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Strategic complementarity in games☆

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ABSTRACT

The lattice-theoretic approach has had a significant impact in all fields of economics, being progressively incorporated into the standard toolbox. This paper presents a selective survey with an emphasis on basic tools, some important results, and applications in industrial organization, dynamic games, games of incomplete information, and mechanism design. Frontier theoretical research employing lattice-theoretic methods continues to be developed in areas such as mean-field games and information design.

1. Introduction

The study of games of strategic complementarities has proved extremely useful in all fields of economics, generating an enormous amount of literature. Lattice-theoretic methods provide the appropriate toolbox to deal with economic problems when complementarities are involved. The theory of supermodular games and monotone comparative statics exploits both order and monotonicity properties in contrast to classic convex analysis (Topkis, 1978, 1979). It has been further developed and applied to economics by Vives (1985, 1990) and Milgrom and Roberts (1990). As we will see, the theory continues to be extended with frontier research in, for example, dynamic games, Bayesian games, and mechanism and information design.

Monotone comparative statics analysis provides conditions under which optimal solutions to optimization problems move monotonically with a parameter. This allows the characterization of games of strategic complementarities, where a player's best response increases in the (totally ordered) actions of the rivals. The approach is powerful since it provides a framework for identifying the critical properties of the payoffs and action spaces that deliver the desired results (e.g., existence and uniqueness of equilibrium without requiring quasiconcavity of payoffs or smoothness assumptions and comparative statics of equilibria). It allows for the incorporation of complex strategy spaces, such as those with indivisibilities and function spaces, arising in dynamic and Bayesian games. It can also deal with multiple equilibria, and the approach extends well beyond games of strategic complementarities.

This paper presents a selective survey with examples of the power of the approach taking off from the papers of Vives (1985, 1990) with applications to industrial organization, dynamic games, Bayesian games, and mechanism and information design.[1] The emphasis is on basic tools, some important results and recent applications, leaving aside most of the work in fields such as cooperative games, organization theory, macroeconomics, international trade, and political economy.

The plan of the paper is as follows. Section 2 introduces the theory, basic results, and some extensions. Section 3 provides applications to industrial organization including multimarket oligopoly, entry, and innovation. Section 4 deals with dynamic games surveying overlapping generations, Markov games, and mean-field games. Section 5 studies games of incomplete information, characterization of equilibria, and applications to global games, information frictions, and data complementarities. Section 6 considers mechanism design and information design, and Section 7 concludes.

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[1] Previous reviews of the literature include Vives (1999, 2005, 2018) and Amir (2005). The paper draws in particular from Vives' surveys.

[2] See Tarski (1955), Topkis (1978, 1979, 1998), Vives (1990, 1999), Milgrom and Roberts (1990) and Milgrom and Shannon (1994) for more general and extensive treatments of supermodularity tools and supermodular games.

[3] In a more general formulation of supermodular games, strategy spaces are only required to be complete (compact) lattices and payoff functions need only be continuous with conditions (i) and (ii) stated in non-differential terms.

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2. Supermodularity in games: tools and results

2.1. An introduction to the theory

This section presents a brief overview of the main results of the theory of monotone comparative statics and supermodular games.[2] Consider the smooth game $G := \langle N, (S_i, u_i)_{i\in N}\rangle$, where $N := \{1, 2, \dots, n\}$ is the set of players, S_i is player i's strategy space, a compact cube in $\mathbb{R}^m$, and $u_i : S \to \mathbb{R}$ is her twice continuously differentiable payoff function, where $S := \times_{i\in N} S_i$. Let $s_{-i} \in S_{-i} := \times_{j\in N\setminus\{i\}} S_j$ denote the strategy profile of all players except i, and s_{ih} denote the hth component of the strategy s_i of player i.[3] Then, G is smooth supermodular if, for every player $i \in N$, $u_i(s_i, s_{-i})$

(i) is supermodular in s_i: $\partial^2 u_i(s_i, s_{-i})/\partial s_{ih}\partial s_{ik} \geq 0$ for every $h, k \in \{1, \dots, m\}$ with $h \neq k$ and every $s \in S$, and
(ii) has increasing differences in (s_i, s_{-i}): $\partial^2 u_i(s_i, s_{-i})/\partial s_{ih}\partial s_{jk} \geq 0$ for every $s \in S$, every $i, j \in N$ with $i \neq j$, and every $h, k \in \{1, \dots, m\}$.

G is smooth strictly supermodular if the inequality in condition (ii) holds strictly. Condition (i) says that the different dimensions of each player i's strategy are complements (to each other) in increasing i's payoff. Condition (ii) is a strategic complementarity property: it says that (each dimension of) a player i's strategy and (each dimension of) a rival j's strategy are complements in increasing i's payoff.[4]

Denote by $\Psi(s_{-i})$ player i's best-reply correspondence, and by $\overline{\Psi}_i(s_{-i}) := \sup \Psi(s_{-i})$ and $\underline{\Psi}_i(s_{-i}) := \inf \Psi(s_{-i})$ its largest and smallest elements (which always exist in supermodular games), respectively. In a strictly supermodular game, any selection from the best-reply correspondence is increasing. $\overline{\Psi}(s) := (\overline{\Psi}_1(s_{-1}), \overline{\Psi}_2(s_{-2}), \dots, \overline{\Psi}_n(s_{-n}))$ and $\underline{\Psi}(s) := (\underline{\Psi}_1(s_{-1}), \underline{\Psi}_2(s_{-2}), \dots, \underline{\Psi}_n(s_{-n}))$ are the extremal best-reply maps.[5]

2.1.1. Nash equilibrium existence and structure of the equilibrium set

Tarski's fixed-point theorem applied on $\overline{\Psi}(s)$ and $\underline{\Psi}(s)$ implies that the equilibrium set of a supermodular game has a largest $\overline{s} := \sup S^-$ and a smallest $\underline{s} := \inf S^+$ element (Topkis, 1979), where $S^- := \{s \in S : \overline{\Psi}(s) \geq s\}$ is the lower contour set of the largest best-reply map and $S^+ := \{s \in S : \underline{\Psi}(s) \leq s\}$ the upper contour set of the smallest best-reply map.[6] Also, the equilibrium set is a complete lattice (Vives, 1985, 1990; Zhou, 1994).[7]

Last, the extremal equilibria of a symmetric supermodular game (exchangeable against permutations of the players) are symmetric.[8] Particularly, if the game is strictly supermodular with one-dimensional strategy spaces (i.e., $m = 1$), then *all* equilibria are symmetric. The same holds if strategy spaces are completely ordered.

If the payoff to a player has positive spillovers (i.e., it is increasing in the strategies of the other players), then the largest (resp., smallest) equilibrium in a supermodular game is the Pareto best (resp., worst) equilibrium. It follows that equilibria can be Pareto ranked in games with strategic complