $b = c = 1$, $f = 0.01$

(b) linear demand, constant MC: $a = 2$, $b = c = 1$, $f = 0.05$



(c) linear demand, linear-quadratic costs (increasing MC): $a = 2$, $b = c_1 = 1$, $c_2 = 5$, $f = 0.05$

(d) linear demand, linear-quadratic costs (decreasing MC): $a = 10$, $b = 1$, $c_1 = 9$, $c_2 = -1.5$, $f = 0.01$



Note: Black lines represent values in equilibrium; blue represent values in the (entry-controlling) planner's solution.

[50]When the market is so small (*i.e.*, production costs are large relative to demand) that it can accommodate entry by only one firm absent overlapping ownership, then a high level of overlapping ownership is not necessary for the efficiency gains from IRS to materialize. However, overlapping ownership does not harm welfare either, since only one firm enters with or without overlapping ownership (and when only one firm enters, there is no room for overlapping ownership to soften product market competition).

5.4 Entry cost effect on entry

We have seen so far that overlapping ownership is beneficial under substantial IRS. On the other hand, an increase in the extent of overlapping ownership tends to harm welfare under DRS—especially when overlapping ownership is high to start with. At the same time, the rising entry costs documented by Gutiérrez et al. (2021) tend to reduce entry. But how do the two forces—increasing overlapping ownership and entry costs—interact? Is their combined impact on competition worse than the sum of the two effects, or does overlapping ownership mitigate the suppression of entry caused by rising entry costs?

Proposition 6 studies the effect of the entry cost on entry, as well as how this effect depends on the extent of overlapping ownership (with the level of entry held fixed). Note that λ affects the slope $d\widehat{n}^*(\lambda)/df$ directly but also through its effect on $n^*(\lambda)$. We are interested in the direct effect so we keep $n^*(\lambda)$ fixed as we vary λ.

Proposition 6. Ignore the integer constraint on n (so that entry is given by $\widehat{n}^*(\lambda)$). Then

(i) entry is decreasing in the entry cost,

(ii) if λ increases and other parameters x (e.g., demand, cost parameters) change infinitesimally with $\widehat{n}^*(\lambda)$ staying fixed and $\partial^2\Psi(n,\lambda)/(\partial x\partial n) = 0$ (e.g., (f,λ) infinitesimally changes in direction $\mathbf{v} \coloneqq (-(d\widehat{n}^*(\lambda)/d\lambda)/(d\widehat{n}^*(\lambda)/df),1))$, then $|d\widehat{n}^*(\lambda)/df|$ changes in direction given by $\operatorname{sgn}\left\{\partial^2\Psi\left(n,\lambda\right)/(\partial\lambda\partial n)|_{n=\widehat{n}^*(\lambda)}\right\}$.

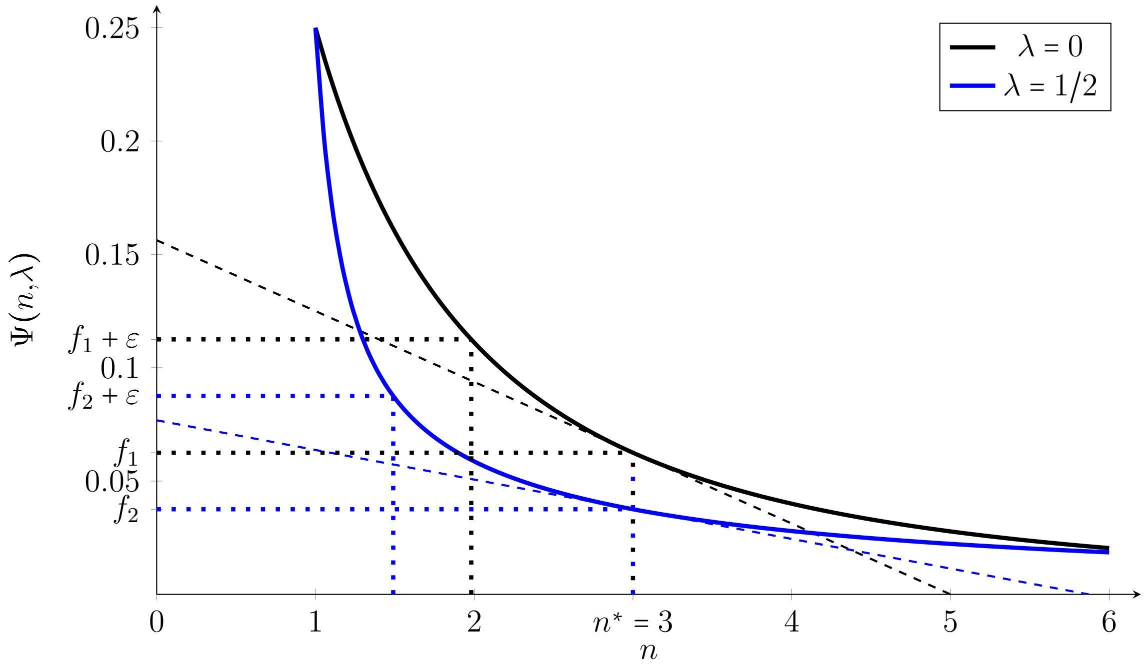
As long as $\Psi(n,\lambda)$ is decreasing in n, the results of Proposition 6 are not specific to Cournot competition. Part (ii) says that if an increase in λ makes the internalized profit in the quantity-setting stage equilibrium more (resp. less) strongly decreasing in the number of firms, then an increase in the entry cost needs to be met with a smaller (resp. larger) increase (resp. decrease) in the number of firms for the condition $\Psi\left(\widehat{n}^*(\lambda),\lambda\right) = f$ to continue to hold.

Figure 4 explains the reasoning behind this. There are initially $n^* = 3$ firms, which can be a result of $\lambda = 0$ and $f = f_1$, or $\lambda = 1/2$ and $f = f_2$. Also, an increase in λ from 0 to 1/2 makes the internalized profit less strongly decreasing in n (*i.e.*, $\partial^2\Psi\left(n,\lambda\right)/(\partial\lambda\partial n) > 0$).[51]

[51]For example, at $n = 3$, the slope of $\Psi(n,1/2)$ (see the tangent blue dashed line) is smaller in absolute value than the slope of $\Psi(n,0)$ (see the tangent black dashed line). $\Psi(n,\lambda)$ decreases less strongly with n when $\lambda = 1/2$ than when $\lambda = 0$.

Thus, an increase in the entry cost by ε will decrease entry by more when $\lambda = 1/2$ (and initially $f = f_2$) compared to when $\lambda = 0$ (and initially $f = f_1$).

Figure 4: Entry cost effect on entry mediated by λ under linear demand and constant MC



Note: $a = 2$, $b = 1$, $c = 1$. The black and blue solid lines represent $\Psi(n,0)$ and $\Psi(n,1/2)$, respectively. The black and blue dashed lines are tangent to the corresponding solid lines at $n = n^*$.

Numerical result 3 provides conditions under which the cross derivative of $\Psi(n,\lambda)$ is positive, which by Proposition 6 implies that overlapping ownership exacerbates the negative effect of the entry cost on entry.

Numerical Result 3. Under CESL demand and constant MC, $\partial^2\Psi(n,\lambda)/(\partial\lambda\partial n) > 0$ if (i) $E \in (1,1.7]$ and $n \in [2,7]$, or (ii) $E < 1$ and $n \in [2,8]$.[52]

Under CESL demand and constant MC, markets with low entry are particularly susceptible to further decreases in entry when there is overlapping ownership. In such markets, apart from the direct effect it has on entry, overlapping ownership also makes entry more strongly decreasing in the entry cost. This means that overlapping ownership could exacerbate the negative macroeconomic implications of rising entry costs in moderately or severely concentrated markets.[53]

[52]Empirical estimates for various markets place E in the range specified in Numerical Result 3. See for example Duso and Szücs (2017), Mrázová and Neary (2017), and Bergquist and Dinerstein (2020).

[53]Remember that by Proposition 2, when $\Delta > 0$, a decrease in the number of firms causes consumer surplus to fall. Also, in section 5.2 we saw that increasing MC and high overlapping ownership tend to make entry insufficient, in which case a decrease in the number of firms will also reduce total surplus.

6 Extensions and robustness

In this section, we present extensions and discuss the robustness of our results.

6.1 Asymmetric overlapping ownership or costs

Our model with symmetric overlapping ownership is rich enough to capture the main forces and allows us to study several issues.[54] Although the assumption of a unique Edgeworth coefficient of effective sympathy λ across all firm pairs is a simplification, an increase in λ in comparative statics exercises captures a particularly relevant phenomenon. It can for instance represent the expansion of an investment fund's holdings across all firms in an industry, as has recently been the trend that has spurred the antitrust interest in overlapping ownership. This is also the reason why proposed policies have emphasized the industry-wide holdings of each investor rather than only individual stock trades. For instance, Posner et al. (2017) and Posner (2021) propose that an investor holding shares of more than one firm in an oligopoly be not allowed to own more than 1% of the market shares unless they commit to being purely passive.

While a model with asymmetric overlapping ownership would allow for the analysis of additional issues (e.g., on the effects of a change in the ownership links between only a subset of firms), we expect our main insights on the effects of a *uniform* increase in overlapping ownership (*i.e.*, an increase in λ's across all firm pairs) to hold in a model with asymmetric overlapping ownership.[55] Particularly, similar forces will be at play. A uniform increase in λ's will tend to (i) directly increase the price and profits (*i.e.*, with the number of firms held fixed), (ii) indirectly affect welfare through its impact on entry, which will depend on the balance of the three forces identified in section 5.1.

Yet, in a model with asymmetric cost functions, there will be an additional force, which can make overlapping ownership more attractive from a welfare standpoint. Although with symmetric costs a uniform increase in λ affects production efficiency (through its

[54]The assumption of a symmetric overlapping ownership structure greatly facilitates the analysis. First, with an asymmetric overlapping ownership structure, the Cournot game is neither symmetric nor even aggregative, and the analysis would require very strong assumptions. Second, there would be an extensive multiplicity of equilibria in the entry stage. Even when there are two groups of firms, one with and one without overlapping ownership, there would often be equilibria where (i) firms from only the first group enter, (ii) firms from only the second group enter, and (iii) firms from both groups enter. Third, there would be no single measure of overlapping ownership, as is λ in our model.

[55]Our main insights also hold in an extension of the model, discussed in section 6.4, where there is also a competitive fringe of maverick firms (without overlapping ownership).

effect on entry) based on the type of returns to scale, with asymmetric costs an increase in overlapping ownership can also induce cost efficiencies by causing production to shift towards low-cost firms.[56]

6.2 Endogenous overlapping ownership

Although standard in the overlapping ownership literature, the assumption that the ownership structure is exogenous may seem unrealistic. However, even if λ is affected by parameters of the model (e.g., if common ownership tends to be higher in markets with larger demand), the analysis is still valid if there are factors that affect λ but are unrelated to the demand and production costs in the specific market that firms consider entering. For example, an increase in the level of overlapping ownership may be due to an increase in stock market participation (by small investors, who invest mostly through diversified investment funds). Stock market participation is arguably affected mostly by factors other than the demand and production costs in a specific market under analysis. These factors can be increased financial literacy, the rise of fintech, and lower fund management fees.

6.3 Product differentiation and alternative modes of competition

We now discuss how our main insights still apply when the product market competition stage is not a homogeneous product Cournot market. In doing so, we emphasize the fundamental forces behind our results, which are present in most forms of product market competition.

Consumer surplus As long as (i) consumers benefit from increased competition—and, in the case of differentiated products, enhanced product variety—induced by higher entry, and (ii) overlapping ownership (iia) tends to harm consumer surplus by undermining product market competition (*i.e.*, ignoring its effects on entry), and (iib) suppresses entry (*i.e.*, the internalization of the entry externality channel dominates), overlapping

[56]To see why, fix the number of firms and let the ownership structure be symmetric. Notice that each firm i's equilibrium quantity is such that if i marginally increases its output, the resulting negative externality on the other firms weighed by λ will be equal to the increase in i's own profit. That is, $\partial\pi_i/\partial q_i = -\lambda P'(Q^*)Q^*_{-i}$ for $q_i = q_i^*$. Low-cost firms with larger output q_i^* impose a smaller (in absolute value) marginal externality, $P'(Q^*)Q^*_{-i}$, on the other firms (because $Q^*_{-i} = Q^* - q_i^*$). Thus, low-cost firms have a smaller marginal externality to internalize, so they tend to decrease their quantities by less as λ increases. Therefore, as λ increases, the market shares of low-cost firms increase pushing down the average production cost in the market. This effect is highlighted in Azar and Vives (2021).

ownership will still tend to harm consumer surplus. Part (i) can be expected to hold in many markets without substantial IRS. On the other hand, with substantial IRS, the cost-savings that come with decreased entry and are passed on to the consumer may more than compensate for the consumer surplus lost due to increased market power and less product variety (that also comes with lower entry). Thus, our finding that IRS tend to make overlapping ownership enhance welfare through entry should also apply to other forms of competition. We also expect part (iia) to hold with differentiated goods.[57] For example, the influence of overlapping ownership under homogeneous product Cournot is similar to its impact under Bertrand competition with differentiated goods (e.g., see López and Vives, 2019).

Last, whether part (iib) holds will depend on the balance of the same three channels identified in section 5.1 (which are not specific to homogeneous product Cournot competition). Nevertheless, the direction of the change in the entry externality $\Xi(n,\lambda)$ and the magnitudes of the different channels depend on the market structure.[58] For example, (keeping the number of firms fixed) overlapping ownership may increase profits in a differentiated products market by less than it does in the Cournot game, since with differentiated products, firms' pricing and production decisions have smaller effects (to be internalized due to overlapping ownership) on other firms; in the extreme case of independent monopolies, overlapping ownership would not affect profits at all. For the same reason, the magnitude of the entry externality will also be diminished. Overall, as long as overlapping ownership increases the pricing/quantity-setting stage profits at a decreasing rate (without decreasing the entry externality too much), a further increase in overlapping ownership when its level is already high should decrease entry.

Total surplus The effects on total surplus will depend on (i) the direct effect (*i.e.*, ignoring effects on entry) of overlapping ownership on total surplus (see section 6.1 for a discussion on when the direct effect may be positive), (ii) its effect on entry, as discussed above, and (iii) whether entry is excessive or insufficient.

[57] With substitute goods, overlapping ownership induces firms to internalize their negative externality on other firms partly. Thus, they should price less aggressively and produce less, harming consumer surplus, as overlapping ownership expands.

[58] The three channels are also present in the circular-market model with common ownership of Sato and Matsumura (SM; 2020)—although the authors discuss only the first two channels. The direction of the third channel's effect is not ambiguous in their model, where the magnitude of the entry externality monotonically increases with the extent of overlapping ownership.

With regard to part (iii), notice that with differentiated products, entry can benefit consumers not only by leading to lower prices and higher output but also by expanding product variety. However, firms may not internalize the benefit of product variety on consumers. To see this, as in Spence (1976), let the representative consumer's gross benefits be given by $U\left(\sum_{i=1}^{N} f(q_i)\right)$, where $f(0) = 0$, $f',U' > 0 > U'',f''$ globally. Then, the products are substitutes and consumers have a preference for variety with $E_f(q) \coloneqq f'(q)q/f(q) \in (0,1)$. In a symmetric equilibrium with n firms, $\mathrm{TS}_n = U(nf(q_n)) - nC(q_n) - nf$, and $\Pi(n,\lambda) = P_n(Q_n)q_n - C(q_n)$, where $P_n(Q) \coloneqq U'(nf(Q/n))f'(Q/n)$ is each firm's price when n firms symmetrically produce total quantity Q, and

$$\frac{d\,\mathrm{TS}_n}{dn}\bigg|_{n=\widehat{n}^*(\lambda)} \propto \lambda\underbrace{\frac{\Xi(n,\lambda)}{\Pi(n,\lambda)}}_{=1-\frac{f}{\Pi(n,\lambda)}>0} + \underbrace{\left(1 - \frac{E_C(q_n) - 1}{\frac{P_n(Q_n)}{AC(q_n)} - 1}\right)}_{>0 \text{ given } P_n(Q_n)>C'(q_n)} \underbrace{\frac{\partial q_n}{\partial n}\frac{n}{q_n}}_{\substack{<0 \text{ under} \\ \text{business-stealing}}} + \underbrace{\frac{1 - E_f(q_n)}{E_f(q_n)\left(1 - \frac{AC(q_n)}{P_n(Q_n)}\right)}}_{>0}\Bigg|_{n=\widehat{n}^*(\lambda)}.$$

The first two terms are exactly as in a homogeneous product market. Thus, the same insights apply in terms of how returns to scale and overlapping ownership mediate the relationship between equilibrium and socially optimal entry. Notice also that—just like the original expression in equation (4)—the expression for $d\,\mathrm{TS}_n/dn|_{n=\widehat{n}^*(\lambda)}$ does not depend on the mode of competition. For example, it holds under both Cournot and Bertrand competition. The last term captures the fact that entry benefits consumers due to their preference for variety, which is not internalized by the marginal entrant. This acts as an additional—to overlapping ownership—force pushing towards insufficient entry, which makes the indirect effect (on total surplus) of an entry-suppressing expansion of overlapping ownership negative.

6.4 Entry under the presence of maverick firms

We have examined the effects of overlapping ownership under a symmetric overlapping ownership structure. In that context, overlapping ownership can suppress entry by inducing firms to internalize the negative externality that their entry would have on other firms. However, if there are also potential entrants without ownership ties—which we call maverick firms, then limited entry by commonly-owned firms may spur entry by maverick ones.[59] This could enhance the incentives of a commonly-owned firm to enter.

In section B.9 of the Appendix we model the maverick firms as a competitive fringe that in the first stage (where oligopolists enter) submit an aggregate supply schedule. We show that the (prospect of) entry by maverick firms essentially changes the demand faced by the commonly-owned firms by depressing it and making it more elastic. With demand adjusted accordingly, the results of the previous sections on the effects of overlapping ownership on entry and the price continue to hold (with the number of firms n not counting maverick firm entry), as does the comparison between the equilibrium and socially optimal levels of entry. Since demand is depressed, we expect lower levels of entry by commonly-owned firms. Also, given that higher (resp. lower) entry by commonly-owned firms leads to lower (resp. higher) entry by maverick firms, we expect entry to be less sensitive to overlapping ownership due to the presence of maverick firms.[60] This is indeed verified in section B.9.

6.5 Post-entry overlapping ownership

Post-entry overlapping ownership applies to the case of a new industry that is to be mostly populated by start-ups without overlapping ownership that will develop ownership links after entry.[61] In this case, firms do not internalize the negative externality their entry has on other firms, as in the standard Cournot model with free entry. Thus, modulo the integer constraint on the number of firms, firms enter until the individual gross profit is equal to the entry cost, so in equilibrium the net profit is zero. Nevertheless, when deciding whether to enter, they take into account how overlapping ownership will affect product market outcomes. Naturally, an increase in the degree of post-entry overlapping ownership spurs entry, since it tends to increase profits by softening pricing competition (see Proposition 2). However, section B.10 in the Appendix (which studies the model with

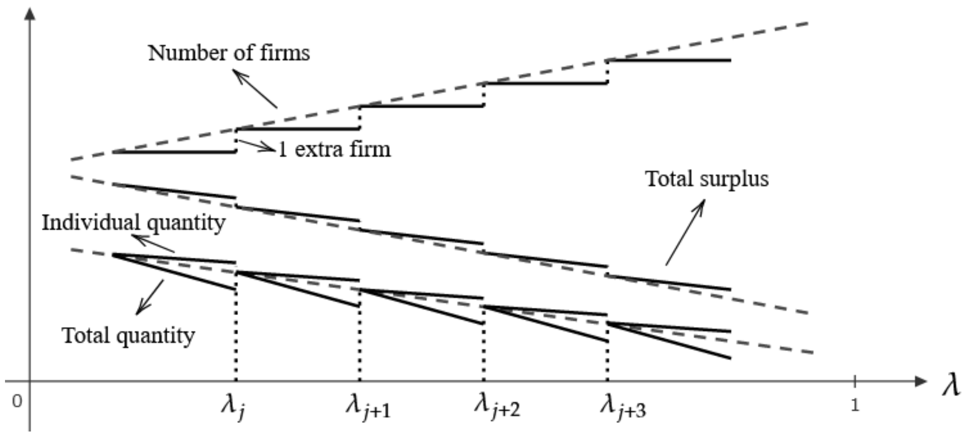
[59] Indeed, Eldar and Grennan (2021) argue that by softening competition, common ownership of public firms gives start-ups the opportunity to enter. However, they also provide evidence that—acknowledging this opportunity—VCs concentrate their activities on markets with extensive common ownership. This causes VC-induced common ownership to increase with public-firm common ownership.

[60] That is because all channels through which λ affects entry will diminish in magnitude when there are maverick firms. First, quantity-setting stage profit will not increase as strongly with λ, because maverick firms will produce more when oligopolists reduce production as λ increases. In other words, when maverick firms are present, the demand that the oligopolists face is more elastic, so the externality that one oligopolist imposes on the others by producing—thereby pushing down the price—is lower. Thus, there is a smaller externality to be internalized (as λ increases) in the quantity-setting stage, so the effect of λ on Cournot profit is milder. Similarly, the entry externality is also smaller, since by entering an oligopolist limits maverick entry.

[61] The post-entry overlapping ownership case can also be interpreted to address pre-entry overlapping ownership when it does not induce firms to internalize their entry externality.

post-entry overlapping ownership) shows that the anti-competitive effect of overlapping ownership prevails causing the price to increase and consumer, as well as total surplus to fall. The results on the effects of post-entry overlapping ownership are schematically summarized in Figure 5.

Figure 5: Equilibrium with post-entry overlapping ownership for varying λ



Note: The solid lines represent equilibrium values under the integer constraint; from bottom to top they represent the behavior of the total quantity, individual quantity, total surplus, and number of firms. The dashed lines represent equilibrium values when we ignore the integer constraint; from bottom to top the first line represents the behavior of both the total and the individual quantity, the second of total surplus, and the third of the number of firms. The solid total quantity line is drawn for the case $\Delta > 0$. The solid individual quantity is drawn above the dashed one given that q_n is decreasing in n (see Proposition 2). To draw the solid total surplus line above the dashed one, we assume that the total surplus is single-peaked in n, and that $n^*(\lambda) \geq n^o(\lambda)$ (see Proposition 13 in the Appendix). Only the signs of the slopes of the lines and the directions of the jumps are part of the result; the curvatures of lines and spacing of the jumps have been chosen for simplicity in depiction.

The comparison between the equilibrium and the optimal level of entry also changes: the result that entry tends to be excessive under business-stealing competition generalizes. Entry is never insufficient by more than one firm as in the standard Cournot model with free entry (see Mankiw and Whinston, 1986; Amir et al., 2014).

7 Conclusion

In this paper, we have studied the effects of overlapping ownership in a Cournot oligopoly with free entry. Potential entrants are established firms with overlapping ownership and decide whether to enter a new industry or product market.

We have shown that overlapping ownership significantly differs from collusion. In infinitely repeated oligopoly games, the possibility of entry limits the attainable collusive outcomes, since by colluding incumbents increase prices thereby making entry more attractive to outsider firms. However, pre-entry overlapping ownership functions as an additional channel for the competitive effects of overlapping ownership. Our main finding is that in most relevant cases—and particularly when overlapping ownership is already widespread, an expansion of overlapping ownership harms welfare. That is because as the level of overlapping ownership increases, not only (i) does competition among active firms soften, but also (ii) fewer firms are active in the market with entry being lower than socially optimal. At the same time, (iii) entry becomes more sensitive to the entry cost (*i.e.*, it decreases faster with the entry cost). Overlapping ownership can mostly be beneficial only under strong increasing returns to scale (IRS), in which case industry monopolization (induced by overlapping ownership) generates considerable cost-savings.

These results have important implications for competition policy. They suggest that regulators should be most concerned about overlapping ownership among firms that operate (or may consider operating) in markets and industries with decreasing returns to scale (DRS) or weak IRS. Particularly, markets with DRS and already high and increasing overlapping ownership warrant the most regulatory scrutiny. On the other hand, overlapping ownership among firms operating in markets with substantial IRS seems less problematic—if not beneficial.

For example, overlapping ownership among generic drug manufacturers or between a generic and a brand name should be more of a concern compared to overlapping ownership among pharmaceuticals that focus on drug discovery. That is because drug discovery—in contrast to generic drug manufacturing—requires sizable R&D costs (and is thus characterized by strong IRS), the duplication of which can be avoided when there is extensive overlapping ownership.[62] Particularly, apart from direct R&D costs, the cost of entry can also be understood to include the opportunity cost associated with foregoing R&D of drugs for other conditions. Then, overlapping ownership will induce each pharmaceutical to focus on drug discovery for a different condition rather than compete with other pharmaceuticals in the development of drugs for the same condition. Li et al. (2023) provide evidence pointing in this direction.[63] On the other

[62]Henderson and Cockburn (1996) even find IRS in the R&D phase (*i.e.*, that larger firms are more cost-efficient in R&D). Of course, one should also consider the uncertainty associated with drug discovery, which may render the duplication of R&D costs (by firms testing different drugs for the same condition) socially desirable. Still, such duplication can also be performed by a single firm. Also, as long as different treatments for the same condition are close substitutes, the social benefit from increased variety will not justify the duplication of R&D costs.

[63]Namely, they find that venture capital firms that fund multiple pharmaceutical startups hold back

hand, overlapping ownership involving generic manufacturers is more likely to soften competition and suppress generic entry without generating important cost efficiencies, thereby harming patients and welfare. Indeed, Newham et al. (2022), Xie and Gerakos (2020), and Xie (2021) find that common ownership between a brand firm and potential generic entrants suppresses generic entry.

In more detail, we derive the following three results. First, an increase in overlapping ownership affects entry through three separate channels. It increases the degree of internalization of the negative externality of entry on other firms' profits, which tends to limit entry. However, it also increases equilibrium profits in the quantity-setting stage, which tends to increase entry. Last, it changes the magnitude of the entry externality on other firms' profits; this channel can affect entry in either direction. In markets that are dominated by a small number of firms or where there already is extensive overlapping ownership—as is currently the case in several U.S. markets, a further increase in the level of overlapping ownership will suppress entry.

Second, apart from entry we also study welfare since, as Bar-Isaac (2016) notes, the extent of entry is not a sufficient statistic for welfare and the efficiency of a market. Particularly, a decrease in entry caused by an expansion of overlapping ownership will be welfare-damaging or enhancing when entry is insufficient or excessive (from a welfare standpoint), respectively, in equilibrium. We find that entry tends to be excessive under IRS. However, entry is insufficient in markets with DRS and high levels of overlapping ownership.

Third, given the negative macroeconomic implications of rising entry costs documented by Gutiérrez et al. (2021) in the U.S. economy over the past 20 years, we are interested in how overlapping ownership mediates the negative effect of entry costs on entry. We find that, under common assumptions, overlapping ownership exacerbates the negative impact of an increase in entry costs on entry.

Further, we derive the following testable implications for markets that existing firms with overlapping ownership consider entering. First, for low levels of overlapping ownership, an increase in overlapping ownership will (i) increase entry if there is low market concentration, but (ii) it will decrease entry under high market concentration. Second, for high levels of overlapping ownership, further increases in it will suppress entry. Thus, entry

projects, withhold funding, and redirect innovation at lagging startups—partly to prevent R&D cost duplication.

will either depend negatively on overlapping ownership or have an inverted-U relationship with it. Third, unless there are increasing returns to scale, an increase in the extent of overlapping ownership will increase the price. Fourth, increases in the entry cost can suppress entry more in industries with higher levels of overlapping ownership. Fifth, entry by commonly-owned firms is more responsive to the level of overlapping ownership in industries where the prospect of entry by firms without ownership ties to incumbents is less salient.

Finally, given that the extent to which ownership ties affect firm conduct is an open empirical question, our results suggest a test of the common ownership hypothesis. If the common ownership hypothesis fails completely (*i.e.*, common ownership does not affect firm behavior), then entry (and other market outcomes) should be independent of common ownership. If the common ownership hypothesis is only *partially* correct in the sense that common ownership influences pricing behavior but does *not* cause the entry externality to be internalized, then entry is expected to increase with the level of common ownership. If, on the other hand, common ownership induces firms to internalize the entry externality without affecting their pricing behavior, then entry should decrease with overlapping ownership.[64] Finally, if the common ownership hypothesis is correct (*i.e.*, common ownership affects firm conduct in both ways), then entry is expected to either depend negatively on common ownership or have an inverted-U relationship with it.

Future research could study the effects of overlapping ownership through entry (into a new market) when possible entrants interact in multiple (possibly interdependent) markets. This opens the gate to potential collusive strategies even with independent markets.

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[64]This could be the case if entry decisions are made by top executives who internalize the (common) shareholders' interests, while pricing decisions are made by low-level management who are not incentivized to do so. For instance, Ruiz-Pérez (2019) finds that in the U.S. airline industry, common ownership matters for entry decisions but not for pricing behavior. On the other hand, Park and Seo (2019) find that common ownership makes U.S. airlines internalize the effects of their pricing decisions on the competitors' profits.

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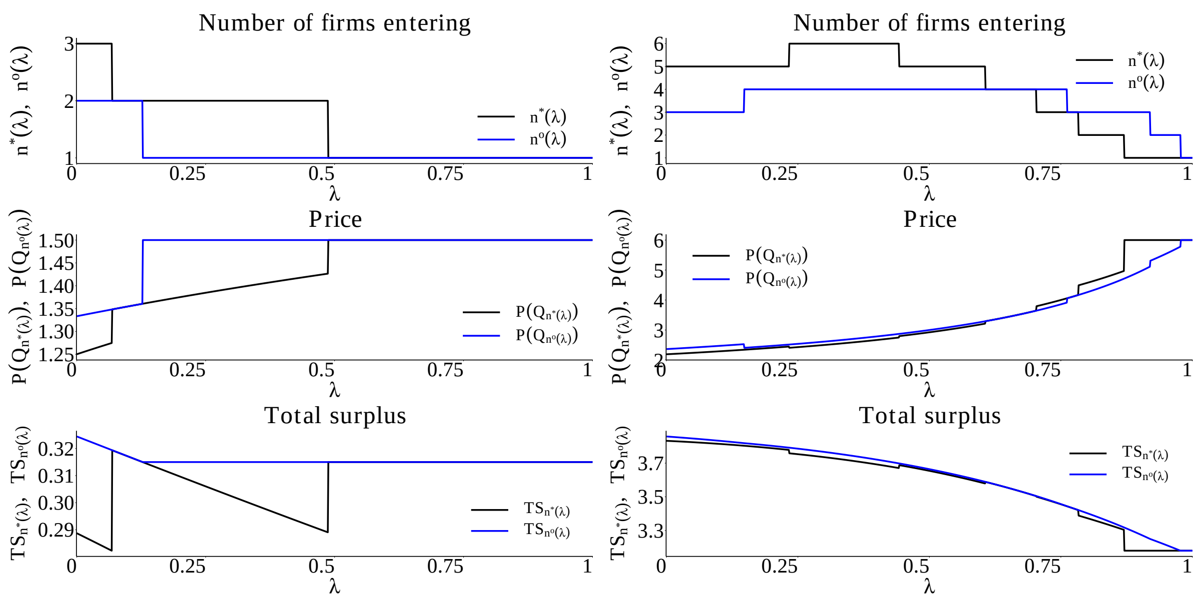
A Appendix

A.1 Additional simulation results

Figure 6a shows that for high enough levels of the entry cost, a planner that regulates overlapping ownership (but not entry) may choose positive levels of it. Figure 6b shows that even under constant MC, entry can be insufficient (by more than one firm) for high levels of overlapping ownership.

Figure 6: Equilibrium and planner outcomes for varying λ

(a) linear demand, constant MC: $a = 2$, $b = c = 1$, $f = 0.06$ **(b)** CESL demand, constant MC: $a = 1$, $b = 1$, $E = 9/5$, $c = 2$, $f = 0.03$

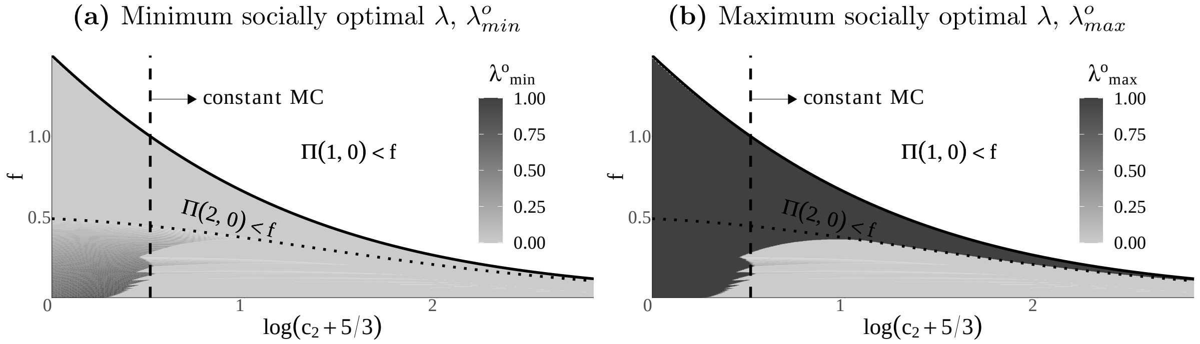


Note: Black lines represent values in equilibrium; blue represent values in the (entry-controlling) planner's solution.

Figure 7 plots the socially optimal level(s) $\lambda^o \coloneqq \arg\max_{\lambda \in [0,1]} \mathrm{TS}_{n^*(\lambda)}$ of overlapping ownership under linear demand and linear-quadratic costs as a function of the entry cost f and the level c_2 of decreasing, constant, or increasing MC.[65]

[65]It can be checked that $\mathrm{sgn}\{\widehat{n}^*(\lambda) - \widehat{n}^o(\lambda)\}$, $\Pi(n,\lambda)$, TS_n, and $\widehat{n}^*(\lambda)$ depend on a, c_1, and f only through $(a - c_1)^2/f$ (see the discussion in the Derivation of Numerical result 2 in Appendix B). Thus, $a = 3$, $c_1 = 1$ in Figure 2 and Figure 7 in the appendix is, modulo the integer constraint on n, without loss of generality in the following sense. If plotted without the integer constraint, then for any a,c_1,f such that $(a - c_1)^2/f = 80$, Figure 2 will give the same result about when entry is excessive or insufficient. Similarly, Figure 7 without the integer constraint (*i.e.*, for $\widehat{\lambda}^o \coloneqq \arg\max_{\lambda \in [0,1]} \mathrm{TS}_{\widehat{n}^*(\lambda)}$) will give the same result for any a and c_1 with the scaling of the f-axis adjusted. For instance, if a is equal to 2 instead of 3, then the f-axis scale will be divided by $(3-1)^2/(2-1)^2 = 2$ (e.g., where 0.5 under $a = 3$ on the f-axis, we will have 0.25 under $a = 2$).

Figure 7: Socially optimal level(s) $\lambda^o \coloneqq \arg\max_{\lambda\in[0,1]} \mathrm{TS}_{n^*(\lambda)}$ of overlapping ownership under linear demand and linear-quadratic costs as a function of the entry cost f and the level c_2 of decreasing, constant, or increasing MC



Note: $a = 3$, $b = c_1 = 1$, $c_2 \in [-2/3,15]$. For better readability, an increasing transformation is applied on the x-axis (*i.e.*, c_2). On the dashed line, there is constant MC (*i.e.*, $c_2 = 0$). On the left (resp. right) of the line, marginal cost is decreasing (resp. increasing). In the white region above the solid line, the net monopoly profit is negative (*i.e.*, $\Pi(1,0) < f$), and thus no firm enters for every λ. Above the dotted line, the net duopoly profit is negative (*i.e.*, $\Pi(2,0) < f$). Thus, in the region between the solid and the dotted line, only one firm enters absent overlapping ownership (*i.e.*, for $\lambda = 0$). In the region on the left of (and including) the dashed line (and below the solid line), one firm enters if $\lambda = 1$ (to see why, look at the derivation of Remark 5.3 in the proof of Proposition 5). In some cases, λ^o is not a singleton. In these cases, every optimal $\lambda \in \lambda^o$ induces entry by only one firm, in which case the exact value of λ does not matter. Panel (a) plots the minimum among all optimal levels of overlapping ownership, while panel (b) plots the maximum one. For example, in the region between the dotted and solid lines, both $0 \in \lambda^o$ and $1 \in \lambda^o$ lead to entry by a single firm, maximizing total surplus.

A.2 Some commonly used conditions

Lemma 1 below provides necessary and sufficient conditions for some of our standard assumptions. The proof is elementary and therefore omitted.

Lemma 1. The following hold:

(i) $\Delta(Q,Q_{-i}) > 0$ on L for every $\lambda \in [0,1)$ if and only if $C''(q) \geq 0$ for every $q < \overline{Q}$.

(ii) $E_{P'}(Q) < (1+\lambda)/H_n$ for every $n \in [2,+\infty)$ (resp. $n \in [1,2]$) and every $\lambda \in [0,1]$ if and only if $E_{P'}(Q) < 2$. (resp. $E_{P'}(Q) < 1$).

In the proofs to come, it will be useful to remember that if $\Delta > 0$ (resp. $\Delta < 0$), then

$$\begin{aligned} &(1+\lambda+\Delta/n)/H_n = 1 + H_n^{-1} - \Lambda_n^{-1} C''(q)/P'(Q) \\ \overset{\text{(resp. }\geq)}{\leq}\ &(1+\lambda+\Delta/\Lambda_n)/H_n = 1 + H_n^{-1} + [(1-\lambda)(1-H_n) - C''(q)/P'(Q)]/(\Lambda_n H_n) \\ \overset{\text{(resp. }\geq)}{\leq}\ &(1+\lambda+\Delta)/H_n = (2 - C''(q)/P'(Q))/H_n, \end{aligned}$$

where $\Lambda_n \coloneqq 1 + \lambda(n-1) = nH_n$. Also, $E_{P'}(Q) < \frac{1+\lambda+\Delta(Q,Q_{-i})}{1-(1-\lambda)(1-s_i)}$ on L implies that for any

$n \in [1, +\infty)$ and any $Q < \overline{Q}$, $E_{P'}(Q) < (1 + \lambda + \Delta(Q, (n-1)Q/n))/H_n$. Thus, part (ii) of the maintained assumption implies that when $\Delta < 0$, $E_{P'}(Q)$ is also lower than $(1 + \lambda + \Delta/\Lambda_n)/H_n$ and $(1 + \lambda + \Delta/n)/H_n$ in the symmetric equilibrium.

A.3 Proofs of section 3

Where clear we may simplify notation (e.g., omitting the subscript n).

Proof of Proposition 1 Wlog we can constrain attention to quantity profiles $\boldsymbol{q} \in \{\boldsymbol{x} \in [0,\overline{q}]^n : \sum_{i\in\mathcal{F}} x_i \leq \overline{Q}\}$. Also, the best response of firm i depends on $\boldsymbol{q}_{-i}$ only through Q_{-i}. Denote by $r(Q_{-i})$ the best response correspondence of a firm (the same for all firms). If it is a differentiable function, its slope is given by $r_i'(Q_{-i}) = -1 + \Delta(Q,Q_{-i})/[1 + \lambda + \Delta(Q,Q_{-i}) - (s_i + \lambda(1-s_i))E_{P'}(Q)]$, for $q_i = r(Q_{-i})$. The proof is then similar to that of Theorem 2.1 in Amir and Lambson (AL; 2000).[66]

Case $\boldsymbol{\Delta} > \mathbf{0}$: We first prove statement (a).

Existence of symmetric equilibrium: Firm i's problem is equivalent to choosing the total quantity to be given by the correspondence $R : [0,\overline{Q}] \to [0,\overline{Q}]$ defined as

$$R(Q_{-i}) \coloneqq \underset{Q\in[Q_{-i},Q_{-i}+\overline{q}]}{\arg\max} \{P(Q)[Q - (1-\lambda)Q_{-i}] - C(Q - Q_{-i})\} = r(Q_{-i}) + Q_{-i}.$$

taking Q_{-i} as given. The maximand above is strictly supermodular since $\Delta > 0$, so by Theorem A.1 in AL every selection from $R(Q_{-i})$ is non-decreasing in Q_{-i}. Thus, every selection of the correspondence $B : [0,(n-1)\overline{q}] \rightrightarrows [0,(n-1)\overline{q}]$ given by $B(Q_{-i}) \coloneqq (n-1)R(Q_{-i})/n$ is also non-decreasing in Q_{-i}. By Tarski's intersection point theorem (Theorem A.3 in AL), B has a fixed point, which is a symmetric equilibrium.

Non-existence of asymmetric equilibria: Suppose by contradiction that an asymmetric equilibrium exists, and denote it by $\widetilde{\boldsymbol{q}}$. Then, any permutation of $\widetilde{\boldsymbol{q}}$ should also be an equilibrium, and since $\widetilde{\boldsymbol{q}}$ is asymmetric there exists a permutation $\widehat{\boldsymbol{q}}$ with a firm i such that $\widehat{q}_i > \widetilde{q}_i$. But $\widetilde{Q} = \widehat{Q}$, so $\widehat{Q}_{-i} < \widetilde{Q}_{-i}$. Thus, $R(\widehat{Q}_{-i}) = R(\widetilde{Q}_{-i}) = \widetilde{Q} \geq \widetilde{Q}_{-i} > \widehat{Q}_{-i} \implies R(\widehat{Q}_{-i}) > \widehat{Q}_{-i}$, so $\widetilde{Q} = R(\widehat{Q}_{-i})$ makes the first derivative of the firm's objective non-negative, that is $P(\widetilde{Q}) + P'(\widetilde{Q})[\widetilde{Q} - (1-\lambda)\widehat{Q}_{-i}] - C'(\widetilde{Q} - \widehat{Q}_{-i}) \geq 0$. Also, since the firm's action space is not bounded from above, it trivially holds that $P(\widetilde{Q}) + P'(\widetilde{Q})[\widetilde{Q} - (1-\lambda)\widetilde{Q}_{-i}] - C'(\widetilde{Q} - \widetilde{Q}_{-i}) \leq 0$.

[66] The proof of uniqueness under $\Delta > 0$ is not considered in AL but is also an extension of standard results.

The last two inequalities imply

$$-(1-\lambda)P'\left(\widetilde{Q}\right) - \frac{C'(\widetilde{Q}-\widetilde{Q}_{-i}) - C'(\widetilde{Q}-\widehat{Q}_{-i})}{\widetilde{Q}_{-i}-\widehat{Q}_{-i}} \leq 0. \tag{5}$$

Last, since every selection from $R(Q_{-i})$ is non-decreasing in Q_{-i}, it follows from $R\left(\widehat{Q}_{-i}\right) = R\left(\widetilde{Q}_{-i}\right) = \widetilde{Q}$ that $R(Q_{-i}) = \widetilde{Q}$ for all $Q_{-i} \in [\widehat{Q}_{-i},\widetilde{Q}_{-i}]$. Therefore, in (5) we can let $\widehat{Q}_{-i} \to \widetilde{Q}_{-i}$, which gives $\Delta(\widetilde{Q},\widetilde{Q}_{-i}) \leq 0$, a contradiction.

For part (b) it remains to show that at most one symmetric equilibrium exists. $E_{P'} < (1+\lambda+\Delta)/H_n$ on L—which holds given that $E_{P'} < (1+\lambda+\Delta/n)/H_n$ and $\Delta > 0$ on L—implies that $\partial^2\left(\pi_i + \lambda\sum_{j\neq i}\pi_j\right)\Big/(\partial q_i)^2 < 0$, so that $r(Q_{-i})$ is a differentiable function. At a symmetric quantity profile we have $r'(Q_{-i}) = -1+\Delta(Q,Q_{-i})/(1+\lambda+\Delta(Q,Q_{-i})-H_nE_{P'}(Q))$. Symmetric equilibria are solutions to $g(q) \equiv r((n-1)q) - q = 0$. Thus, there will be at most one symmetric equilibrium if $g' < 0$, that is, if for any $q \in [0,\overline{Q}/n)$,

$$-\frac{1+\lambda-H_nE_{P'}(nq)}{1+\lambda+\Delta(nq,(n-1)q)-H_nE_{P'}(nq)} < \frac{1}{n-1} \iff E_{P'}(nq) < \frac{1+\lambda+\Delta(nq,(n-1)q)/n}{H_n}$$

which is true, since by assumption it is true on L.

Case $\boldsymbol{\Delta} < \mathbf{0}$: We first prove part (a) for $m = n$. $\Delta < 0$ and $E_{P'}(Q) < \frac{2-C''(Q-Q_{-i})/P'(Q)}{1-(1-\lambda)(1-s_i)}$ implies that the objective function of each firm is strictly concave in its quantity (in the part where $P(Q) > 0$). Thus, for Q_{-i} such that $r(Q_{-i}) > 0$, $r(Q_{-i})$ is a differentiable function with slope $r_i'(Q_{-i}) = -1 + \Delta/(2 - C''(q_i)/P'(Q) - (s_i + \lambda(1-s_i))E_{P'}(Q)) < -1$ given $\Delta < 0$. Thus, again $g' < 0$ since $r' < -1 < (n-1)^{-1}$ for every $n \geq 2$. Also, $g(0) \geq 0$ and $\lim_{q\to\infty} g(q) = -\infty$, so by continuity of g there exists a unique symmetric equilibrium.

We now prove part (a) for $m < n$. Let q_m be the symmetric equilibrium quantity produced by each firm when m firms are in the market. The m firms are clearly best-responding by producing q_m each. Also, $r'(Q_{-i}) < -1$ (when $r(Q_{-i}) > 0$) implies that $r(mq_m) = r((m-1)q_m + q_m) \leq \max\{r((m-1)q_m) - q_m, 0\} = 0$, since by definition of q_m, $r((m-1)q_m) = q_m$. Thus, the non-producing firms are also best-responding.

To show part (b) assume by contradiction that there is an equilibrium $\widetilde{\boldsymbol{q}}$ of a different type. Then there exist firms i and j such that $\widetilde{q}_i \neq \widetilde{q}_j$, $\widetilde{q}_i > 0$, $\widetilde{q}_j > 0$ in that equilibrium. Wlog let $\widetilde{q}_i > \widetilde{q}_j$. Given that $R'(Q_{-i}) = r'(Q_{-i}) + 1 < 0$ (when $R(Q_{-i}) > Q_{-i}$) it follows that $R(\widetilde{Q}_{-i}) = R(\widetilde{Q}_{-j}) \implies \widetilde{Q}_{-i} = \widetilde{Q}_{-j} \implies \widetilde{q}_i = \widetilde{q}_j$, a contradiction. **Q.E.D.**

Proof of Corollary 1.1 $\Delta(Q,Q_{-i}) = 1-\lambda+c_2/b$, constant over L. $E_{P'}(Q)=0$, also constant. Last, we have that $1+\lambda+\Delta(Q,Q_{-i}) = 2+c_2/b$. The result then follows from Proposition 1. Notice also that $Q_n = (a-c_1)/[b(H_n+1)+c_2/n]$, which is positive since $a > c_1$ and $c_2 > -2bc_1/a > -2b$. $\Pi(n,\lambda) = (a-c_1)^2\,(bnH_n + c_2/2)\,/\,[bn(H_n+1)+c_2]^2$ is also positive. Last, $C'(q_n) = [bc_1(H_n+1)+ac_2/n]/[b(H_n+1)+c_2/n]$ is positive given $c_2 > -2bc_1/a$, so in equilibrium marginal cost is positive. **Q.E.D.**

Proof of Proposition 2 (i) From the pricing formula (1) the Implicit Function Theorem gives $\partial Q_n/\partial\lambda = -(n-1)Q/[n+\Lambda-C''(Q/n)/P'(Q)-\Lambda E_{P'}(Q)] < 0$. For fixed n, total surplus changes with λ in the same direction as total quantity: $d\,\text{TS} = P(Q)dQ - \sum_{i=1}^{n} C'(q)dq = (P(Q)-C'(q))\,dQ$. Differentiating $\Pi\,(n,\lambda)$ with respect to λ we get

$$\frac{\partial \Pi\,(n,\lambda)}{\partial \lambda} = P'(Q_n)\frac{Q_n}{n}\frac{\partial Q_n}{\partial \lambda} + (P(Q_n)-C'(q_n))\,\frac{\partial Q_n}{\partial\lambda}\frac{1}{n} = P'(Q_n)\frac{Q_n}{n}\frac{\partial Q_n}{\partial\lambda}\frac{n-\Lambda_n}{n},$$

which is positive for $\lambda < 1$, where the second equality follows from the pricing formula (1).

(ii) Using the pricing formula (1) we get

$$\begin{aligned}\frac{\partial \Pi\,(n,\lambda)}{\partial n} &= P'(Q_n)\frac{Q_n}{n}\frac{\partial Q_n}{\partial n} - Q_n P'(Q_n) H_n \frac{n\frac{\partial Q_n}{\partial n} - Q_n}{n^2} \\ &\propto -\left[(1-\lambda)\left(H_n^{-1}-1\right) + n + \Lambda_n - H_n^{-1}C''\,(q_n)\,/P'\,(Q_n) - \Lambda_n E_{P'}\,(Q_n)\right] < 0,\end{aligned}$$

where the inequality follows from what we have seen in section A.2.

(iii) $\partial q_n/\partial n = \partial(Q_n/n)/\partial n = n^{-1}\partial Q_n/\partial n - Q_n/n^2 \propto -\,(1+\lambda-H_nE_{P'}(Q))$.

(iv) From the pricing formula (1) the Implicit Function Theorem gives $\partial Q_n/\partial n = q_n\Delta/(n(1+\lambda+\Delta/n-H_nE_{P'}\,(Q_n))) \propto \Delta$. **Q.E.D.**

A.4 Proofs of sections 4 and 5

Proof of Proposition 3 The derivative of $\Psi(n,\lambda)$ with respect to n is equal to

$$\begin{aligned}\frac{\partial\Psi(n,\lambda)}{\partial n} &= \lambda\,(\Pi\,(n,\lambda)-\Pi\,(n-1,\lambda)) + \Lambda_n\frac{\partial \Pi\,(n,\lambda)}{\partial n} - (\Lambda_n-1)\left.\frac{\partial\Pi\,(\nu,\lambda)}{\partial\nu}\right|_{\nu=n-1} \\ &\propto E_{\Delta\Pi,n} - \left(\frac{\Lambda_n-1}{\Lambda_n} + \frac{n-1}{\Lambda_n}\frac{\left.\frac{\partial\Pi(\nu,\lambda)}{\partial\nu}\right|_{\nu=n-1}}{\Pi\,(n,\lambda)-\Pi\,(n-1,\lambda)}\right) < 0,\end{aligned}$$

and the result obtains given Proposition 1. **Q.E.D.**

Proof of Proposition 4 The derivative of $\Psi(n,\lambda)$ with respect to λ is given by

$$\frac{\partial\Psi(n,\lambda)}{\partial\lambda} = (n-1)\left(\Pi(n,\lambda) - \Pi(n-1,\lambda)\right) + \Lambda_n \frac{\partial\Pi(n,\lambda)}{\partial\lambda} - (\Lambda_n - 1)\frac{\partial\Pi(n-1,\lambda)}{\partial\lambda}$$
$$\propto -\frac{\lambda\left(\frac{\partial\Pi(n,\lambda)}{\partial\lambda} - \frac{\partial\Pi(n-1,\lambda)}{\partial\lambda}\right)}{\Pi(n,\lambda) - \Pi(n-1,\lambda)} - \frac{1}{\lambda}\frac{\lambda\frac{\partial\Pi(n,\lambda)}{\partial\lambda}\Big/\Pi(n,\lambda)}{\frac{\Pi(n,\lambda)-\Pi(n-1,\lambda)}{\Pi(n,\lambda)}(n-1)} - 1.$$

The result follows by the Implicit Function Theorem given Proposition 3. **Q.E.D.**

Proof of Proposition 5 We have $\partial\,\mathrm{TS}_n/\partial n = \Pi(n,\lambda) - f - \Lambda_n Q_n P'(Q_n)\partial q_n/\partial n$. Given $\Psi(\widehat{n}^*(\lambda),\lambda) = f$, $d\,\mathrm{TS}_n/dn|_{n=\widehat{n}^*(\lambda)}$ is equal to (denote $\Pi_n(n,\lambda) \equiv \partial\Pi(n,\lambda)/\partial n$)

$$-\phi(\widehat{n}^*(\lambda),\lambda)\,\lambda\widehat{n}^*(\lambda)\Pi_n(\widehat{n}^*(\lambda),\lambda) - \Lambda_{\widehat{n}^*(\lambda)}Q_{\widehat{n}^*(\lambda)}P'\left(Q_{\widehat{n}^*(\lambda)}\right)\frac{\partial q_n}{\partial n}\bigg|_{n=\widehat{n}^*(\lambda)},$$

and the result follows from single-peakedness of total surplus in n (and given $P' < 0$), if we substitute in $\Pi(n,\lambda)/\partial n$ and $\partial q_n/\partial n$ from the proof of Proposition 2. For Remark 5.3, notice that $\Delta < 0$ on L implies $C''(q) < 0$ for every $q < \overline{Q}$. By Proposition 2, Q_n is decreasing in n, and thus, so is consumer surplus. Also, $n\Pi(n,\lambda) \equiv P(Q_n)Q_n - nC(q_n) < P(Q_n)Q_n - C(Q_n) \leq P(q_1)q_1 - C(q_1) = \Pi(1,\lambda)$, where the first inequality follows from $C'' < 0$, and the second from q_1 being the monopolist's optimal quantity. Thus, both consumer surplus and industry profits are maximized for $n = 1$, so $n^o(\lambda) = 1$. Last, $n\Pi(n,\lambda) < \Pi(1,\lambda)$ for $n = 2$ and $\lambda = 1$ implies that $\Psi(2,1) = 2\Pi(2,1) - \Pi(1,1) < 0$, so $n^*(1) = 1$. **Q.E.D.**

Proof of Proposition 6 We have that $d\widehat{n}^*(\lambda)/df = \left(\partial\Psi(n,\lambda)/\partial n\right)^{-1}\big|_{n=\widehat{n}^*(\lambda)}$, and part (ii) follows if we take the directional derivative of $d\widehat{n}^*(\lambda)/df$. **Q.E.D.**

B Additional material

B.1 Individual firm's objective function under overlapping ownership

Here we briefly describe settings of common and cross ownership which can give rise to the Cournot-Edgeworth λ oligopoly model that we study.

B.1.1 A model of corporate control under common ownership

There is a finite set $\mathcal{J}$ of investors. For each $j \in \mathcal{J}$, β_{ji} denotes investor j's share of firm i, γ_{ji} captures the extent of her control over firm i, and $u_j(\boldsymbol{q}) \coloneqq \sum_{i \in \mathcal{F}} \beta_{ji}\pi_i(\boldsymbol{q})$ is her total portfolio profit, where π_i firm i's profit function. O'Brien and Salop (2000) assume that the manager of firm i maximizes a weighted average of the shareholders' portfolio profits; that is, given $\boldsymbol{q}_{-i}$ she maximizes

$$\sum_{j \in \mathcal{J}} \gamma_{ji} u_j(\boldsymbol{q}) \propto \pi_i(\boldsymbol{q}) + \sum_{k \in \mathcal{F} \setminus \{i\}} \lambda_{ik}\pi_k(\boldsymbol{q}),$$

where $\lambda_{ik} \coloneqq \sum_{j \in J} \gamma_{ji}\beta_{jk} \Big/ \sum_{j \in J} \gamma_{ji}\beta_{ji}$. A common assumption on γ is proportional control, that is $\gamma_{ji} = \beta_{ji}$ for every $j \in \mathcal{J}$ and every $i \in \mathcal{F}$. For appropriate ownership and control structures β and γ it will be that $\lambda_{ik} = \lambda$, fixed for every pair of firms i,k. One such ownership and control structure (β,γ) is described in section B.1.3.

B.1.2 Firm objectives under cross ownership

Firm objectives under cross ownership are also described in Gilo et al. (2006) and López and Vives (2019). Assume that we start with each firm i being held by shareholders who do not hold shares of any of the other firms. Then, each *firm* i buys share $\alpha \in [0, 1/(N-1))$ of every other firm $k \in \mathcal{F} \setminus \{i\}$ *without control rights.* In other words, each firm i acquires a claim to share α of the *total earnings* of every other firm. The total earnings of each firm i now include the profit directly generated by firm i and firm i's earnings from its claims over the other firms' total earnings.

We end up with each firm i being controlled by its initial shareholders, each of whom only holds claims to firm i's total earnings. The controlling shareholders collectively hold a claim to share $(1-(N-1)\alpha)$ of firm i's total earnings. All controlling shareholders of firm i agree that firm i should seek to maximize its total earnings.

For every $\boldsymbol{q}$, the total earnings $\widetilde{\pi}_i(\boldsymbol{q})$ of each firm i are then given by the solution to the system of equations

$$\widetilde{\pi}_i(\boldsymbol{q}) = \overbrace{\pi_i(\boldsymbol{q})}^{\substack{\text{firm } i\text{'s earnings from} \\ \text{the profit directly} \\ \text{generated by firm } i}} + \overbrace{\alpha \sum_{k \in \mathcal{F} \setminus \{i\}} \widetilde{\pi}_k(\boldsymbol{q})}^{\substack{\text{firm } i\text{'s earnings from its} \\ \text{claims over each firm } k\text{'s,} \\ k \neq i, \text{ total earnings}}}, \quad \text{for each } i \in \mathcal{F}.$$

Solving the system of equations we find that each firm i's objective is to maximize

$$\widetilde{\pi}_i(\boldsymbol{q}) \propto \pi_i(\boldsymbol{q}) + \lambda \sum_{k \in \mathcal{F} \setminus \{i\}} \pi_k(\boldsymbol{q}) \qquad \text{where } \lambda \coloneqq \alpha/[1 - (N-2)\alpha] \in [0,1).$$

B.1.3 An example of post-entry overlapping ownership

Post-entry overlapping ownership can for example arise in the form of common ownership as described below. Let all firms be newly-established and the set of investors $\mathcal{J}$ be partitioned into $\{J_0\} \cup \cup_{i \in \mathcal{F}} \{J_i\}$ with $|J_i| = |J_0| = m$ for every $i \in \mathcal{F}$. Before entry each firm i is (exclusively) held by the set J_i of entrepreneurs with $\beta_{ji} = 1/m$ for every $j \in J_i$; there is no common ownership before entry, so when considering entry, the entrepreneurs of each firm unanimously agree to maximize their own firm's profit.[67] After entry, the set J_0 of investors, who previously held no shares of any firm, buy firm shares. Each investor $j \in J_0$ now holds share $\beta'_{ji} = \sigma/m$ of each firm i that has entered, and each entrepreneur $j \in J_i$ holds share $\beta'_{ji} = (1-\sigma)/m$ of her firm for some $\sigma \in [0,1]$. That is, after entry each entrepreneur sells the same amount of shares to the investors, who are now uniformly invested in all firms in the industry. Consider the O'Brien and Salop (2000) model and for every firm i that has entered let $\gamma'_{ji} = \widetilde{\gamma}/m$ be the control each investor $j \in J_0$ has over firm i for some $\widetilde{\gamma} \in [0,1]$, and $\gamma'_{ji} = (1-\widetilde{\gamma})/m$ the control each entrepreneur $j \in J_i$ has over her firm i.[68] After entry, the manager of each firm i maximizes

$$\pi_i(\boldsymbol{q}) + \lambda \sum_{k \neq i} \pi_k(\boldsymbol{q}), \quad \text{where } \lambda = \frac{\widetilde{\gamma}\sigma}{\widetilde{\gamma}\sigma + (1-\widetilde{\gamma})(1-\sigma)} = \frac{1}{1 + (\widetilde{\gamma}^{-1} - 1)(\sigma^{-1} - 1)} \in [0,1].$$

[67]This relies on the fact that a firm's entrepreneurs only hold shares of their firm both before and after entry. Common ownership develops after entry not through a firm's entrepreneurs investments in other firms but because outside investors invest in multiple firms.

[68]For every other pair of entrepreneur j and firm i, $\beta'_{ji} = \gamma'_{ji} = 0$.

Here λ is increasing in the common owners' level of holdings σ and control $\widetilde{\gamma}$. Under proportional control $\sigma = \widetilde{\gamma}$, and $\lambda = \left[1 + (\sigma^{-1} - 1)^2\right]^{-1}$.

B.2 Pricing-stage equilibria under parametric assumptions

CESL demand is of the form

$$P(Q) = \begin{cases} a + bQ^{1-E} & \text{if } E > 1 \\ \max\{a - b\ln Q, 0\} & \text{if } E = 1 \\ \max\{a - bQ^{1-E}, 0\} & \text{if } E < 1 \end{cases}$$

for parameters $a \geq 0$ and $b > 0$. For $E = 0$ this reduces to linear demand, while for $a = 0$ and $E > 1$ it reduces to constantly elastic demand with elasticity $\eta = (E-1)^{-1}$.

Claim 1 provides the equilibria under parametric assumptions on the demand and cost functions. The total quantity is decreasing in the level of overlapping ownership, λ.

Claim 1. Under CESL demand and constant returns to scale the total equilibrium quantity in the pricing stage is

$$Q_n = \begin{cases} \left[\frac{b(1-H_n(E-1))}{c-a}\right]^{\frac{1}{E-1}} & \text{if } E \in (1,2) \text{ and } c > a \\ e^{\frac{a-c-bH_n}{b}} & \text{if } E = 1 \\ \left[\frac{a-c}{b(1+H_n(1-E))}\right]^{\frac{1}{1-E}} & \text{if } E < 1 \text{ and } a > c, \end{cases}$$

where $H_n := \Lambda_n/n$, $\Lambda_n := 1 + \lambda(n-1)$. Under linear demand and linear-quadratic costs, it is $Q_n = \frac{a-c_1}{b(1+H_n)+c_2/n}$.

B.3 Stability of pricing stage equilibrium

Proposition 7 examines the local asymptotic stability of the pricing stage equilibrium in the sense of the myopic continuous adjustment process, as described in al Nowaihi and Levine (1985).

Proposition 7. If $\Delta > 0$, then the pricing stage equilibrium is locally stable.

Proposition 7' studies stability with the maintained assumption relaxed.

Proposition 7'. Assume $\Delta > 0$ but drop the assumption that $E_{P'}(Q) < (1 + \lambda + \Delta(Q,Q_{-i})/n)/H_n$ on L, so that multiple symmetric equilibria may exist. Then, a pricing stage equilibrium is locally stable if and only if $E_{P'}(Q) < (1 + \lambda + \Delta(Q,Q_{-i})/n)/H_n$ in that equilibrium.

Remark B.1. For $\lambda = 0$ we recover the sufficient local (in)stability conditions implied by Theorems 3, 4, and 5 of al Nowaihi and Levine (1985).[69]

Under $\Delta > 0$, when we drop the condition $E_{P'} < (1 + \lambda + \Delta/n)/H_n$ on L guaranteeing uniqueness, multiple symmetric equilibria may exist, some of which stable and some unstable. These two sets of equilbria are differentiated by a local version of the dropped condition. An equilibrium is stable if and only if the dropped condition holds *in that equilibrium.*

B.4 Additional comparative statics of pricing stage equilibrium

Let P and C be three times differentiable. Denote by $E_{P''}(Q) \coloneqq P'''(Q)Q/P''(Q)$ the elasticity of the curvature of inverse demand.

Proposition 8. The following hold:

(i) If $\Delta > 0$ and also for every $Q < \overline{Q}$, $E_{P'}(Q) < 2$, $E_{P'}(Q)\left[E_{P'}(Q) + E_{P''}(Q)\right] \geq -2$ and for every $q < \overline{q}$, $C''(q), C'''(q) \geq 0$,[70] then $(\partial Q_n)^2/(\partial\lambda\partial n) < 0$.

(ii) $\partial^2 q_n/(\partial\lambda\partial n)$ can be negative or positive (and change sign as λ and/or n changes). For example, for constant MC and CESL demand

$$\operatorname{sgn}\left\{\frac{\partial^2 q_n}{\partial\lambda\partial n}\right\} = \operatorname{sgn}\left\{n + \lambda(n-1) - 3 - \left(H_n(n-1) - 1\right)E\right\}.$$

Under the assumptions of part (i), the negative effect of overlapping ownership on the total quantity is strongest in industries with a large number of firms, which would otherwise be the most competitive ones.

[69]al Nowaihi and Levine (1985) deal with a possibly asymmetric equilibrium; they provide analogous conditions where expressions such as Δ vary across firms.

[70]If $P''(Q) = 0$, cancel P'' in $E_{P'}(Q)$ with the one in $E_{P''}(Q)$. Under CESL demand, $E_{P'}(Q)\left[E_{P'}(Q) + E_{P''}(Q)\right] \geq -2$ holds if and only if $E \leq 2$.

Now we study how aggregate industry profits depend on the number of firms.

$$\mu_n \coloneqq 1 - \frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\frac{1 - H_n}{\eta(Q_n) - H_n}.$$

Proposition 9. Let $\lambda < 1$. Then, the following statements hold:

(i) if $\mu_n \leq 0$, aggregate industry profits are decreasing in n,

(ii) if $\mu_n > 0$, aggregate industry profits are decreasing (resp. increasing) in n if $E_C(q_n) \overset{(\text{ resp. } >)}{<} \mu_n^{-1}$,

(iii) if $C''(q) < 0$ for every $Q \in [0,Q_n]$, then monopoly maximizes aggregate industry profits, $\Pi(1,\lambda) > n\Pi(n,\lambda)$.

Remark B.2. If $\Delta < 0$, then $\partial Q_n/\partial n > 0$, so $\mu_n < 1$, and thus, aggregate industry profits are decreasing in n if $E_C(q_n) \leq 1$. If for example $C'' < 0$ globally (consistent with $\Delta < 0$), then indeed $E_C(q_n) < 1$.

Remark B.3. If $\lambda = 1$ and $C''(q) > 0$ for every $q \in [0,q_n]$, aggregate industry profits are increasing in n.

Consider the extreme case of $\lambda = 1$ and notice the following. Condition $\Delta > 0$ requires decreasing returns to scale, so that aggregate gross profits increase with n (*i.e.*, $n\Pi(n,1) > (n-1)\Pi(n-1,1)$ for any n) due to savings in variable costs as production is distributed across more firms, even though the total quantity increases (see Proposition 2), and thus price decreases with the number of firms. Intuitively, aggregate gross profits increasing in n for $\lambda = 1$ is tied to the uniqueness of the (symmetric) equilibrium in the pricing stage. Since firms jointly maximize aggregate profits, the latter should increase with n for firms to strictly prefer to spread production evenly. On the other hand, under constant returns to scale aggregate profits are constant in n; increasing the number of firms simply changes how the firms can jointly produce the fixed level of total output that maximizes joint profits.[71] Last, under increasing returns to scale it is an equilibrium for all production to be concentrated in a single firm.

[71] As argued already, in this case, the are infinitely many equilibria of the pricing stage, all with the same total quantity.

Claim 2. Under linear demand and linear-quadratic costs with $c_1 = 0$

$$\frac{\partial\left[n\Pi\left(n,\lambda\right)\right]}{\partial n} = \frac{c_2}{2bn} - \frac{b(1-\lambda)+c_2}{b(n+\Lambda_n)+c_2}\left(1-H_n\right) \quad \text{with} \quad \frac{\partial^2\left[n\Pi\left(n,\lambda\right)\right]}{\partial\lambda\partial n} > 0.$$

(i) for $\lambda = 0$, $\text{sgn}\left\{\partial\left[n\Pi\left(n,0\right)\right]/\partial n\right\} = \text{sgn}\left\{c_2 - b(n-1)\right\}$,

(ii) for $\lambda = 1$, $\partial\left[n\Pi\left(n,1\right)\right]/\partial n > 0$,

(iii) if $c_2 > b(n-1)$, then $\partial\left[n\Pi\left(n,\lambda\right)\right]/\partial n > 0$ for every $\lambda \in [0,1]$,

(iv) if $c_2 < b(n-1)$, then there exists $\lambda^* \in (0,1)$ such that $\partial\left[n\Pi\left(n,\lambda\right)\right]/\partial n \overset{(\text{ resp. } <)}{>} 0$ if and only if $\lambda \overset{(\text{ resp. } <)}{>} \lambda^*$.

In the decreasing returns to scale case of Claim 2 we see that λ and n are complements in increasing aggregate industry profits. Particularly, for λ high enough aggregate industry profits are increasing in the number of firms. This is because with λ high, entry does not reduce the price as much (see point (iii-b) of Proposition 2), so the cost-saving effect of entry under decreasing returns to scale dominates.

B.5 Concavity of total surplus in the number of firms

Lemma 2. TS_n is globally strictly concave in n if for every n

$$\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\left[1-\lambda-H_n\left(\left(\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}-1\right)\left(1-E_{P'}(Q_n)\right)+\frac{\partial^2 Q_n}{(\partial n)^2}\left(\frac{\partial Q_n}{\partial n}\right)^{-1}n-1\right)\right] > \frac{1-\lambda}{n}.$$

Under constant marginal costs and $E_{P'}(Q_n) < 2$ for every n, this is true if $E'_{P'}(Q) \equiv \partial E_{P'}(Q)/\partial Q$ is not too high; particularly, $E'_{P'} \leq 0$ is sufficient, and thus so is CESL demand.

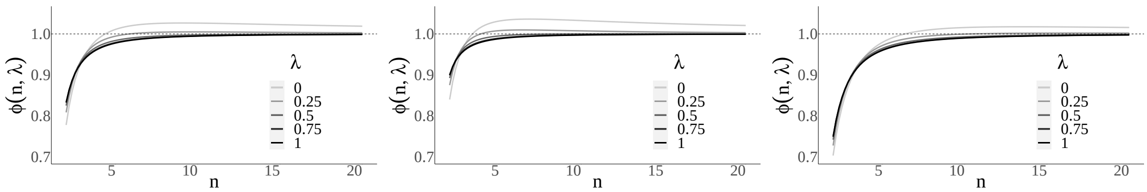
Remark B.4. More generally, all else constant, the condition of Lemma 2 is satisfied if the elasticity of the slope of Q_n with respect to n, $\frac{\partial^2 Q_n}{(\partial n)^2}\left(\frac{\partial Q_n}{\partial n}\right)^{-1} n$, is not too high. Also, remember that $\frac{\partial Q_n}{\partial n}\frac{n}{Q_n} \in (0,1)$ under the assumptions of Proposition 2(iii-a), so all else constant, in that case the condition is satisfied if $E_{P'}(Q)$ is not too high.

B.6 Numerical results showing that ϕ is close to 1

The numerical results of Figure 8 verify that $\phi\left(n,\lambda\right)$ is indeed close to 1, especially for $n \geq 3$.

Figure 8: $\phi(n,\lambda)$ under linear demand and linear-quadratic costs

(a) $b = 1$, $c_2 = 1$ **(b)** $b = 2$, $c_2 = 1$ **(c)** $b = 1$, $c_2 = 2$



Note: It can be checked that $\phi(n,\lambda)$ is invariant with respect to c_1 and the demand parameter a.

B.7 Derivation of Numerical Results

Under CESL demand and constant returns to scale, given Claim 1 we find that

$$\Pi(n,\lambda) = \begin{cases} \frac{1}{n}\left[a + b\left[\frac{b(1-H_n(E-1))}{c-a}\right]^{\frac{1-E}{E-1}} - c\right]\left[\frac{b(1-H_n(E-1))}{c-a}\right]^{\frac{1}{E-1}} & \text{if } E \in (1,2) \text{ and } c > a \\ \frac{1}{n}\left[a - b\ln\left(e^{\frac{a-c-bH_n}{b}}\right) - c\right]e^{\frac{a-c-bH_n}{b}} & \text{if } E = 1 \\ \frac{1}{n}\left[a - b\left[\frac{a-c}{b(1+H_n(1-E))}\right]^{\frac{1-E}{1-E}} - c\right]\left[\frac{a-c}{b(1+H_n(1-E))}\right]^{\frac{1}{1-E}} & \text{if } E < 1 \text{ and } a > c, \end{cases}$$

$$= \begin{cases} \frac{H_n(E-1)b^{\frac{1}{E-1}}}{n}\left[\frac{1-H_n(E-1)}{c-a}\right]^{\frac{2-E}{E-1}} & \text{if } E \in (1,2) \text{ and } c > a \\ \frac{bH_n}{n}e^{\frac{a-c-bH_n}{b}} & \text{if } E = 1 \\ \frac{H_n(1-E)}{nb^{\frac{1}{1-E}}}\left[\frac{a-c}{1+H_n(1-E)}\right]^{\frac{2-E}{1-E}} & \text{if } E < 1 \text{ and } a > c, \end{cases}$$

Derivation of Numerical Result 1 Parameters a, b and c only affect the magnitudes of $d\widehat{n}^*(\lambda)/d\lambda$ and $dQ_{\widehat{n}^*(\lambda)}/d\lambda$, and not their signs. The result then is obtained in a way analogous to the one described in the Derivation of Numerical Result 3. ∎

Derivation of Numerical Result 2 Notice that, given a fixed n, $\phi(n,\lambda)$ is independent of a, c_1, and f. Also, $E_{P'}(Q_n) = 0$ and $\Delta = 1 - \lambda + c_2/b$ always (independently of a, c_1, and f). Thus, the expressions in Proposition 5 depend on a, c_1, and f only through their effect on $\widehat{n}^*(\lambda)$. Also, if we look at the expression for $\Pi(n,\lambda)$ in the proof of Corollary 1.1, it is easy to see that $\widehat{n}^*(\lambda)$ depends on a, c_1, and f only though $(a - c_1)^2/f$.[72] Further, $\Pi(2,0) \geq f$ if and only if $(a - c_1)^2/f \geq 2(3b + c_2)^2/(2b + c_2)$, so the values that b and c_2 can take that make the net monopoly profit non-negative also depend on a, c_1, and f only through $(a - c_1)^2/f$. Finally, $\Delta \geq 0$ is satisfied for every λ if and only if $c_2 \geq 0$, which does not depend on a, c_1, or f. Thus, without loss of generality,

[72]Namely, $\widehat{n}^*(\lambda)$ is increasing in $(a - c_1)^2/f$.

we can let $a = 1$, $c_1 = 0$ and only vary b, c_2, and f in the simulations. We then numerically check that for every $(b,c_2,f) \in \{(b,c_2,f) : \exists(t_1,t_2,t_3) \in \{0,1,\ldots,9\}^3$ such that $b = 0.01 + 1.11t_1, c_2 = 100t_2/9, f = 0.001 + [(2b + c_2)/(2(3b + c_2)^2) - 0.001]t_3/9\}$ (*i.e.*, for 1,000 parametrizations) there exists a threshold $\overline{\lambda}$ as claimed by solving for $\widehat{n}^*(\lambda)$ and $\widehat{n}^o(\lambda)$ for every $\lambda \in \{0,0.05,0.1,\ldots,0.95\}$. ■

Derivation of Numerical Result 3 It is easy to see that the signs of derivatives of $\Psi(n,\lambda)$ are independent of a, b and c. Thus, we can wlog set (i) $a = b = 1$ and $c = 2$ for the case $E \in (1,2)$, and (ii) $a = 2$, $b = c = 1$ for the case $E < 1$.

For $E > 1$ we run the following R code:

```
# load packages #
library(Deriv)
library(optimx)

# define functions #
Lambda = function(n,lambda) {1 + lambda*(n-1)}
H = function(n,lambda) {(1 + lambda*(n-1))/n}

Pi = function(n,lambda,E,a,b,c) { H(n,lambda)*(E-1)*b^(1/(E-1))*
( (1-H(n,lambda)*(E-1))/(c-a) )^((2-E)/(E-1))/n }
Psi = function(n,lambda,E,a,b,c) {  Pi(n,lambda,E,a,b,c)-lambda*(n-1)*
( Pi(n-1,lambda,E,a,b,c)-Pi(n,lambda,E,a,b,c)) }

# symbolically differentiate Psi #
Deriv_wrt_n_Psi = Deriv(Psi,"n")
Deriv_wrt_nlambda_Psi = Deriv(Deriv_wrt_n_Psi,"lambda")

# define function that creates grid of starting points for optimization #
grid = function(density_n,min_n,max_n,density_l,min_l,max_l,
density_E,min_E,max_E) {
  output = matrix(nrow = (density_n+1)*(density_l+1)*(density_E+1), ncol = 3)
  row_number = 1
  for (i in seq(from = min_n, to = max_n, by = (max_n-min_n)/density_n)) {
    for (j in seq(from = min_l, to = max_l, by = (max_l-min_l)/density_l)) {
```

```
      for (k in seq(from = min_E, to = max_E, by = (max_E-min_E)/density_E)) {
        output[row_number,] = c(i,j,k)
        row_number = row_number + 1
      }
    }
  }
  return(output)
}

# minimize cross derivative of Psi from multiple starting points #

minima =  multistart(parmat = grid(15,2,7,15,0,1,30,1.001,1.7),
      fn = function(x) {Deriv_wrt_nlambda_Psi(x[1],x[2],x[3],1,1,2)},
      method = c("L-BFGS-B"), lower = c(2,0,1.001), upper = c(7,1,1.7))
```

The code returns that

$$\min_{(n,\lambda,E)\in[2,7]\times[0,1]\times[1.001,1.7]} \frac{\Psi(n,\lambda)}{\partial\lambda\partial n} \approx 2.31 \cdot 10^{-6} > 0,$$

which is reached for $n = 7$, $\lambda = 0$ and $E = 1.001$.

In the case of $E < 1$ we similarly find that

$$\min_{(n,\lambda,E)\in[2,8]\times[0,1]\times[-1000,0.999]} \frac{\Psi(n,\lambda)}{\partial\lambda\partial n} \approx 1.11 \cdot 10^{-7} > 0,$$

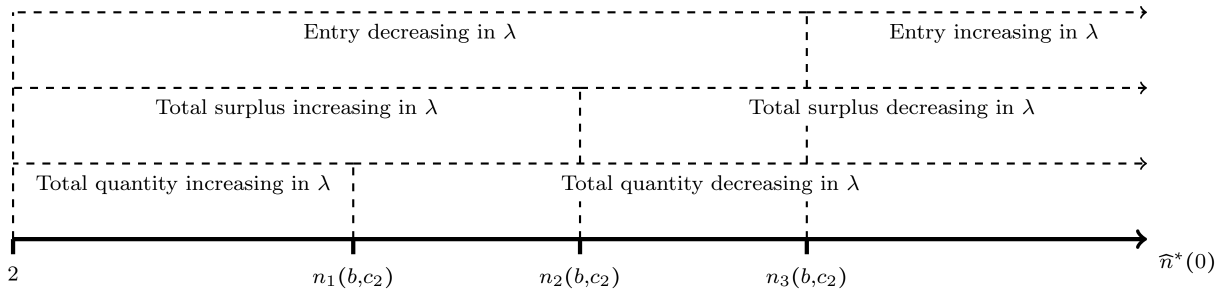
which is reached for $n = 8$, $\lambda = 0$ and $E = 0.999$. In additional simulations, allowing E to be even lower than -1000 does not change the result. ■

B.8 Additional results on the linear-quadratic model

Claim 3 studies how entry, the total quantity and total surplus change with overlapping ownership around $\lambda = 0$. Figure 9 summarizes the results.

Claim 3. Ignore the integer constraint on n (so that entry is given by $\widehat{n}^*(\lambda)$). Let demand be linear and cost be linear-quadratic with $a > c_1 \geq 0$, $-b \neq c_2 > -2bc_1/a$, and assume $\widehat{n}^*(0) \geq 2$. Then, there exist thresholds $n(b,c_2) \in \mathbb{R}^3$ (that depend on b and c_2) with $n_3(b,c_2) > n_2(b,c_2) > \max\{n_1(b,c_2), 2\}$ such that starting from $\lambda = 0$:

Figure 9: Comparative statics around $\lambda = 0$ under linear demand and linear-quadratic cost



Note: See Claim 3 for precise statement. $n_1(b,c_2) > 2$ only under significantly decreasing MC.

(i) entry is locally increasing (resp. decreasing) in λ if $\widehat{n}^*(0) \overset{(\text{resp. } <)}{>} n_3(b,c_2)$,

(ii) the total surplus is locally increasing (resp. decreasing) in λ if $\widehat{n}^*(0) \overset{(\text{resp. } >)}{<} n_2(b,c_2)$,

(iii) if $c_2 > -3b/2$, then $n_1(b,c_2) < 2$ and the total quantity is locally decreasing in λ,

(iv) if $c_2 < -3b/2$, then $n_1(b,c_2) > 2$ and the total quantity is locally increasing (resp. decreasing) in λ if $\widehat{n}^*(0) \overset{(\text{resp. } >)}{<} n_1(b,c_2)$,

Part (i) of the Corollary extends our finding that if without overlapping ownership many (resp. few) firms enter, then marginally increasing overlapping ownership will increase (resp. decrease) entry.

Part (ii) shows that marginally increasing λ above 0 increases total surplus if and only if entry is low. Particularly, the direct (negative) effect of an increase in λ on total surplus is dominated by the alleviation of excessive entry (since for $\lambda = 0$ entry is excessive) due to the increase in λ. We thus obtain another sufficient condition: if absent overlapping ownership, entry would be low, then a planner that regulates overlapping ownership (but not entry) should choose a positive level of it.

Parts (iii) and (iv) show that introducing a small amount of overlapping ownership may only increase the total quantity when MC is significantly decreasing (which means that the Cournot market is quasi-anticompetitive) and entry is low. In that case, the softening of pricing competition due to the increase in overlapping ownership is dominated by the concurrent decrease in entry—which tends to increase the total quantity since the market is quasi-anticompetitive. This yields a sufficient condition for consumer surplus to be maximized by some $\lambda > 0$. As shown in Figure 3d, this condition is not necessary, since with decreasing MC a positive level of overlapping ownership can be optimal under a consumer surplus standard even when overlapping ownership decreases the total quantity around $\lambda = 0$.

B.9 Free entry under pre-entry overlapping ownership and the presence of maverick firms

This section presents a model of free entry with pre-entry overlapping ownership under the presence of maverick firms.

For simplicity, model the maverick firms as a competitive fringe that in the first stage (where oligopolists enter) submit an aggregate supply schedule. Namely, there is a set $\mathcal{F}_m$ of infinitesimal firms. Firm $i \in \mathcal{F}_m$ chooses to either be inactive or produce one (infinitesimal) unit of the good at cost $\chi(i)$.[73] Thus, the aggregate supply function by the maverick firms in the third stage $S : \mathbb{R}_+ \to \mathbb{R}_+$ is given by $S(p) := \int_{i \in \mathcal{F}_m} I(\chi(i) \le p)\, di$. $S : \mathbb{R}_+ \to \mathbb{R}_+$ with $S(p) = 0$ for every $p \in [0, \underline{p}]$ and $S'(p) > 0$ for every $p > \underline{p}$ where $\underline{p} \ge 0$. Then, the price $p > 0$ in the competitive equilibrium among the maverick firms will be implicitly given by $P^{-1}(p) = Q + S(p)$, where Q is the total quantity produced by the oligopolists.[74] This means that in the second stage, the oligopolists are essentially faced with inverse demand $\widetilde{P} : \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$\widetilde{P}(Q) = \begin{cases} P\left(Q + \omega^{-1}(Q)\right) \in \left(\underline{p}, P(Q)\right) & \text{if } P(Q) > \underline{p} \\ P(Q) & \text{if } P(Q) \le \underline{p} \end{cases}$$

where $\omega : \mathbb{R}_{++} \to \mathbb{R}_+$ is given by $\omega(y) := P^{-1} \circ S^{-1}(y) - y$.[75] $\omega^{-1}(Q)$ gives the quantity supplied in the competitive equilibrium among the maverick firms when the oligopolists produce Q. For example, in the case of (i) linear demand $P(Q) = \max\{a - bQ, 0\}$, (ii) linear maverick aggregate supply schedule $S(p) = \max\{(p - \underline{p})/b_m, 0\}$ with $b_m > 0$ and $\underline{p} \ge 0$, and (iii) constant MC (for the oligopolists), $C(q) = cq$, with $a > c \ge \underline{p}$,[76] for any

[73] This cost can be thought to include any applicable entry costs. Since maverick firms are infinitesimal and each supply an infinitesimal quantity, their entry cost is also infinitesimal.

[74] We assume that $S(p) > P^{-1}(p)$ for p large enough.

[75] To see this substitute $p = P\left(Q + \omega^{-1}(Q)\right)$ in $P^{-1}(p) = Q + S(p)$, which gives

$$Q + \omega^{-1}(Q) = Q + S \circ P\left(Q + \omega^{-1}(Q)\right) \iff P^{-1} \circ S^{-1} \circ \omega^{-1}(Q) - \omega^{-1}(Q) = Q,$$

which is true by definition of ω.

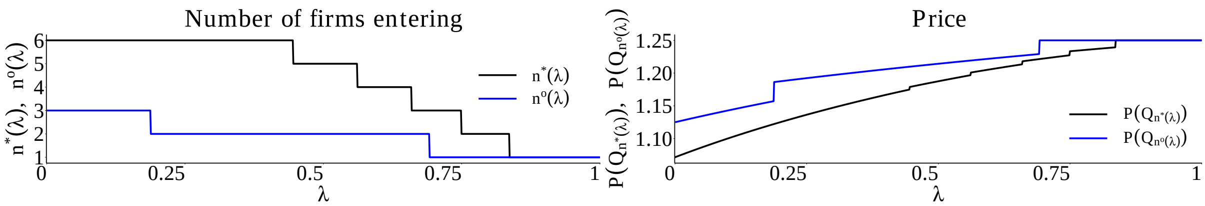
[76] For $c = \underline{p}$, the most efficient maverick firms is as efficient as the oligopolists.

$Q \in [0, (a-c)/b]$, $\widetilde{P}$ is given by[77]

$$\widetilde{P}(Q) = a - \overbrace{\frac{a - \underline{p}}{1 + b_m/b}}^{<a} - \overbrace{\frac{b}{1 + b/b_m}}^{<b} Q.$$

The (prospect of) entry by maverick firms essentially changes the demand faced by the commonly-owned firms by depressing it and making it more elastic. If in the paper wherever P we read $\widetilde{P}$, the results on the effects of overlapping ownership on entry and the price continue to hold (with the number of firms n not counting maverick firm entry). A comparison of Figure 10 with Figure 3a in the paper (the two figures use the same parametrization but in the former maverick firms are added) shows entry to be less sensitive to overlapping ownership due to the presence of the maverick firms, as argued in the paper.

Figure 10: Equilibrium with pre-entry overlapping ownership under the presence of maverick firms for varying λ



Note: Lines represent values in equilibrium; linear demand, constant MC: $a = 2$, $b = c = 1$, $f = 0.01$; linear maverick aggregate supply schedule: $b_m = \underline{p} = 1$.

Last, the total surplus $\widetilde{\text{TS}}(\boldsymbol{q})$ now includes the maverick firms' surplus, where $\boldsymbol{q}$ still is the quantity profile of the oligopolists. Denote by $\widetilde{\text{TS}}_n$ the pricing stage equilibrium total surplus when n commonly-owned firms enter. Equation (4) also applies in the case with maverick firms but with $\widetilde{\Xi}(n,\lambda) \coloneqq (n-1)\left(\widetilde{\Pi}(n-1,\lambda) - \widetilde{\Pi}(n,\lambda)\right)$, $\widetilde{\Pi}(n,\lambda) \coloneqq \widetilde{P}(Q_n)q_n - C(q_n)$ and $\widetilde{P}$ replacing Ξ, Π and P. Q_n, q_n are still the quantities produced by the commonly-owned firms in the pricing stage equilibrium where n of them enter. $\widehat{n}^*(\lambda)$ is now pinned down by $\widetilde{\Pi}(\widehat{n}^*(\lambda),\lambda) - \lambda\widetilde{\Xi}(\widehat{n}^*(\lambda),\lambda) = f$.

Provided $\widetilde{P}(Q) \geq \underline{p}$ or equivalently $P(Q) \geq \underline{p}$,[78] total surplus now includes the maverick

[77]The inverse demand $\widetilde{P}$ for higher Q does not play a role since the commonly-owned firms will never produce more than $(a-c)/b$. To derive $\widetilde{P}$, solve for it in $(a - \widetilde{P}(Q))/b = Q + (\widetilde{P}(Q) - \underline{p})/b_m$.

[78]Otherwise, wherever $\widetilde{P}(Q)$ substitute $\underline{p}$, and the equation reduces to $\widetilde{\text{TS}}(\boldsymbol{q}) = \text{TS}(\boldsymbol{q})$.

firms' surplus and is thus given by

$$\begin{aligned}\widetilde{\mathrm{TS}}(\boldsymbol{q}) &:= \overbrace{\int_0^{Q+S(\widetilde{P}(Q))} \left(P(X) - \widetilde{P}(Q)\right) dX}^{\text{consumer surplus}} + \overbrace{\int_{\underline{p}}^{\widetilde{P}(Q)} S(p)dp}^{\text{maverick firms' surplus}} + \overbrace{\widetilde{P}(Q)Q - \sum_{i=1}^{n} C(q_i) - nf}^{\text{commonly-owned firms' profits}} \\ &= \mathrm{TS}(\boldsymbol{q}) + \underbrace{\int_Q^{Q+S(\widetilde{P}(Q))} P(X)dX - S\left(\widetilde{P}(Q)\right)\widetilde{P}(Q)}_{\geq 0;\text{ consumer surplus “due to” maverick firms' production}} + \underbrace{\int_{\underline{p}}^{\widetilde{P}(Q)} S(p)dp}_{\geq 0;\text{ maverick firms' surplus}}\end{aligned}$$

where $\boldsymbol{q}$ still the quantity profile of the oligopolists and $\mathrm{TS}(\boldsymbol{q}) \equiv \int_0^Q P(X)dX - \sum_{i=1}^n C(q_i) - nf$ the total surplus without maverick firms. For any fixed quantity profile of the oligopolists, total surplus is higher when the maverick firms are present (and produce) compared to when they are not. We have that

$$\begin{aligned}\frac{d\widetilde{\mathrm{TS}}_n}{dn} =& P(Q_n)\left(n\frac{\partial q_n}{\partial n} + q_n\right) - C(q_n) - nC'(q_n)\frac{\partial q_n}{\partial n} - f \\ &+ \begin{bmatrix} \left(1 + S'(\widetilde{P}(Q_n))\widetilde{P}'(Q_n)\right) P\left(Q_n + S(\widetilde{P}(Q_n))\right) - P(Q_n) \\ -S'(\widetilde{P}(Q_n))\widetilde{P}'(Q_n)\widetilde{P}(Q_n) - S(\widetilde{P}(Q_n))\widetilde{P}'(Q_n) + S(\widetilde{P}(Q_n))\widetilde{P}'(Q_n) \end{bmatrix} \frac{\partial Q_n}{\partial n} \\ =& \widetilde{\Pi}(n,\lambda) - f - (1 + \lambda(n-1))\, Q_n \widetilde{P}'(Q_n)\frac{\partial q_n}{\partial n},\end{aligned}$$

where $\widetilde{\mathrm{TS}}_n$ is the pricing stage equilibrium total surplus when n commonly-owned firms enter, $\widetilde{\Pi}(n,\lambda) := \widetilde{P}(Q_n)q_n - C(q_n)$, and Q_n, q_n are still the quantities produced by the commonly-owned firms in the pricing stage equilibrium where n of them enter.

Whether there is excessive or insufficient entry by commonly-owned firms will depend on the same forces identified in the previous section but with adjusted magnitude since P is replaced by $\widetilde{P}$. Notice that excessive or insufficient entry is based on a planner that controls the entry of oligopolists and allows them and the maverick firms to produce freely. Importantly, given the production decisions of the oligopolists, the maverick firms' production level maximizes total surplus since the maverick firms are perfect competitors.

B.10 Free entry under post-entry overlapping ownership

In the last section overlapping ownership develops before entry, thus directly affecting the incentives of firms to enter. In this section we study the case where potential entrants have no prior overlapping ownership, but after they enter the market and before they pick quantities in the second stage they develop overlapping ownership, so that they have an Edgeworth coefficient of effective sympathy $\lambda \in [0,1]$. Now, the only channel through which overlapping ownership affects entry is by increasing profits in the post-entry game. Firms expect this and therefore entry increases with overlapping ownership.

This can be interpreted as a long-run equilibrium whereby start-up firms (or already existing firms but without overlapping ownership) enter the industry and then develop overlapping ownership through time. Appendix B.1 describes explicitly how post-entry overlapping ownership can arise. Also, given that the extent to which overlapping ownership affects corporate conduct is an open empirical question, this section can also be interpreted as studying pre-entry overlapping ownership when it affects pricing but does *not* cause firms to internalize their entry externality.

The exogeneity of λ is important with post-entry overlapping ownership, since the incentives of firms to allow for ownership ties after entry are not modeled. For instance, if the number of shares that investors buy from the entrepreneurs depended on the extent of entry—since the latter affects profits, then λ would be a function of n. Although the exogeneity of λ is restrictive, if firms become publicly traded after entry (at least in the long-run), they indeed have limited control over their ownership ties, since for instance investment funds are free to buy shares of all firms.

B.10.1 The entry stage

Each firm only looks at its own profit to decide whether to enter as there is no overlapping ownership when it does so.[79] $\boldsymbol{q}_n$ is a free entry equilibrium production profile if and only if

$$\Pi(n,\lambda) \geq f > \Pi(n+1,\lambda)$$

as in Mankiw and Whinston (1986). If overlapping ownership develops only after firms enter, it affects the incentives of firms to enter only through its effect on product market outcomes. We assume that there exists n such that $\Pi(n,\lambda) < f$ for any λ.

B.10.2 Existence and uniqueness of equilibrium

Proposition 10 studies the existence and uniqueness of a free entry equilibrium.

Proposition 10. $\Pi(n,\lambda)$ is decreasing in n and a unique free entry equilibrium exists.

[79]Formally, if a firm does not enter, its payoff is 0; if it does, it is $(1+\lambda(n-1))(\Pi(n,\lambda)-f)$. Thus, it is optimal for an n-th firm to enter if and only if $\Pi(n,\lambda) \geq f$.

In equilibrium, firms enter until profits have fallen so much that if an additional firm enters, gross profit will no longer cover the entry cost. $\widehat{n}^*(\lambda)$ is uniquely pinned down by $\Pi(\widehat{n}^*(\lambda),\lambda) = f$ and $n^*(\lambda) = \max\{n \in \mathbb{N} : \Pi(n,\lambda) \geq f\} = \lfloor \widehat{n}^*(\lambda) \rfloor$.

B.10.3 Overlapping ownership effects

Proposition 11 studies the effects of overlapping ownership.

Proposition 11. Ignore the integer constraint on n (so that entry is given by $\widehat{n}^*(\lambda)$). Then

(i) the number of firms entering is increasing in λ,

(ii) individual quantity, total quantity, and total surplus are decreasing in λ,

(iii) if $C'' \geq 0$, then the MHHI is increasing in λ.

Remark B.5. There exists a set of thresholds $\mathcal{L} \coloneqq \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$, $\lambda_1 < \lambda_2 < \cdots < \lambda_k$, such that

(a) for every $\lambda \in \mathcal{L}$, $\Pi(n^*(\lambda),\lambda) = f$, and $n^*(\lambda) = \widehat{n}^*(\lambda)$,

(b) for λ between two consecutive thresholds $n^*(\lambda)$ remains constant and everything behaves as in the Cournot game with a fixed number of firms.

When we take into account the integer constraint, the number of firms is a step function of λ, and individual quantity decreases with jumps down. The total quantity has a decreasing trend with jumps up (resp. down) for the values of λ at which an extra firm enters under $\Delta > 0$ (resp. $\Delta < 0$). Also, total surplus tends to decrease with λ.[80]

Importantly, even when there is free entry of firms—so that increases in λ lead to the entry of new firms as incumbents suppress their quantities, if the entering firms develop overlapping ownership after entering (up to the level the incumbents have), consumer and total surplus tend to decrease with λ, as in the symmetric case with a fixed number of

[80]To compare total surplus under the integer constraint on n, $TS_{n^*(\lambda)}$, to its value when we ignore the integer constraint, $TS_{\widehat{n}^*(\lambda)}$, notice the following. For λ between two consecutive thresholds, $\lambda \in (\lambda_k,\lambda_{k+1})$, it holds that $\widehat{n}^*(\lambda) > n^*(\lambda)$. Thus, given that total surplus is single-peaked in n, if there is (weakly) excessive entry under the integer constraint, ignoring the integer constraint exacerbates excess entry. Therefore, between two λ thresholds $TS_{\widehat{n}^*(\lambda)} < TS_{n^*(\lambda)}$, and for λ equal to a thresholds $TS_{n^*(\lambda)}$ has a jump down. But if under the integer constraint entry is insufficient by 1 firm (which is possible), $n^*(\lambda) = n^o(\lambda) - 1$, then the above does not follow.

firms. Also, if one looks at HHI, it will seem as if competition rises as λ increases, which can even be the case with MHHI, although the latter will increase with λ if we slightly strengthen our assumptions. Last, for appropriate levels of λ a small increase in λ can spur the entry of an extra firm causing the total quantity to rise.

The fact that the price increases with λ is to be expected. Remember that an increase in λ is met with an increase in n so that the zero profit condition $\Pi(\widehat{n}^*(\lambda),\lambda) = f$ is satisfied. When the Cournot market is quasi-anticompetitive ($\Delta < 0$), both the increase in λ and the increase in n cause the price to increase. When the Cournot market is quasi-competitive ($\Delta > 0$), the increase in λ tends to increase the price, while the increase in n tends to decrease it. The former effect dominates. For example, assume non-increasing MC and by contradiction that after an increase in λ enough additional firms enter the market to keep the price at its level before the increase in λ (or even make it lower). Then, after the increase in λ (i) each firm has a lower share of the market, (ii) the price has not increased, and (iii) the average (variable) cost of production has not decreased (due to non-increasing MC and individual quantity has decreased). Thus, individual profit has decreased, violating the zero profit condition. The result still holds under increasing MC, since under $\Delta > 0$,

$$\left|\frac{\overbrace{\partial\Pi(n,\lambda)/\partial\lambda}^{+}}{\underbrace{\partial\Pi(n,\lambda)/\partial n}_{-}}\right| = \left|\frac{\overbrace{(1-H_n)\frac{\partial Q_n}{\partial\lambda}}^{-}}{\underbrace{(1-H_n)\frac{\partial Q_n}{\partial n}}_{+}+\underbrace{H_n\frac{Q_n}{n}}_{+}}\right| < \left|\frac{\partial Q_n/\partial\lambda}{\partial Q_n/\partial n}\right| = \left|\frac{\overbrace{dP(Q_n)/d\lambda}^{+}}{\underbrace{dP(Q_n)/dn}_{-}}\right|.$$

This means that for individual profit to stay unchanged after an increase in λ, fewer firms need to enter compared to the number of firms that need to enter for the price to remain unchanged after the increase in λ.

The mechanism behind the effect of λ on entry is akin to the impact of collusion on entry in the dynamic stochastic oligopoly model of Fershtman and Pakes (2000), where firms freely enter, set prices, and invest in quality. In their model, for example, a potential entrant only looks at its profit to decide whether to enter foreseeing the possibility of future collusion with an incumbent monopolist. This possibility increases entry incentives (*i.e.* it increases the threshold of quality that the incumbent needs to achieve to deter entry) compared to the equilibrium without collusion. This in turn causes the incumbent monopolist to invest more in quality when future collusion is possible. Overall, the collusive equilibrium features on average higher prices but also more entry and higher qualities and consumer surplus.

B.10.4 Entry cost effect on entry

Proposition 12 studies the effect of the entry cost on entry, as well as how this effect depends on the extent of overlapping ownership. It mirrors Proposition 6 with the role of internalized profit $\Psi(n,\lambda)$ now assumed by profit $\Pi(n,\lambda)$.

Proposition 12. Ignore the integer constraint on n (so that entry is given by $\widehat{n}^*(\lambda)$). Then

(i) entry is decreasing in the entry cost,

(ii) if λ increases and other parameters x (e.g., demand, cost parameters) change infinitesimally so that $\widehat{n}^*(\lambda)$ stays fixed and $\partial^2\Pi(n,\lambda)/(\partial x \partial n) = 0$ (e.g., (f,λ) infinitesimally changes in direction $\mathbf{v} \coloneqq (-(d\widehat{n}^*(\lambda)/d\lambda)/(d\widehat{n}^*(\lambda)/df),1))$, then $|d\widehat{n}^*(\lambda)/df|$ changes in direction given by $\operatorname{sgn}\left\{ \partial^2\Pi\left(n,\lambda\right)/(\partial\lambda\partial n)|_{n=\widehat{n}^*(\lambda)}\right\}$.

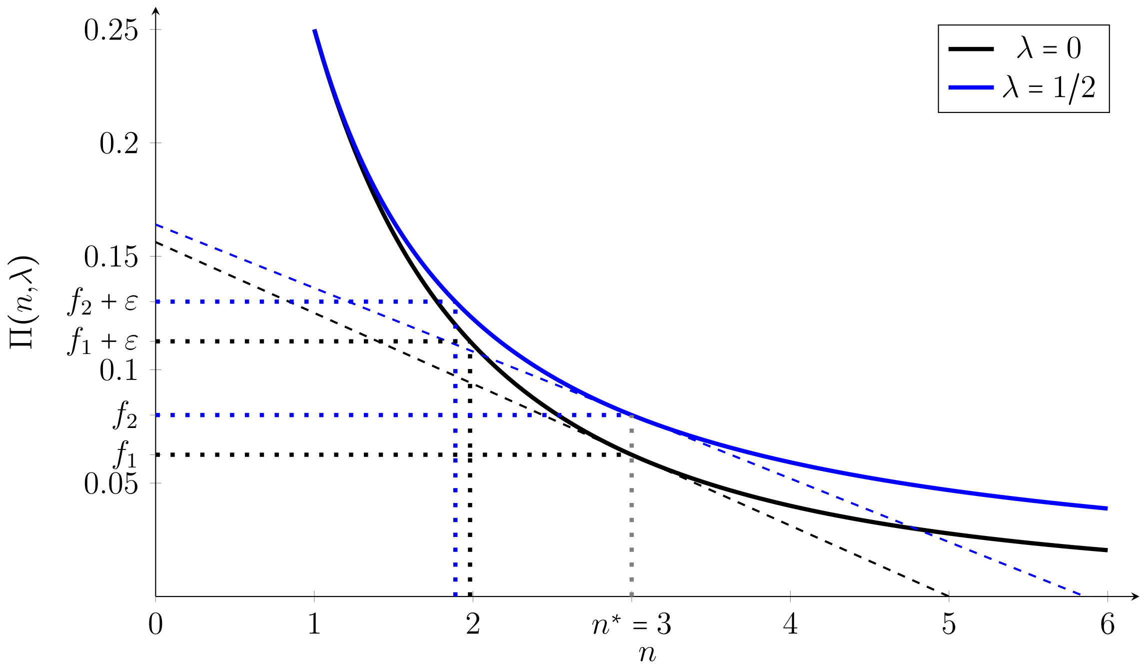
As long as individual profit is decreasing in n, the results of Proposition 12 are not specific to Cournot competition. Part (ii) says that if an increase in λ makes individual profit in the pricing stage equilibrium more (resp. less) strongly decreasing in the number of firms, then an increase in the entry cost needs to be met with a smaller (resp. larger) increase in the number of firms for the zero profit entry condition to continue to hold.

Figure 11 explains the reasoning behind this result. There are initially $n^* = 3$ firms in equilibrium, which can be a result of $\lambda = 0$ and $f = f_1$, or $\lambda = 1/2$ and $f = f_2 > f_1$. Also, for $n \leq 3$, an increase of λ from 0 to 1/2 makes profit less strongly decreasing in n (*i.e.*, $\partial^2\Pi\left(n,\lambda\right)/(\partial\lambda\partial n) > 0$). Thus, an increase in the entry cost by ε will decrease entry by more when $\lambda = 1/2$ (and initially $f = f_2$) compared to when $\lambda = 0$ (and initially $f = f_1$).

Claim 4 provides sufficient conditions for the cross derivative of $\Pi(n,\lambda)$ to be negative (resp. positive), which by Proposition 12 implies that overlapping ownership alleviates (resp. exacerbates) the negative effect of the entry cost on entry.

Claim 4. Assume constant MC.

Figure 11: Entry cost effect on entry mediated by λ under linear demand and constant MC



Note: $a = 2$, $b = 1$, $c = 1$. The black and blue solid lines represent $\Pi(n,0)$ and $\Pi(n,1/2)$, respectively. The black and blue dashed lines are tangent to the corresponding solid lines at $n = n^*$.

(i) If $\partial E_{P'}(Q)/\partial Q \geq 0$, $E_{P'}(Q_n) \in [0,1]$ and $n \geq 5+E_{P'}(Q_n)$, then $\partial^2\Pi(n,\lambda)/(\partial\lambda\partial n) < 0$ for every $\lambda \in (0,1)$.

(ii) If $\partial E_{P'}(Q)/\partial Q \leq 0$, $E_{P'}(Q_n) \leq 0$ and $n \leq 6/(2 - E_{P'}(Q_n))$, then $\partial^2\Pi(n,\lambda)/(\partial\lambda\partial n) > 0$ for every $\lambda \in (0,1)$.

Claim 4 encompasses CESL demand. Therefore, under CESL demand with $E \in [0,1]$ and constant MC, in markets with not too low entry ($n \geq 6$ is sufficient), overlapping ownership makes entry less strongly decreasing in the entry cost. This means that as long as it does not induce firms to internalize the entry externality, overlapping ownership could alleviate the negative macroeconomic implications of rising entry costs documented by Gutiérrez et al. (2021) in the U.S. over the past 20 years. The sufficient condition of part (ii) requires $n \leq 3$, as is the case in Figure 11.

The conditions in part (i) of Claim 4 overlap with those of Numerical result 3, which deals with the case of pre-entry overlapping ownership. Thus, under the same parameterization, whether overlapping ownership exacerbates or alleviates the negative effect of the entry cost on entry will depend on the form of overlapping ownership. If overlapping ownership is present prior to entry thus making firms internalize the entry externality, then it exacerbates the effect. If it develops after entry, it alleviates the effect.

B.10.5 Equilibrium entry versus the socially optimal level of entry

The derivative of equilibrium total surplus with respect to n is given by

$$\begin{aligned}\frac{d\,\mathrm{TS}_n}{dn} &= P(Q_n)\left(n\frac{\partial q_n}{\partial n} + q_n\right) - C(q_n) - nC'(q_n)\frac{\partial q_n}{\partial n} - f \\ &= \Pi(n,\lambda) - f + n\left(P(Q_n) - C'(q_n)\right)\frac{\partial q_n}{\partial n},\end{aligned}$$

and therefore

$$\left.\frac{d\,\mathrm{TS}_n}{dn}\right|_{n=\widehat{n}^*(\lambda)} = \overbrace{\Pi\left(\widehat{n}^*(\lambda),\lambda\right) - f}^{=0} + n\left(P(Q_n) - C'(q_n)\right)\left.\frac{\partial q_n}{\partial n}\right|_{n=\widehat{n}^*(\lambda)} \propto \left.\frac{\partial q_n}{\partial n}\right|_{n=\widehat{n}^*(\lambda)},$$

so that with TS_n single-peaked in n, under business-stealing (resp. business-enhancing) competition entry is excessive (resp. insufficient). The results of Mankiw and Whinston (1986) and Amir et al. (2014) generalize to the case of post-entry overlapping ownership. Proposition 13 shows that indeed with business-stealing competition and under the integer constraint, entry is never insufficient by more than one firm.

Proposition 13. The following statements hold:

(i) if $\Delta > 0$ and $E_{P'}(Q) < 2$ on L, then $n^*(\lambda) \geq n^o(\lambda) - 1$,

(ii) if $\Delta < 0$, then $n^*(\lambda) \geq n^o(\lambda) = 1$.

Remark B.6. Under a consumer surplus standard

(i) if $\Delta > 0$, then $n^o(\lambda) = \infty$ (since Q_n is increasing in n), so $n^*(\lambda) < n^o(\lambda)$,

(ii) if $\Delta < 0$, then $n^o(\lambda) = 1$ (since Q_n is decreasing in n), so $n^*(\lambda) \geq n^o(\lambda)$.

Under a consumer surplus standard, entry is insufficient (resp. excessive) when returns to scale are at most mildly increasing (resp. sufficiently increasing).

B.11 Results with (possible) multiplicity of equilibria

This section provides results with the maintained assumption $\Delta > 0$ on L but dropping the assumption that $E_P' < (1 + \lambda + \Delta/n)/H_n$ on L. The second-order condition (SOC) of the firm's problem, that is $E_{P'} < (1 + \lambda + \Delta)/H_n$, will still be assumed to hold strictly in any symmetric pricing stage equilibrium. Then, the Cournot game equilibrium set may consist of multiple symmetric equilibria. Propositions under this relaxed version of the maintained assumption will be marked with an apostrophe (').

B.11.1 Pricing stage equilibrium

Proposition 2' studies the comparative statics of pricing stage equilibria.

Proposition 2'. Let $\Delta > 0$ on L. Then, at extremal equilibria:[81]

(i) total and individual quantity, and total surplus (resp. individual profit) are non-increasing (resp. non-decreasing) in λ,

(ii) individual profit is non-increasing in n,

(iii) total quantity is non-decreasing in n.

Under $\Delta > 0$, when we drop the condition $E_{P'} < (1 + \lambda + \Delta/n)/H_n$ on L guaranteeing uniqueness, the results of Proposition 2 still hold weakly for extremal equilibria. They also hold strictly but only locally around stable equilibria.[82] As observed in AL, a discrete change (e.g., in the integer number n of firms) may even lead to a change in the number of equilibria rendering it hard to make meaningful comparisons between non-extremal equilibra.

B.11.2 Free entry under post-entry overlapping ownership

Proposition 10' studies the existence of a free entry equilibrium.

Proposition 10'. Let $\Delta > 0$ on L. Then, at extremal equilibria profit is non-increasing in n and a free entry equilibrium where in the pricing stage firms play an extremal equilibrium exists.

If for example there is a multiplicity of pricing stage equilibria for every n, there will exist at least two free entry equilibria: one where the minimum pricing stage equilibrium is played and one where the maximum pricing stage equilibrium is played.[83]

[81] By extremal equilibria we mean the equilibrium with minimum quantity among all equilibria and the equilibrium with maximum quantity among all equilibria.

[82] Namely, parts (i)-(iv) of Proposition 2 hold locally in any stable equilibrium (with n treated as a continuous variable in parts (ii)-(iv)).

[83] Extremal equilibria correspond to extremal equilibrium profits. Namely, the minimum (resp. maximum) equilibrium quantity corresponds to the maximum (resp. minimum) equilibrium profit.

Proposition 13' compares equilibrium entry to the socially optimal level of entry considering also the case of business-enhancing competition. To economize on notation, we are still using q_n, $n^*(\lambda)$ and $n^o(\lambda)$ to denote equilibrium values in a specific extremal equilibrium even though multiple equilibria may exist.

Proposition 13'. Let $\Delta > 0$ on L. Let the same type of extremal equilibrium (*i.e.*, minimum or maximum) be played in the pricing stage of the free entry equilibrium and the planner's solution. Then,

(i) if $q_{n^o(\lambda)-1} \geq q_{n^o(\lambda)}$, then $n^*(\lambda) \geq n^o(\lambda) - 1$.

(ii) if $q_{n^o(\lambda)+1} \geq q_{n^o(\lambda)}$, then $n^*(\lambda) \leq n^o(\lambda)$.

Remark B.7. Proposition 13' and part (ii) of Proposition 13 extend the results of Amir et al. (2014) to the case of post-entry overlapping ownership.

Under $\Delta > 0$, when competition is locally business-stealing, equilibrium entry is not insufficient by more than one firm as in the case without overlapping ownership. On the other hand, if competition is locally business-enhancing, entry is not excessive.

B.12 Free entry with pre-entry overlapping ownership: a more tractable framework

In this section, we make the free entry model with pre-entry overlapping ownership more tractable by ignoring the integer constraint on n. The way we do this is *not* just by letting (2) hold with equality. Instead, now each "infinitesimal" firm considers whether to enter or not examining a differential version of (3).[84] Consider firm i of "size" $\varepsilon > 0$ and let $n \in \mathbb{R}_+$ be the number of other firms entering. Firm i's payoff if it enters is $(\varepsilon + \lambda n)(\Pi(n+\varepsilon,\lambda) - f)$, while if it does not, it is $\lambda n(\Pi(n,\lambda) - f)$. The difference is

$$\varepsilon \Pi(n+\varepsilon,\lambda) + \lambda n\left[\Pi(n+\varepsilon,\lambda) - \Pi(n,\lambda)\right] - \varepsilon f.$$

[84] Of course, the firm is infinitesimal only for the purpose of the algebra. The firm understands the (marginal) effect of its entry on market outcomes, and in the pricing stage firms still compete à la Cournot but with the symmetric equilibrium solution extended to $n \in \mathbb{R}_{++}$.

Notice that for $\varepsilon = 1$ we recover the case with an integer number of firms. Dividing this expression by ε and letting $\varepsilon \to 0$ gives

$$\Pi(n,\lambda) + \lambda n \frac{\partial \Pi(n,\lambda)}{\partial n} - f.$$

Therefore, $\boldsymbol{q}_n$ is a free entry equilibrium if

$$\underbrace{\Pi(n,\lambda)}_{\substack{\text{own profit} \\ \text{from entry}}} + \lambda \underbrace{n \frac{\partial \Pi(n,\lambda)}{\partial n}}_{\substack{\text{entry externality} \\ \text{on other firms}}} = \underbrace{f}_{\substack{\text{entry} \\ \text{cost}}} \quad \text{and} \tag{6}$$

$$(1+\lambda)\frac{\partial \Pi(n,\lambda)}{\partial n} + \lambda n \frac{\partial^2 \Pi(n,\lambda)}{(\partial n)^2} < 0. \tag{7}$$

Naturally, we only consider the free entry equilibrium and planner's solution with $n \in \mathbb{R}_+$; we denote the number of firms in the two solutions by $n^*(\lambda)$ and $n^o(\lambda)$, respectively. The entry externality is now measured by $n\partial\Pi(n,\lambda)/\partial n$. (6) says that the marginal firm entering is exactly indifferent between entering or not. (7) guarantees that an extra infinitesimal firm does not want to enter, and given that $\partial\Pi(n,\lambda)/\partial n < 0$, can equivalently be written as

$$1 + \lambda - \lambda E_{\partial\Pi/\partial n, n}(n,\lambda) > 0, \qquad \text{where} \quad E_{\partial\Pi/\partial n, n}(n,\lambda) := -\frac{\frac{\partial^2 \Pi(n,\lambda)}{(\partial n)^2}}{\frac{\partial \Pi(n,\lambda)}{\partial n}} n$$

is the elasticity of the slope of individual profit with respect to n. Also, given that $\partial\Pi(n,\lambda)/\partial n < 0$, $\lambda > 0$ implies through (6) that the entering firms make positive net profits in equilibrium. For $\lambda = 0$, (6) reduces to the standard zero profit condition.

Provided that (7) holds for every n, the (unique) equilibrium level of entry $n^*(\lambda)$ is pinned down by

$$\Pi\left(n^*(\lambda),\lambda\right) + \lambda n^*(\lambda) \left.\frac{\partial \Pi(n,\lambda)}{\partial n}\right|_{n=n^*(\lambda)} = f.$$

Assume that $\Pi(1,\lambda) + \lambda\partial\Pi(n,\lambda)/\partial n\big|_{n=1} > f$ so that more than 1 firm enters, and

$$\lim_{n\to\infty} \left[\Pi(n,\lambda) + \lambda\partial\Pi(n,\lambda)/\partial n\right] < f.$$

Proposition 14 guarantees that the left-hand side of (6) is decreasing in n, thus ensuring the existence of a unique equilibrium.

Proposition 14. If for every n such that $\Pi(n,\lambda) + \lambda\partial\Pi(n,\lambda)/\partial n \geq f$ it holds that $1 + \lambda - \lambda E_{\partial\Pi/\partial n,n}(n,\lambda) > 0$, then a unique Cournot equilibrium with free entry exists.

Proposition 15. Fix a value for λ and consider the unique symmetric Cournot equilibrium with free entry, where $\partial\Pi(n,\lambda)/\partial n|_{n=n^*(\lambda)} < 0$ and $1 + \lambda - \lambda E_{\partial\Pi/\partial n,n}(n^*(\lambda),\lambda) > 0$.

(i) The number of firms locally changes with λ with direction given by[85]

$$\operatorname{sgn}\left\{\frac{dn^*(\lambda)}{d\lambda}\right\} = \operatorname{sgn}\left\{ \overbrace{E_{\partial\Pi/\partial n,\lambda}(n^*(\lambda),\lambda)}^{\substack{\text{change in magnitude}\\ \text{of entry externality}}} + \overbrace{\frac{1}{\lambda}\frac{E_{\Pi,\lambda}(n^*(\lambda),\lambda)}{E_{\Pi,n}(n^*(\lambda),\lambda)}}^{\substack{\text{increase in own}\\ \text{profit from entry}}} - \overbrace{1}^{\substack{\text{increase in internalization}\\ \text{of entry externality}}} \right\}.$$

(ii) The total quantity changes with λ with direction given by

$$\operatorname{sgn}\left\{\frac{dQ_{n^*(\lambda)}}{d\lambda}\right\} = \operatorname{sgn}\left\{\frac{E_{\partial\Pi/\partial n,\lambda}(n^*(\lambda),\lambda) + \frac{1}{\lambda}\frac{E_{\Pi,\lambda}(n^*(\lambda),\lambda)}{E_{\Pi,n}(n^*(\lambda),\lambda)} - 1}{1 + \lambda - \lambda E_{\partial\Pi/\partial n,n}(n^*(\lambda),\lambda)} \frac{\Delta(Q_{n^*(\lambda)},(n-1)\,q_{n^*(\lambda)})}{n^*(\lambda) - 1} - 1\right\}$$

where

$$E_{\partial\Pi/\partial n,\lambda}(n,\lambda) := -\frac{\frac{\partial^2\Pi(n,\lambda)}{\partial\lambda\partial n}}{\frac{\partial\Pi(n,\lambda)}{\partial n}}\lambda, \quad E_{\Pi,n}(n,\lambda) := -\frac{\frac{\partial\Pi(n,\lambda)}{\partial n}}{\Pi(n,\lambda)}n > 0, \quad E_{\Pi,\lambda}(n,\lambda) := \frac{\frac{\partial\Pi(n,\lambda)}{\partial\lambda}}{\Pi(n,\lambda)}\lambda > 0$$

are, respectively, the elasticity with respect to λ of the slope of individual profit with respect to n, the elasticity of profit with respect to n, and the elasticity of profit with respect to λ.

Corollary 15.1. In addition to the assumptions of Proposition 15, assume constant returns to scale. Then

$$\operatorname{sgn}\left\{\frac{dn^*(\lambda)}{d\lambda}\right\} = \operatorname{sgn}\left\{\begin{array}{l} \left(n - 1 + 2\lambda - \dfrac{\Lambda_n(2n - \Lambda_n E_{P'}(Q_n))}{n - \Lambda_n}\right)(n + \Lambda_n - \Lambda_n E_{P'}(Q_n)) \\ \quad +\lambda(2n - \Lambda_n)(2 - E_{P'}(Q_n)) - \dfrac{\lambda\Lambda_n(n - \Lambda_n)Q_n E'_{P'}(Q_n)}{n + \Lambda_n - \Lambda_n E_{P'}(Q_n)} \end{array}\right|_{n=n^*(\lambda)}\right\},$$

(i) for $\lambda = 0$, given $E_{P'}(Q_{n^*(0)}) < 2$, $dn^*(\lambda)/d\lambda \overset{(\text{resp. }>)}{<} 0$ if and only if $n^*(0) \overset{(\text{resp. }>)}{<} 2 + \sqrt{3 - E_{P'}(Q_{n^*(0)})}$.

[85]For $\lambda = 0$ cancel the λ in the second term with the one in $E_{\Pi,\lambda}(n,\lambda)$.

(ii) If $E'_{P'}(Q_{n^*(\lambda)}) \leq 0$ and $E_{P'}\left(Q_{n^*(\lambda)}\right) > \left[2n - \left(H_n^{-1} - 1\right)(n - 1 + 2\lambda)\right]/\Lambda_n$, then $dn^*(\lambda)/d\lambda > 0$.

(iii) If $E_{P'}\left(Q_{n^*(\lambda)}\right) < \max\left\{2, \left[2n - \left(H_n^{-1} - 1\right)(n + 1 + 2\lambda)\right]/\Lambda_n\right\}$ and $E'_{P'}(Q_{n^*(\lambda)}) \geq 0$, then $dn^*(\lambda)/d\lambda < 0$.

(iv) If $\lim_{\lambda \to 1^-} E_{P'}(Q_{n^*(\lambda)}) < 2$ (and $E'_{P'}$ bounded), then $dn^*(\lambda)/d\lambda < 0$ for λ close to 1.

(v) Under linear demand

$$\operatorname{sgn}\left\{\frac{dn^*(\lambda)}{d\lambda}\right\} = \operatorname{sgn}\left\{\left(n - 1 + 2\lambda - \frac{2n\Lambda_n}{n - \Lambda_n}\right)(n + \Lambda_n) + 2\lambda(2n - \Lambda_n)\bigg|_{n=n^*(\lambda)}\right\}.$$

(vi) If $E_{\partial\Pi/\partial n,n}(n^*(\lambda),\lambda) \leq 2$, $E_{P'}(Q_{n^*(\lambda)}) < 2$, $E'_{P'}(Q_{n^*(\lambda)}) \geq 0$ and $n^*(\lambda) \geq 2$, then the total quantity decreases with λ.

Claim 5. Under linear demand and constant marginal costs $E_{\partial\Pi/\partial n,n}(n,\lambda) \leq 2$ for every $\lambda \in [0,1]$ and $n \geq 1$.

Corollary 15.1 shows that under reasonable assumptions overlapping ownership can spur entry. Proposition 16 shows that the effect of overlapping ownership on the magnitude of the entry externality is ambiguous in our setting. Proposition 17 shows that with pre-entry overlapping ownership both possibilities of excessive and insufficient entry are possible.

Proposition 16. Assume that $\Delta > 0$ and $E_{P'}(Q_n) < 1 + \frac{\Lambda_n}{n-\Lambda_n}\Big/\left(E_{Q_n,n}(n,\lambda) + \frac{\Lambda_n}{n-\Lambda_n}\right)$ for $n = n^*(\lambda)$. The direction of the change (due to the change in λ) in the magnitude of the entry externality, $\operatorname{sgn}\left\{E_{\partial\Pi/\partial n,\lambda}(n,\lambda)\right\}$, is given by

$$\operatorname{sgn}\left\{\overbrace{\frac{E_{Q_n,\lambda}(n,\lambda)}{E_{Q_n,n}(n,\lambda)}}^{+} - \frac{\overbrace{E_{\partial Q_n/\partial n,\lambda}(n,\lambda)}^{-\text{ under Prop. 2(ii)}} + \overbrace{\frac{n-1}{n-\Lambda_n}\left[\left(E_{Q_n,n}(n,\lambda)\right)^{-1} - 1\right]}^{+}}{\left[E_{Q_n,n}(n,\lambda) + \frac{\Lambda_n}{n-\Lambda_n}\right]\left[1 + \frac{\frac{\Lambda_n}{n-\Lambda_n}}{E_{Q_n,n}(n,\lambda) + \frac{\Lambda_n}{n-\Lambda_n}} - E_{P'}(Q_n)\right]}\Bigg|_{n=n^*(\lambda)}\right\}$$

evaluated at $n = n^*(\lambda)$, where

$$E_{\partial Q_n/\partial n,\lambda}(n,\lambda) \coloneqq \frac{\frac{\partial^2 Q_n}{\partial\lambda\partial n}}{\frac{\partial Q_n}{\partial n}}\lambda, \quad E_{Q_n,n}(n,\lambda) \coloneqq \frac{\partial Q_n}{\partial n}\frac{n}{Q_n} > 0, \quad E_{Q_n,\lambda}(n,\lambda) \coloneqq -\frac{\partial Q_n}{\partial \lambda}\frac{\lambda}{Q_n} > 0.$$

Under constant marginal costs

$$\operatorname{sgn}\left\{E_{\partial\Pi/\partial n,\lambda}(n,\lambda)\right\} = \operatorname{sgn}\left\{\begin{array}{r} 2\Lambda_n^2\left(E_{P'}(Q_n)\right)^2 + \left[n\Lambda_n(n-\Lambda_n-1)-2n^2-\Lambda_n^2\right]E_{P'}(Q_n) \\ -n(n-\Lambda_n)\left(n+\Lambda_n-6\right) - \dfrac{\Lambda_n(n-\Lambda_n)^2 Q_n E'_{P'}(Q_n)}{n+\Lambda_n-\Lambda_n E_{P'}(Q_n)} \end{array}\right|_{n=n^*(\lambda)}\right\},$$

which can be negative or positive. For linear demand, $\operatorname{sgn}\left\{E_{\partial\Pi/\partial n,\lambda}(n,\lambda)\right\} = \operatorname{sgn}\{6-(n+\Lambda_n)\}$.

Proposition 17. Consider the Cournot model with free entry and pre-entry overlapping ownership. Assume that $\text{TS}(\boldsymbol{q}_n)$ is globally concave in n, and $\lambda < 1$. Then in equilibrium there is excessive (insufficient) entry if and only if

$$E_{P'}\left(Q_{n^*(\lambda)}\right) \overset{(\text{ resp. } >)}{<} H_n^{-1}\left\{1+\lambda\left(1-\frac{\Delta(Q_n,(n-1)\,q_n)}{(1-\lambda)\,(1+\lambda(n-1))}\right)\right\}\Bigg|_{n=n^*(\lambda)}.$$

The results of this section very closely resemble the ones we obtain under the integer constraint. Therefore, the gain in tractability from dropping the constraint as described above comes at a minimal cost.

C Proofs of additional results

Where clear we may simplify notation, for example omitting the subscript n,λ for equilibrium objects. We may also write for example Q_n instead of $Q_{n^*(\lambda)}$, n instead of $n^*(\lambda)$. Also, we write $\Pi_n(n,\lambda) \equiv \partial\Pi(n,\lambda)/\partial n$, $\Pi_\lambda(n,\lambda) \equiv \partial\Pi(n,\lambda)/\partial\lambda$, $\Pi_{n\lambda}(n,\lambda) \equiv \partial^2\Pi(n,\lambda)/(\partial n\partial\lambda)$, $\Pi_{nn}(n,\lambda) \equiv \partial^2\Pi(n,\lambda)/(\partial n)^2$.

C.1 Proof of section B.2

Proof of Claim 1 Under CESL demand and constant marginal costs the pricing formula $P(Q_n) - C'(q_n) = -H_n Q_n P'(Q_n)$ gives

$$\begin{aligned} a + b(Q_n)^{1-E} - c &= H_n b(E-1)(Q_n)^{1-E} && \text{if } E > 1 \\ a - b\ln Q_n - c &= H_n b && \text{if } E = 1 \\ a - b(Q_n)^{1-E} - c &= H_n b(1-E)(Q_n)^{1-E} && \text{if } E < 1 \end{aligned}$$

and the result follows. In the case $E > 1$, $E < 2$ and $c > a$ guarantee that there is an interior equilibrium. Notice that if $a > c$, then the profit per unit $P(Q) - AC(q) \geq a - c > 0$ is positive and bounded away from zero for every $Q \geq q \geq 0$, and thus there is no equilibrium. In the case $E < 1$, if $a \leq c$, then in the unique equilibrium $Q_n = 0$.

For linear demand and linear-quadratic costs the pricing formula $P(Q_n) - C'(q_n) = -H_n Q_n P'(Q_n)$ gives $a - bQ_n - c_1 - c_2(Q_n/n) = H_n b Q_n$, and the result follows. **Q.E.D.**

C.2 Proofs of section B.3

Proof of Propositions 7 and 7' For simplicity, we use the notation Q_n and q_n to refer to values in a specific equilibrium even if that equilibrium is not unique.

Let $\widetilde{\pi}_i(\boldsymbol{q}) \coloneqq \pi_i(\boldsymbol{q}) + \lambda \sum_{j \neq i} \pi_j(\boldsymbol{q})$. A linearization of the adjustment process around an equilibrium production profile $\boldsymbol{q}_n$ gives

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix} = \overbrace{\begin{bmatrix} k_1 \frac{\partial^2 \widetilde{\pi}_1(\boldsymbol{q})}{(\partial q_1)^2} & k_1 \frac{\partial^2 \widetilde{\pi}_1(\boldsymbol{q})}{\partial q_1 \partial q_2} & \cdots & k_1 \frac{\partial^2 \widetilde{\pi}_1(\boldsymbol{q})}{\partial q_1 \partial q_n} \\ k_2 \frac{\partial^2 \widetilde{\pi}_2(\boldsymbol{q})}{\partial q_2 \partial q_1} & k_2 \frac{\partial^2 \widetilde{\pi}_2(\boldsymbol{q})}{(\partial q_2)^2} & \cdots & k_2 \frac{\partial^2 \widetilde{\pi}_2(\boldsymbol{q})}{\partial q_2 \partial q_n} \\ \vdots & \vdots & \cdots & \vdots \\ k_n \frac{\partial^2 \widetilde{\pi}_n(\boldsymbol{q})}{\partial q_n \partial q_1} & k_n \frac{\partial^2 \widetilde{\pi}_n(\boldsymbol{q})}{\partial q_n \partial q_2} & \cdots & k_n \frac{\partial^2 \widetilde{\pi}_n(\boldsymbol{q})}{(\partial q_n)^2} \end{bmatrix}}^{=:A} \begin{bmatrix} q_1 - q_n \\ q_2 - q_n \\ \vdots \\ q_n - q_n \end{bmatrix},$$

where for $i \neq j$

$$\frac{\partial^2 \widetilde{\pi}_i(\boldsymbol{q})}{\partial q_i \partial q_j} = P'(Q)\left(1 + \lambda - E_{P'}(Q)\left((1-\lambda)s_i + \lambda\right)\right),$$
$$\frac{\partial^2 \widetilde{\pi}_i(\boldsymbol{q})}{(\partial q_i)^2} = P'(Q)\left(2 - \frac{C_i''(q_i)}{P'(Q)} - E_{P'}(Q)\left((1-\lambda)s_i + \lambda\right)\right) < 0$$

are evaluated at the equilibrium production profile $\boldsymbol{q}_n$. The second derivative with respect to q_i evaluated at $\boldsymbol{q}_n$ is negative given that $E_{P'}(Q_n) < (1 + \lambda + \Delta(Q_n,(n-1)q_n))/H_n$.

Notice that $\frac{\partial^2 \widetilde{\pi}_i(\boldsymbol{q})}{\partial q_i \partial q_j}$ does not depend on the identity of firm j as long as $i \neq j$, so that the off-diagonal elements in each row are equal. From Theorem 2(i) in al Nowaihi and Levine (1985)—which also follows from Hosomatsu (1969)—it follows that all eigenvalues of A are real.

Also, we have that $\partial^2 \widetilde{\pi}_i(\boldsymbol{q})/(\partial q_i \partial q_j)\big|_{\boldsymbol{q}=\boldsymbol{q}_n} = P'(Q_n)(1 + \lambda - H_n E_{P'}(Q_n))$. We distinguish two cases.

Case 1: If $E_{P'}(Q_n) \leq (1+\lambda)/H_n$, then that combined with $\Delta(Q_n,(n-1)q_n) > 0$ imply

$$\left.\frac{\partial^2 \widetilde{\pi}_i(q)}{(\partial q_i)^2}\right|_{\boldsymbol{q}=\boldsymbol{q}_n} < \left.\frac{\partial^2 \widetilde{\pi}_i(q)}{\partial q_i \partial q_j}\right|_{\boldsymbol{q}=\boldsymbol{q}_n} \leq 0,$$

for every $i \neq j$, and it follows from Theorem 2(ii-a) in al Nowaihi and Levine (1985)—also in Hosomatsu (1969)— that all eigenvalues of A are negative. From standard stability theory, we then have that the equilibrium is locally stable.

Case 2: For $E_{P'}(Q_n) > (1+\lambda)/H_n$ we get $\partial^2 \widetilde{\pi}_i(q)/(\partial q_i \partial q_j)|_{\boldsymbol{q}=\boldsymbol{q}_n} > 0$, and it follows from Theorem 2(ii-b) in al Nowaihi and Levine (1985) that all eigenvalues of A are negative it and only if

$$\sum_{i=1}^{n} \frac{\left.\frac{\partial^2 \widetilde{\pi}_i(q)}{\partial q_i \partial q_j}\right|_{\boldsymbol{q}=\boldsymbol{q}_n}}{\left.\frac{\partial^2 \widetilde{\pi}_i(q)}{\partial q_i \partial q_j}\right|_{\boldsymbol{q}=\boldsymbol{q}_n} - \left.\frac{\partial^2 \widetilde{\pi}_i(q)}{(\partial q_i)^2}\right|_{\boldsymbol{q}=\boldsymbol{q}_n}} < 1, \quad \text{or equivalently}$$

$$-n\frac{1+\lambda - H_n E_{P'}(Q_n)}{1-\lambda - \frac{C''(q_n)}{P'(Q_n)}} < 1 \Longleftrightarrow -[1+\lambda - H_n E_{P'}(Q_n)] < \Delta(Q_n,(n-1)q_n)/n,$$

Again the result follows from standard stability theory. **Q.E.D.**

C.3 Proofs of section B.4

Proof of Proposition 8 $\operatorname{sgn}\left\{\frac{d^2Q}{d\lambda dn}\right\}$ is equal to

$$= -\operatorname{sgn}\left\{\begin{array}{r}\left[Q+(n-1)\dfrac{dQ}{dn}\right][n+\Lambda - C''(Q/n)/P'(Q) - \Lambda E_{P'}(Q)] - (n-1)Q\times \\ \left[1+\lambda - \dfrac{C'''(Q/n)\left(\frac{dQ}{dn}-\frac{Q}{n}\right)\frac{P'(Q)}{n} - C''(Q/n)P''(Q)\frac{dQ}{dn}}{(P'(Q))^2} - \lambda E_{P'}(Q) - \Lambda E'_{P'}(Q)\dfrac{dQ}{dn}\right]\end{array}\right\}$$

$$= -\operatorname{sgn}\left\{\begin{array}{l}\dfrac{n}{n-1}[2-E_{P'}(Q)] + \dfrac{dQ}{dn}\dfrac{n}{Q}[n+\Lambda+\Lambda E_{P'}(Q)[E_{P'}(Q)+E_{P''}(Q)]] \\ +\left(\dfrac{dQ}{dn}\dfrac{n}{Q}-1\right)\dfrac{C'''(Q/n)Q/n}{P'(Q)} - \dfrac{C''(Q/n)}{P'(Q)}\left[\dfrac{n}{n-1}+\dfrac{dQ}{dn}\dfrac{n}{Q}(1-E_{P'}(Q))\right]\end{array}\right\}.$$

If also $C'', C''' \geq 0$ and $E_{P'}(Q)[E_{P'}(Q)+E_{P''}(Q)] \geq -2$, then $\operatorname{sgn}\{d^2Q/(d\lambda dn)\} = -$ given that $n \geq 2$ and $E_{P'}(Q) < 2$; the latter two imply $\frac{dQ}{dn}\frac{n}{Q} \in (0,1)$.

Also,

$$\frac{d^2q}{d\lambda dn} = \frac{d\left(\frac{dq}{d\lambda}\right)}{dn} = \frac{d\left[\left(\frac{dQ}{d\lambda}\right)\frac{1}{n}\right]}{dn} = \frac{1}{n}\left[\frac{d^2Q}{d\lambda dn} - \frac{1}{n}\frac{dQ}{d\lambda}\right]$$

$$\propto \frac{1}{n}\left\{\begin{array}{l} -\dfrac{Q(n-1)}{n}\left[\begin{array}{r}\left(\dfrac{dQ}{dn}\dfrac{n}{Q}-1\right)\dfrac{C'''(q)q}{P'(Q)} - \dfrac{C''(q)}{P'(Q)}\left[\dfrac{n}{n-1}+\dfrac{dQ}{dn}\dfrac{n}{Q}(1-E_{P'}(Q))\right] \\ +\dfrac{n}{n-1}[2-E_{P'}(Q)] + \dfrac{dQ}{dn}\dfrac{n}{Q}[n+\Lambda+\Lambda E_{P'}(Q)[E_{P'}(Q)+E_{P''}(Q)]]\end{array}\right] \\ \qquad - \dfrac{(n+\Lambda - C''(q)/P'(Q) - \Lambda E_{P'}(Q))^2}{n}\dfrac{dQ}{d\lambda}\end{array}\right\}$$

$$\propto -\left(\frac{dQ}{dn}\frac{n}{Q}-1\right)\frac{C'''(q)q}{P'(Q)}+\frac{C''(q)}{P'(Q)}\left[\frac{1}{n-1}+\frac{dQ}{dn}\frac{n}{Q}\left(1-E_{P'}(Q)\right)\right]-\frac{n}{n-1}\left[2-E_{P'}(Q)\right]$$
$$-\frac{dQ}{dn}\frac{n}{Q}\left[n+\Lambda+\Lambda E_{P'}(Q)\left[E_{P'}(Q)+E_{P''}(Q)\right]\right]+n+\Lambda-\Lambda E_{P'}(Q).$$

Under CESL demand, for $Q<\overline{Q}$ the elasticity $E_{P''}(Q)$ of the curvature is then given by

$$E_{P''}(Q)\equiv\frac{QP'''(Q)}{P''(Q)}=\begin{cases}\frac{Qb(E+1)E(1-E)Q^{-(E+2)}}{-bE(1-E)Q^{-(E+1)}}=-(E+1) & \text{if } E\neq 1\\ \frac{-Q2b/Q^3}{b/Q^2}=-2 & \text{if } E=1\end{cases},$$

so $E_{P''}(Q)=-(E+1)$. Thus, if marginal costs are linear and demand is CESL, we get

$$\frac{d^2q}{d\lambda dn}\propto\frac{n(n-3)}{n-1}+\Lambda-\left(\Lambda-\frac{n}{n-1}\right)E-\frac{1-\lambda}{n+\Lambda-\Lambda E}\left[n+\Lambda-\Lambda E\right]$$
$$\propto n-3+\lambda(n-1)-\left(\frac{\Lambda(n-1)}{n}-1\right)E=n+\Lambda-4-\left(\frac{\Lambda(n-1)}{n}-1\right)E.$$

Q.E.D.

Proof of Proposition 9 (i-ii) Given what we see in the proof of Proposition 10, for aggregate industry profits we have that

$$\frac{\partial\left[n\Pi\left(n,\lambda\right)\right]}{\partial n}=P(Q_n)\frac{Q_n}{n}-C(q_n)+nP'(Q_n)\left(\frac{Q_n}{n}\right)^2\left[\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\left(1-H_n\right)+H_n\right]$$
$$\propto-\left[\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\left(1-H_n\right)-\eta(Q_n)\frac{P(Q_n)-C'(q_n)+C'(q_n)\frac{E_C(q_n)-1}{E_C(q_n)}}{P(Q_n)}+H_n\right]$$
$$\overset{(1)}{=}-\left[\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\left(1-H_n\right)-\eta(Q_n)\left(1-\frac{H_n}{\eta(Q_n)}\right)\frac{E_C(q_n)-1}{E_C(q_n)}\right]$$
$$\propto E_C(q_n)\left(1-\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\frac{1-H_n}{\eta(Q_n)-H_n}\right)-1,$$

where $H_n<\eta(Q_n)$ from the pricing formula (1).

(iii) We have that

$$n\Pi(n,\lambda)\equiv P(Q_n)Q_n-nC(q_n)\overset{C''<0}{<}P(Q_n)Q_n-C(Q_n)\leq P(q_1)q_1-C(q_1)=\Pi(1,\lambda),$$

where the last inequality follows by the definition of q_1 being the monopolist's optimal quantity.

To see why Remark B.3 holds notice that for $\lambda = 1$

$$\begin{aligned}\frac{\partial\left[n\Pi\left(n,\lambda\right)\right]}{\partial n} &= P(Q_n)\frac{Q_n}{n} - C(q_n) + nP'(Q_n)\left(\frac{Q_n}{n}\right)^2 \\ &\overset{C''>0}{>} P(Q_n)\frac{Q_n}{n} - C'(q_n)q_n + P'(Q_n)\frac{Q_n^2}{n} \propto \frac{P(Q_n) - C'(q_n)}{P(Q_n)} - \frac{1}{\eta(Q_n)} \overset{(1)}{=} 0.\end{aligned}$$

Q.E.D.

Proof of Claim 2 From Claim 1 it follows that

$$(Q_n, P(Q_n)) = \left(\frac{a}{b\left(1 + H_n\right) + c/n}, a\left(1 - \frac{b}{b\left(1 + H_n\right) + c/n}\right)\right)$$

and

$$\begin{aligned}\frac{\partial\left[n\Pi\left(n,\lambda\right)\right]}{\partial n} =& P(Q_n)\frac{Q_n}{n} - C(q_n) + nP'(Q_n)\left(\frac{Q_n}{n}\right)^2\left[\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\left(1 - H_n\right) + H_n\right] \\ \propto =& \frac{c}{2n} - \frac{b(1-\lambda) + c}{n + \Lambda + c/b}\left(1 - H_n\right),\end{aligned}$$

and the rest follow. **Q.E.D.**

C.4 Proof of section B.5

Proof of Lemma 2 We have seen that the first derivative of equilibrium total surplus with respect to n is given by

$$\frac{d\,\mathrm{TS}_n}{dn} = \Pi(n,\lambda) - f - \Lambda_n Q_n P'(Q_n)\frac{\frac{\partial Q_n}{\partial n} - q_n}{n},$$

so if we denote $\Pi_n(n,\lambda) \equiv \partial\Pi(n,\lambda)/\partial n$, the second derivative is given by

$$\begin{aligned}\frac{d^2\,\mathrm{TS}_n}{(dn)^2} =& \Pi_n(n,\lambda) - \lambda Q_n P'(Q_n)\frac{\frac{\partial Q_n}{\partial n} - q_n}{n} - \Lambda_n\frac{\partial Q_n}{\partial n}P'(Q_n)\frac{\frac{\partial Q_n}{\partial n} - q_n}{n} \\ & - \Lambda_n Q_n P''(Q_n)\frac{\partial Q_n}{\partial n}\frac{\frac{\partial Q_n}{\partial n} - q_n}{n} - \Lambda_n Q_n P'(Q_n)\frac{\left(\frac{\partial^2 Q_n}{(\partial n)^2} - \frac{dq_n}{\partial n}\right)n - \frac{\partial Q_n}{\partial n} + q_n}{n^2} \\ \propto & -\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\left[1 - \lambda - H_n\left(\left(\frac{\partial Q_n}{\partial n}\frac{n}{Q_n} - 1\right)\left(1 - E_{P'}(Q_n)\right) + \frac{\partial^2 Q_n}{(\partial n)^2}\left(\frac{\partial Q_n}{\partial n}\right)^{-1} n - 1\right)\right] \\ & + \frac{1 - \lambda}{n}.\end{aligned}$$

Under constant marginal costs

$$\frac{\partial Q_n}{\partial n} = \frac{1-\lambda}{n+\Lambda-\Lambda E_{P'}\left(Q_n\right)}\frac{Q_n}{n} \implies$$

$$\frac{\partial^2 Q_n}{(\partial n)^2} = (1-\lambda)\left(-\frac{1+\lambda-\lambda E_{P'}\left(Q_n\right)-\Lambda E'_{P'}\left(Q_n\right)\frac{\partial Q_n}{\partial n}}{\left(n+\Lambda-\Lambda E_{P'}\left(Q_n\right)\right)^2}\frac{Q_n}{n}+\frac{\frac{\frac{\partial Q_n}{\partial n}n-Q_n}{n^2}}{n+\Lambda-\Lambda E_{P'}\left(Q_n\right)}\right),$$

$$\frac{\partial^2 Q_n}{(\partial n)^2}\left(\frac{\partial Q_n}{\partial n}\right)^{-1} n = -n\frac{1+\lambda-\lambda E_{P'}\left(Q_n\right)-\Lambda E'_{P'}\left(Q_n\right)\frac{\partial Q_n}{\partial n}}{n+\Lambda-\Lambda E_{P'}\left(Q_n\right)}+\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}-1$$

so that

$$\frac{d^2\,\mathrm{TS}_n}{(dn)^2} \propto -\left\{\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\left[1-\lambda-H_n\left(\begin{array}{r}\left(\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}-1\right)\left(2-E_{P'}(Q_n)\right)\\ -n\frac{1+\lambda-\lambda E_{P'}\left(Q_n\right)-\Lambda E'_{P'}\left(Q_n\right)\frac{\partial Q_n}{\partial n}}{n+\Lambda-\Lambda E_{P'}\left(Q_n\right)}-1\end{array}\right)\right]-\frac{1-\lambda}{n}\right\}$$

$$\propto -\left\{\begin{array}{r} n(1-\lambda)\left(n+\Lambda_n-\Lambda_n E_{P'}\left(Q_n\right)\right)-\left(n+\Lambda_n-\Lambda_n E_{P'}\left(Q_n\right)\right)^2\\ -\Lambda_n\left(\begin{array}{r}-\left(n+\Lambda_n-(1-\lambda)-\Lambda_n E_{P'}\left(Q_n\right)\right)\left(2-E_{P'}(Q_n)\right)\\ -n\left(1+\lambda-\lambda E_{P'}\left(Q_n\right)-\Lambda_n E'_{P'}\left(Q_n\right)\frac{\partial Q_n}{\partial n}\right)-\left(n+\Lambda_n-\Lambda_n E_{P'}\left(Q_n\right)\right)\end{array}\right)\end{array}\right\}.$$

The partial derivative of the expression in the brackets with respect to $E_{P'}(Q_n)$ is given by

$$\begin{aligned}&-\Lambda n(1-\lambda)+2\Lambda\left(n+\Lambda-\Lambda E_{P'}\left(Q_n\right)\right)\\ &-\Lambda\left(\Lambda\left(2-E_{P'}(Q_n)\right)+n+\Lambda-(1-\lambda)-\Lambda E_{P'}\left(Q_n\right)+\Lambda-(1-\lambda)+\Lambda\right)\\ \propto&\lambda n-(3\Lambda-2(1-\lambda))=-\left(2\Lambda-(1-\lambda)\right)<0,\end{aligned}$$

so that, given $E_{P'}(Q_n)<2$, for $d^2\,\mathrm{TS}_n/(dn)^2$ to be negative it is sufficient that

$$\begin{aligned}n(1-\lambda)\left(n+\Lambda-2\Lambda\right)-\left(n+\Lambda-2\Lambda\right)^2&\\ -\Lambda\left(-n\left(1+\lambda-2\lambda-\Lambda E'_{P'}\left(Q_n\right)\frac{\partial Q_n}{\partial n}\right)-\left(n+\Lambda-2\Lambda\right)\right)\geq 0 &\iff\\ 1-\lambda-\frac{\Lambda}{n}E'_{P'}(Q_n)Q_n\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}\geq -\frac{\left(\Lambda+1-\lambda\right)\left(n-\Lambda\right)}{\Lambda n}&,\end{aligned}$$

which is true for $E'_{P'}$ not too high. **Q.E.D.**

C.5 Proof of section B.8

Proof of Claim 3 The total derivative of $\widehat{n}^*(\lambda)$ at $\lambda = 0$ is

$$\frac{d\widehat{n}^*(\lambda)}{d\lambda} = (n-1)\frac{b\left[(n-1)(bn+c_2)^2 - (n+1+c_2/b)(b+c_2/2)(b(2n+1)+2c_2)\right]}{2(b+c_2/2)(bn+c_2)^2}\Bigg|_{n=\widehat{n}^*(0)},$$

where the denominator is positive and the numerator is a third-degree polynomial in n. In part (i), n_3 is the unique real root of the polynomial, which has a negative discriminant. In part (ii), the discriminant is positive, and the result follows with n_3 the highest of the three real roots of the polynomial equation above. Also,

$$\frac{dQ_{\widehat{n}^*(\lambda)}}{d\lambda} = \frac{\partial Q_n}{\partial \lambda} + \frac{\partial Q_n}{\partial n}\frac{d\widehat{n}^*(\lambda)}{d\lambda} = \frac{Q_{\widehat{n}^*(\lambda)}}{n+1+c_2/b}\left[(1+c_2/b)\frac{d\widehat{n}^*(\lambda)}{d\lambda}\frac{1}{n} - (n-1)\right]\Bigg|_{n=\widehat{n}^*(0)}$$

and for $n_1 \equiv (-2b^2 - 5bc_2 - c_2^2)/(2b^2) + \sqrt{(6b^3c_2 + 11b^2c_2^2 + 6bc_2^3 + c_2^4)/b^4}/2$ the corresponding results follow. For $\lambda = 0$, $\Psi(\widehat{n}^*(\lambda),\lambda) = \Pi(\widehat{n}^*(\lambda),\lambda) = f$, we get $d\,\mathrm{TS}_{\widehat{n}^*(\lambda)}/d\lambda \propto dQ_{\widehat{n}^*(\lambda)}/d\lambda - q_{\widehat{n}^*(\lambda)}d\widehat{n}^*(\lambda)/d\lambda$ and for $n_2 \equiv \left(2b - c_2 + \sqrt{8b^2 + 6bc_2 + c_2^2}\right)/(2b)$ the corresponding result follows. It can be checked that $n_3 > n_2 > n_1$. **Q.E.D.**

C.6 Proofs of sections B.10 and B.11

Where clear we may simplify notation, and write for example n instead of $n^*(\lambda)$.

Proof of Proposition 2' (i) Consider $R(Q_{-i})$ as defined in the proof of Proposition 1. For any Q_{-i} the maximand satisfies $\partial^2\left\{P(Q)\left[Q - (1-\lambda)Q_{-i}\right] - C(Q - Q_{-i})\right\}/(\partial\lambda\partial Q) = P'(Q)Q_{-i} \leq 0$. Thus, by Topkis' Monotonicity Theorem (e.g., see Vives, 1999), for any fixed Q_{-i}, $R(Q_{-i})$ is non-increasing in λ in the strong set order, and thus, so is $B(Q_{-i})$ as defined in the proof of Proposition 1. It follows then (e.g., see Chapter 2, Vives, 1999) that the extreme fixed points of $B(Q_{-i})$ (*i.e.*, the total quantity produced by $n-1$ firms in extremal equilibria) are non-increasing in λ, and the result follows.

(ii) Let q_n denote the individual quantity in an extremal equilibrium with n firms. We have then that $\pi(q_n) = q_nP(q_n + (n-1)q_n) - C(q_n) \geq q_{n+1}P(q_{n+1} + (n-1)q_n) - C(q_{n+1}) \geq q_{n+1}P(q_{n+1} + nq_{n+1}) - C(q_{n+1}) = \pi(q_{n+1})$, where the first inequality follows from q_n being a best response of an individual firm, and the second inequality follows from the fact that $(n-1)q_n \leq nq_{n+1}$ by part (iii) below.

(iii) For any fixed Q_{-i}, $B(Q_{-i})$ is non-decreasing in n, so the total quantity produced by $n-1$ firms in an extremal equilibrium is non-decreasing in n (e.g., see Chapter 2, Vives, 1999). We have also seen in the proof of Proposition 1 that when $\Delta > 0$, $R(Q_{-i})$ is non-decreasing in Q_{-i} and the result follows. **Q.E.D.**

Proof of Proposition 10 Given that $\Pi(n,\lambda)$ is decreasing in n by Proposition 2, the result follows given that $\Pi(n,\lambda) < f$ for n large. **Q.E.D.**

Proof of Proposition 10' Given that individual profit in extremal equilibria is non-increasing in n by Proposition 2' the result follows since $\Pi(n,\lambda) < f$ for n large. **Q.E.D.**

Proof of Proposition 11 Given $\Pi\left(\widehat{n}^*(\lambda),\lambda\right) = f$, the Implicit Function Theorem gives

$$\frac{d\widehat{n}^*(\lambda)}{d\lambda} = \frac{(n-1)(H_n^{-1}-1)}{1+H_n+\Lambda_n^{-1}\left[(1-\lambda)(1-H_n)-C''(q_n)/P'(Q_n)\right]-H_n E_{P'}(Q_n)} > 0,$$

where the inequality follows from what we have seen in section A.2.

(ii) The total derivative of the total quantity is then proportional to

$$\begin{aligned}\frac{dQ_{\widehat{n}^*(\lambda)}}{d\lambda} &\propto \frac{\partial Q_n}{\partial \lambda}\frac{\Lambda_n(1+\lambda)+1-\lambda-C''(q_n)/P'(Q_n)-\Lambda_n^2 E_{P'}(Q_n)/n}{(n-1)(n-\Lambda_n)}+\frac{\partial Q_n}{\partial n} \\ &= \frac{Q_n\begin{bmatrix}-\left(\Lambda_n(1+\lambda)+\Delta-\Lambda_n^2 E_{P'}(Q_n)/n\right) \\ +(n-\Lambda_n)\Delta/n\end{bmatrix}}{(n-\Lambda_n)\left(n+\Lambda_n-C''(q_n)/P'(Q_n)-\Lambda_n E_{P'}(Q_n)\right)} = -\frac{\Lambda_n Q_n}{n(n-\Lambda_n)} < 0,\end{aligned}$$

so total quantity decreases with λ, and thus so does individual quantity since the number of firms increases with λ. The total derivative of the total surplus is

$$\begin{aligned}\frac{d\,\mathrm{TS}_{\widehat{n}^*(\lambda)}}{d\lambda} &= P(Q_n)\frac{dQ_{\widehat{n}^*(\lambda)}}{d\lambda}-\frac{d\widehat{n}^*(\lambda)}{d\lambda}C(q_n)-nC'(q_n)\left(\frac{dQ_{\widehat{n}^*(\lambda)}/d\lambda}{n}-\frac{q_n}{n}\frac{d\widehat{n}^*(\lambda)}{d\lambda}\right)-\frac{d\widehat{n}^*(\lambda)}{d\lambda}f \\ &= \frac{dQ_{\widehat{n}^*(\lambda)}}{d\lambda}\left(P(Q_n)-C'(q_n)\right)-\left(P(Q_n)-C'(q_n)\right)q_n\frac{d\widehat{n}^*(\lambda)}{d\lambda} < 0.\end{aligned}$$

where the second equality follows from $\Pi(\widehat{n}^*(\lambda),\lambda) = f$.

(iii) Last, the total derivative of $\mathrm{MHHI}^* = H_{n^*}$ is

$$\frac{d\,\mathrm{MHHI}\left(\boldsymbol{q}_{\widehat{n}^*(\lambda)}\right)}{d\lambda} = \frac{n-1}{n}+\left(\frac{\lambda}{n}-\frac{\Lambda_n}{n^2}\right)\frac{d\widehat{n}^*(\lambda)}{d\lambda} \propto \frac{n-1}{n}\left(\frac{d\widehat{n}^*(\lambda)}{d\lambda}\right)^{-1}+\frac{\lambda n-\Lambda_n}{n^2}$$

$$\propto \frac{\left[1+\lambda+\Delta(Q_n,(n-1)q_n)/n+\left(\frac{1}{n}-\frac{1}{\Lambda_n}\right)\frac{C''(q_n)}{P'(Q_n)}\right]/H_n - E_{P'}(Q_n)}{n(n-\Lambda_n)} > 0,$$

where the inequality is implied by $C'' \geq 0$ combined with the maintained assumption (ii) that requires $E_{P'}(Q_n) < (1+\lambda+\Delta(Q_n,(n-1)q_n)/n)/H_n$. **Q.E.D.**

Proof of Proposition 12 We have that $d\widehat{n}^*(\lambda)/df = \left(\partial\Pi(n,\lambda)/\partial n\right)^{-1}\Big|_{n=\widehat{n}^*(\lambda)}$, and part (ii) follows if we take the directional derivative of $d\widehat{n}^*(\lambda)/df$. **Q.E.D.**

Proof of Claim 4 We have $\partial\Pi(n,\lambda)/\partial n = P'(Q_n)q_n^2\left[\frac{\partial Q_n}{\partial n}\frac{n}{Q_n}(1-H_n)+H_n\right] < 0$, so

$$\frac{\partial^2\Pi(n,\lambda)}{\partial n\partial\lambda} \propto \left\{\begin{array}{r}\left[-(1-E_{P'}(Q_n))\dfrac{\partial Q_n}{\partial n}\dfrac{n}{Q_n}-(2-E_{P'}(Q_n))\dfrac{\Lambda_n}{n-\Lambda_n}\right]\dfrac{\partial Q_n}{\partial\lambda}\dfrac{1}{Q_n}\\ -\dfrac{\partial^2 Q_n}{\partial n\partial\lambda}\dfrac{n}{Q_n}-\dfrac{n-1}{n-\Lambda_n}\left(1-\dfrac{\partial Q_n}{\partial n}\dfrac{n}{Q_n}\right)\end{array}\right\}.$$

Denote $E'_{P'}(Q) \equiv \partial E_{P'}(Q)/\partial Q$. Under constant marginal costs

$$\frac{\partial^2 Q_n}{\partial n\partial\lambda} = \frac{\left\{\begin{array}{r}\left(-Q_n+(1-\lambda)\dfrac{\partial Q_n}{\partial\lambda}\right)(n+\Lambda-\Lambda E_{P'}(Q_n))\\ -(1-\lambda)Q_n\left[n-1-(n-1)E_{P'}(Q_n)-\Lambda E'_{P'}(Q_n)\dfrac{\partial Q_n}{\partial\lambda}\right]\end{array}\right\}}{(n+\Lambda-\Lambda E_{P'}(Q_n))^2}\frac{1}{n}, \quad \text{so that}$$

$$\begin{aligned}\frac{\partial^2\Pi(n,\lambda)}{\partial n\partial\lambda} &\propto \left\{\begin{array}{r}2\Lambda^2(E_{P'}(Q_n))^2+\left[n\Lambda(n-\Lambda-1)-2n^2-\Lambda^2\right]E_{P'}(Q_n)\\ -n(n-\Lambda)(n+\Lambda-6)-\dfrac{\Lambda(n-\Lambda)^2Q_nE'_{P'}(Q_n)}{n+\Lambda-\Lambda E_{P'}(Q_n)}\end{array}\right\}\\ &< E_{P'}(Q_n)\left[2\Lambda^2E_{P'}(Q_n)-n\Lambda-2n^2-\Lambda^2\right] \leq 0,\end{aligned}$$

where the first (resp. second) inequality follows from $n \geq 5+E_{P'}(Q_n)$, $\lambda \in (0,1)$, $E_{P'}(Q_n) \leq 1$, $E'_{P'} \geq 0$ (resp. $0 \leq E_{P'}(Q_n) \leq 1$). Similarly follows part (ii). **Q.E.D.**

Proof of Proposition 13 Part (i): If $n^o(\lambda) \leq 2$, we are done since $n^*(\lambda) \geq 1$ given that monopoly profit is positive. For $n^o(\lambda) \geq 3$ keep in mind that $E_{P'}(Q) < 2$ on L implies that for every $n \in [2,+\infty)$, $E_{P'}(Q) < (1+\lambda)/H_n$ on L. The proof follows the proof of part (a) of Proposition 1 in Amir et al. (ACK; 2014). By definition, $\text{TS}_{n^o(\lambda)} \geq \text{TS}_{n^o(\lambda)-1}$, which implies $\int_{Q_{n^o(\lambda)-1}}^{Q_{n^o(\lambda)}} P(X)dX - n^o(\lambda)C\left(q_{n^o(\lambda)}\right)+(n^o(\lambda)-1)C\left(q_{n^o(\lambda)-1}\right) \geq f$, which then gives $\Pi(n^o(\lambda)-1,\lambda)-f \geq P\left(Q_{n^o(\lambda)-1}\right)q_{n^o(\lambda)-1} - \int_{Q_{n^o(\lambda)-1}}^{Q_{n^o(\lambda)}} P(X)dX + n^o(\lambda)\left(C\left(q_{n^o(\lambda)}\right)-C\left(q_{n^o(\lambda)-1}\right)\right)$, which given $P' < 0$ and that in the Cournot game total quantity is increasing in n, implies

$$\Pi\left(n^o(\lambda)-1,\lambda\right)-f > P\left(Q_{n^o(\lambda)-1}\right)\left(q_{n^o(\lambda)-1}+Q_{n^o(\lambda)-1}-Q_{n^o(\lambda)}\right)$$
$$+n^o(\lambda)\left(C\left(q_{n^o(\lambda)}\right)-C\left(q_{n^o(\lambda)-1}\right)\right) \Longrightarrow$$
$$\Pi\left(n^o(\lambda)-1,\lambda\right)-f > n^o(\lambda)\left(P\left(Q_{n^o(\lambda)-1}\right)-C'\left(\widetilde{q}\right)\right)\left(q_{n^o(\lambda)-1}-q_{n^o(\lambda)}\right),$$

for some $\widetilde{q}\in\left[q_{n^o(\lambda)-1}-q_{n^o(\lambda)}\right]$, where the implication follows by the mean value theorem. As $R(Q_{-i})$ is non-decreasing in Q_{-i}, it follows as in the proof in ACK that there exists $\widetilde{Q}_{-i}\in\left[(n^o(\lambda)-2)q_{n^o(\lambda)-1},(n^o(\lambda)-1)q_{n^o(\lambda)}\right]$ such that $\widetilde{q}\in r\left(\widetilde{Q}_{-i}\right)$ with $R\left(\widetilde{Q}_{-i}\right)\geq Q_{n^o(\lambda)-1}$ and $P\left(R(\widetilde{Q}_{-i})\right)\geq C'(\widetilde{q})$, so that $P\left(Q_{n^o(\lambda)-1}\right)\geq P\left(R(\widetilde{Q}_{-i})\right)\geq C'(\widetilde{q})$.

Given $E_{P'} < (1+\lambda)/H_n$, Proposition 2 implies that $q_{n^o(\lambda)-1} > q_{n^o(\lambda)}$, which combined with the above gives $\Pi\left(n^o(\lambda)-1,\lambda\right)-f\geq 0$. Also, by Proposition 2 $\Pi(n,\lambda)$ is decreasing in n, so it must be $n^*(\lambda)\geq n^o(\lambda)-1$ for the entry condition to be satisfied.

Part (ii): Since $\Pi(1,\lambda) > f$, $n^*(\lambda)\geq 1$. Also, $\Delta < 0$ on L implies that $C''(q) < 0$ for every $q < \overline{Q}$. By Proposition 2 Q_n is decreasing in n, and thus, so is consumer surplus. Also, $n\Pi(n,\lambda)\equiv P(Q_n)Q_n-nC(q_n) < P(Q_n)Q_n-C(Q_n)\leq P(q_1)q_1-C(q_1)=\Pi(1,\lambda)$, where the first inequality follows from $C'' < 0$. Thus, both consumer surplus and industry profits are maximized for $n=1$, so $n^o(\lambda)=1$. **Q.E.D.**

Proof of Proposition 13' For simplicity, we use the notation Q_n, q_n, TS_n, and $\Pi(n,\lambda)$ to refer to values in a specific equilibrium even if that equilibrium is not unique.

(i) The proof works like that of part (i) of Proposition 13. The only differences are that (a) in the Cournot game the total quantity in extremal equilibria is non-decreasing in n (instead of increasing in n), (b) $q_{n^o(\lambda)-1}\geq q_{n^o(\lambda)}$ by assumption (instead of $q_{n^o(\lambda)-1} > q_{n^o(\lambda)}$ implied by conditions on the primitives), and (c) $\Pi(n,\lambda)$ is non-increasing in n in extremal equilibria (instead of decreasing). Still, the weak inequality $\Pi\left(n^o(\lambda)-1,\lambda\right)-f\geq 0$ must hold and given that $\Pi(n,\lambda)$ is non-increasing in n, it must be that $n^*(\lambda)\geq n^o(\lambda)-1$.

(ii) The proof follows the one of part (b) of Proposition 1 in ACK. Since P is decreasing,

$$q_{n^o(\lambda)+1}P\left(Q_{n^o(\lambda)+1}\right) < \int_{n^o(\lambda)q_{n^o(\lambda)+1}}^{(n^o(\lambda)+1)q_{n^o(\lambda)+1}} P(Q)dQ. \tag{8}$$

Also, notice that $V_n(q):=\int_0^{nq}P(Q)dQ-nC(q)$ is concave in q (for every fixed n), since

$V_n'(q) = n\left(P(nq) - C'(q)\right)$, so that

$$V_n''(q) = nP'(nq)\left(n - \frac{C''(q)}{P'(nq)}\right) = nP'(nq)\left(\Delta(nq,(n-1)q) + n - 1 + \lambda\right) < 0.$$

Since $V_n(q)$ is concave in n, it follows that for any n and q,q' such that $q' \geq q$ it holds that

$$V_n(q) - V_n(q') \leq V_n'(q)(q-q') = n\left(P(nq') - C'(q')\right)(q-q'). \tag{9}$$

By definition $\mathrm{TS}_{n^o(\lambda)} \geq \mathrm{TS}_{n^o(\lambda)+1}$, which implies that $\Pi\left(n^o(\lambda)+1,\lambda\right) - f$ is less than or equal to

$$\begin{aligned}
&P\left(Q_{n^o(\lambda)+1}\right)q_{n^o(\lambda)+1} - \int_{Q_{n^o(\lambda)}}^{Q_{n^o(\lambda)+1}} P(X)dX + n^o(\lambda)\left(C\left(q_{n^o(\lambda)+1}\right) - C\left(q_{n^o(\lambda)}\right)\right)\\
&< \int_0^{Q_{n^o(\lambda)}} P(Q)dQ - n^o(\lambda)C\left(q_{n^o(\lambda)}\right) - \left[\int_0^{Q_{n^o(\lambda)+1}} P(Q)dQ - n^o(\lambda)C\left(q_{n^o(\lambda)+1}\right)\right]\\
&= V_{n^o(\lambda)}\left(q_{n^o(\lambda)}\right) - V_{n^o(\lambda)}\left(q_{n^o(\lambda)+1}\right)\\
&\leq n^o(\lambda)\left(P\left(n^o(\lambda)q_{n^o(\lambda)+1}\right) - C'\left(q_{n^o(\lambda)+1}\right)\right)\left(q_{n^o(\lambda)} - q_{n^o(\lambda)+1}\right) \leq 0,
\end{aligned}$$

where the first inequality follows from (8), the second inequality follows from (9), $q_{n^o(\lambda)+1} \geq q_{n^o(\lambda)}$, and the last inequality follows from $q_{n^o(\lambda)+1} \geq q_{n^o(\lambda)+1}$ and $P\left(n^o(\lambda)q_{n^o(\lambda)+1}\right) \geq P\left(Q_{n^o(\lambda)+1}\right) \geq C'\left(q_{n^o(\lambda)+1}\right)$ by the pricing formula (1). Thus, $\Pi\left(n^o(\lambda)+1,\lambda\right) < f$, and given that $\Pi(n,\lambda)$ is non-increasing in n, $n^*(\lambda) \leq n^o(\lambda)$. **Q.E.D.**

C.7 Proofs of section B.12

Proof of Proposition 14 The LHS of (6) is globally decreasing, so (6) has a unique solution given that for $n = 0$ the LHS is at least as high as f and for $n \to \infty$ it is lower than f. Also, (7) is immediately satisfied. **Q.E.D.**

Proof of Proposition 15 and Corollary 15.1 Totally differentiating (6) with respect to λ we get

$$\begin{aligned}
&\Pi_n(n^*(\lambda),\lambda)\left(n^*(\lambda) + (1+\lambda)\frac{dn^*(\lambda)}{d\lambda}\right) + \Pi_\lambda(n^*(\lambda),\lambda)\\
&+\lambda n^*(\lambda)\left(\Pi_{nn}(n^*(\lambda),\lambda)\frac{dn^*(\lambda)}{d\lambda} + \Pi_{n\lambda}(n^*(\lambda),\lambda)\right) = 0,
\end{aligned}$$

which gives

$$\frac{dn^*(\lambda)}{d\lambda} = -\frac{n^*(\lambda)\left(\Pi_n(n^*(\lambda),\lambda) + \lambda\Pi_{n\lambda}(n^*(\lambda),\lambda)\right) + \Pi_\lambda(n^*(\lambda),\lambda)}{(1+\lambda)\Pi_n(n^*(\lambda),\lambda) + \lambda n^*(\lambda)\Pi_{nn}(n^*(\lambda),\lambda)}$$
$$= -\frac{n^*(\lambda)\left(1 + \frac{\Pi_\lambda(n^*(\lambda),\lambda)}{\Pi(n^*(\lambda),\lambda)}\left(\frac{\Pi_n(n^*(\lambda),\lambda)}{\Pi(n^*(\lambda),\lambda)}n^*(\lambda)\right)^{-1} - E_{\partial\Pi/\partial n,\lambda}(n^*(\lambda),\lambda)\right)}{1+\lambda-\lambda E_{\partial\Pi/\partial n,n}(n^*(\lambda),\lambda)}.$$

Given what we see in the proof of Claim 4, $E_{\partial\Pi/\partial n,\lambda}(n,\lambda) - \frac{\Pi_\lambda(n,\lambda)}{\Pi_n(n,\lambda)}\frac{1}{n} - 1$ is equal to

$$\frac{-\left[\lambda(1-E_{P'}(Q_n))\frac{\partial Q_n}{\partial n}\frac{n}{Q_n} + \lambda(2-E_{P'}(Q_n))\frac{\Lambda_n}{n-\Lambda_n} + 1\right]\frac{\partial Q_n}{\partial\lambda}\frac{1}{Q_n} - \frac{\partial^2 Q_n}{\partial n\partial\lambda}\frac{\lambda n}{Q_n} + \frac{2\Lambda_n - n - 1}{n-\Lambda_n}\frac{\partial Q_n}{\partial n}\frac{n}{Q_n} - \frac{2\Lambda_n - 1}{n-\Lambda_n}}{\frac{\partial Q_n}{\partial n}\frac{n}{Q_n} + \frac{\Lambda_n}{n-\Lambda_n}},$$

where for constant marginal costs

$$\frac{\partial Q_n}{\partial n} = \frac{1-\lambda}{n+\Lambda-\Lambda E_{P'}(Q_n)}\frac{Q_n}{n} \xrightarrow{C\text{ linear}}$$

$$\frac{\partial^2 Q_n}{\partial n\partial\lambda} = \frac{\left\{\begin{matrix}\left(-Q_n + (1-\lambda)\frac{\partial Q_n}{\partial\lambda}\right)(n+\Lambda-\Lambda E_{P'}(Q_n)) \\ -(1-\lambda)Q_n\left[n-1-(n-1)E_{P'}(Q_n) - \Lambda E'_{P'}(Q_n)\frac{\partial Q_n}{\partial\lambda}\right]\end{matrix}\right\}}{(n+\Lambda-\Lambda E_{P'}(Q_n))^2}\frac{1}{n},$$

$$\frac{\partial^2 Q_n}{(\partial n)^2} = \frac{\left\{(1-\lambda)\frac{Q_n}{n}\left[\begin{matrix}-\left(1+\lambda-\lambda E_{P'}(Q_n) - \Lambda E'_{P'}(Q_n)\frac{\partial Q_n}{\partial n}\right) \\ +\frac{n+\Lambda-\Lambda E_{P'}(Q_n)}{n}\left(\frac{\partial Q_n}{\partial n}\frac{n}{Q_n} - 1\right)\end{matrix}\right]\right\}}{(n+\Lambda-\Lambda E_{P'}(Q_n))^2} \implies$$

$$\frac{\partial^2 Q_n}{(\partial n)^2} = \frac{\partial Q_n}{\partial n}\frac{\left[\begin{matrix}\frac{n+\Lambda-\Lambda E_{P'}(Q_n)}{n}\left(\frac{\partial Q_n}{\partial n}\frac{n}{Q_n} - 1\right) \\ -\left(1+\lambda-\lambda E_{P'}(Q_n) - \Lambda E'_{P'}(Q_n)\frac{\partial Q_n}{\partial n}\right)\end{matrix}\right]}{n+\Lambda-\Lambda E_{P'}(Q_n)}$$

so that $E_{\partial\Pi/\partial n,\lambda}(n,\lambda) - \frac{\Pi_\lambda(n,\lambda)}{\Pi_n(n,\lambda)}\frac{1}{n} - 1$ has the same sign as

$$\left[\lambda(1-E_{P'}(Q_n))\frac{1-\lambda}{n+\Lambda-\Lambda E_{P'}(Q_n)} + \lambda(2-E_{P'}(Q_n))\frac{\Lambda}{n-\Lambda} + 1\right]\frac{n-1}{n+\Lambda-\Lambda E_{P'}(Q_n)}$$
$$+\lambda\frac{\left\{\begin{matrix}\left(1+(1-\lambda)\frac{n-1}{n+\Lambda-\Lambda E_{P'}(Q_n)}\right)(n+\Lambda-\Lambda E_{P'}(Q_n)) \\ +(1-\lambda)\left[(n-1)(1-E_{P'}(Q_n)) - \frac{\Lambda(n-1)Q_n E'_{P'}(Q_n)}{n+\Lambda-\Lambda E_{P'}(Q_n)}\right]\end{matrix}\right\}}{(n+\Lambda-\Lambda E_{P'}(Q_n))^2}$$

$$
\begin{aligned}
&+\frac{2\Lambda-n-1}{n-\Lambda}\frac{1-\lambda}{n+\Lambda-\Lambda E_{P'}\left(Q_n\right)}-\frac{2\Lambda-1}{n-\Lambda}\\
\propto&\left(n-1+2\lambda-\frac{\Lambda\left(2n-\Lambda E_{P'}\left(Q_n\right)\right)}{n-\Lambda}\right)\left(n+\Lambda-\Lambda E_{P'}\left(Q_n\right)\right)+\lambda(2n-\Lambda)\left(2-E_{P'}\left(Q_n\right)\right)\\
&-\frac{\lambda\Lambda(n-\Lambda)Q_nE'_{P'}\left(Q_n\right)}{n+\Lambda-\Lambda E_{P'}\left(Q_n\right)},
\end{aligned}
$$

which is positive if $E'_{P'}\leq 0$ and $E_{P'}\left(Q_n\right)>\left[2n-\left(n/\Lambda-1\right)\left(n-1+2\lambda\right)\right]/\Lambda$. On the other hand, given $E_{P'}\left(Q_n\right)<2$, $2(n+\Lambda-\Lambda E_{P'}\left(Q_n\right))>\lambda(2n-\Lambda)\left(2-E_{P'}\left(Q_n\right)\right)$, so if $E'_{P'}\geq 0$ and $E_{P'}\left(Q_n\right)<\left[2n-\left(n/\Lambda-1\right)\left(n+1+2\lambda\right)\right]/\Lambda$, then the expression is negative.

If $dn^*(\lambda)/d\lambda\leq 0$, then $Q_{n^*(\lambda)}$ clearly decreases with λ. If $dn^*(\lambda)/d\lambda>0$, then in equilibrium

$$
E_{\partial\Pi/\partial n,\lambda}\left(n,\lambda\right)-\frac{\Pi_\lambda(n,\lambda)}{\Pi_n(n,\lambda)}\frac{1}{n}-1>0
$$

and the directional derivative of the total quantity when (λ,n) changes in direction $\mathbf{v}\coloneqq(1,dn^*(\lambda)/d\lambda)$ is

$$
\begin{aligned}
\nabla_{\mathbf{v}}Q_n=&\frac{\partial Q_n}{\partial\lambda}+\frac{\partial Q_n}{\partial n}\frac{dn^*(\lambda)}{d\lambda}\\
=&\frac{\partial Q_n}{\partial\lambda}-\frac{n^*(\lambda)\left(1+\frac{\Pi_\lambda(n^*(\lambda),\lambda)}{\Pi_n(n^*(\lambda),\lambda)}\frac{1}{n^*(\lambda)}-E_{\partial\Pi/\partial n,\lambda}\left(n^*(\lambda),\lambda\right)\right)}{1+\lambda-\lambda E_{\partial\Pi/\partial n,n}\left(n^*(\lambda),\lambda\right)}\frac{\partial Q_n}{\partial n}\\
\propto&\frac{E_{\partial\Pi/\partial n,\lambda}\left(n,\lambda\right)-\frac{\Pi_\lambda(n,\lambda)}{\Pi_n(n,\lambda)}\frac{1}{n}-1}{1+\lambda-\lambda E_{\partial\Pi/\partial n,n}\left(n,\lambda\right)}\frac{1-\lambda-C''\left(q\right)/P'\left(Q\right)}{n-1}-1,
\end{aligned}
$$

so that under constant marginal costs, $E_{\partial\Pi/\partial n,n}\left(n,\lambda\right)\leq 2$, $E_{P'}(Q)'\geq 0$ and $E_{P'}(Q)<2$,

$$
\begin{aligned}
\operatorname{sgn}\left\{\nabla_{\mathbf{v}}Q_n\right\}\leq&\operatorname{sgn}\left\{E_{\partial\Pi/\partial n,\lambda}\left(n,\lambda\right)-\frac{\Pi_\lambda(n,\lambda)}{\Pi_n(n,\lambda)}\frac{1}{n}-1-(n-1)\right\}\\
=&\operatorname{sgn}\left\{\begin{array}{r}\left(\lambda(n-1)+2\lambda-\dfrac{\Lambda\left(2n+(n-1)(n+\Lambda)-n\Lambda E_{P'}\left(Q_n\right)\right)}{n-\Lambda}\right)\times\\ \left(n+\Lambda-\Lambda E_{P'}\left(Q_n\right)\right)+\lambda(2n-\Lambda)\left(2-E_{P'}\left(Q_n\right)\right)-\dfrac{\lambda\Lambda(n-\Lambda)Q_nE'_{P'}\left(Q_n\right)}{n+\Lambda-\Lambda E_{P'}\left(Q_n\right)}\end{array}\right\}\\
\leq&\operatorname{sgn}\left\{2+\lambda(n-1)+2\lambda-\frac{\Lambda\left(2n+(n-1)(n+\Lambda)-n\Lambda E_{P'}\left(Q_n\right)\right)}{n-\Lambda}\right\}\\
<&\operatorname{sgn}\left\{2+\lambda(n-1)+2\lambda-\frac{\Lambda\left(2n+(n-1)(n+\Lambda)-2n\Lambda\right)}{n-\Lambda}\right\}\\
=&\operatorname{sgn}\left\{2+\lambda(n-1)+2\lambda-\frac{\Lambda(n-\Lambda)\left(n+1\right)}{n-\Lambda}\right\}=\operatorname{sgn}\left\{1+2\lambda-\Lambda n\right\},
\end{aligned}
$$

which is non-positive given $n \geq 2$. **Q.E.D.**

Proof of Claim 5 Under constant marginal costs and linear demand

$$\begin{aligned}
Q_n &= \frac{n(a-c)}{b(n+\Lambda)}, \quad \Pi(n,\lambda) = \left(a - \frac{n(a-c)}{n+\Lambda} - c\right)\frac{a-c}{b(n+\Lambda)} = \frac{\Lambda(a-c)^2}{b(n+\Lambda)^2}, \\
\frac{\partial \Pi(n,\lambda)}{\partial n} &= \frac{(a-c)^2\left(\lambda(n+\Lambda)^2 - 2(n+\Lambda)(1+\lambda)\Lambda\right)}{b(n+\Lambda)^4} = -\frac{(a-c)^2\left(2\Lambda - \lambda(n-\Lambda)\right)}{b(n+\Lambda)^3}, \\
\frac{\partial^2 \Pi(n,\lambda)}{(\partial n)^2} &= \frac{(a-c)^2\left\{-\lambda(1+\lambda)(n+\Lambda)^3 - 3(1+\lambda)(n+\Lambda)^2\left[\lambda(n-\Lambda) - 2\Lambda\right]\right\}}{b(n+\Lambda)^6} \\
&= \frac{2(a-c)^2(1+\lambda)\left[2\Lambda - \lambda(n-\Lambda) + 1 - \lambda\right]}{b(n+\Lambda)^4}, \\
E_{\partial\Pi/\partial n, n}(n,\lambda) &= \frac{2\left[n+\Lambda-(1-\lambda)\right]\left[2\Lambda - \lambda(n-\Lambda) + 1 - \lambda\right]}{(n+\Lambda)\left[2\Lambda - \lambda(n-\Lambda)\right]} \\
&= 2\left(1 - \frac{1-\lambda}{n+\Lambda}\right)\left(1 + \frac{1-\lambda}{2\Lambda - \lambda(n-\Lambda)}\right),
\end{aligned}$$

Thus, under linear demand and constant marginal costs, $E_{\partial\Pi/\partial n, n}(n,\lambda)$ is decreasing in n, and thus bounded from above by

$$2\left(1 - \frac{1-\lambda}{2}\right)\left(1 + \frac{1-\lambda}{2-\lambda(1-1)}\right) = 2\left[1 - \left(\frac{1-\lambda}{2}\right)^2\right] \leq 2.$$

Q.E.D.

Proof of Proposition 16 See the proof of Claim 4. **Q.E.D.**

Proof of Proposition 17 We have seen that the derivative of equilibrium total surplus (in the Cournot game with a fixed number of firms) with respect to n is given by

$$\frac{d\,\mathrm{TS}(q_n)}{dn} = \Pi(n,\lambda) - f - \Lambda_n Q_n P'(Q_n)\frac{\partial q_n}{\partial n}.$$

Given (6) we then have that

$$\left.\frac{d\,\mathrm{TS}(\boldsymbol{q}_n)}{dn}\right|_{n=n^*(\lambda)} = -\,\lambda n^*(\lambda)\Pi_n\left(n^*(\lambda),\lambda\right) - \Lambda_{n^*(\lambda)}Q_{n^*(\lambda)}P'\left(Q_{n^*(\lambda)}\right)\left.\frac{\partial q_n}{\partial n}\right|_{n=n^*(\lambda)}$$

and the result follows as in the proof of Proposition 5. **Q.E.D.**

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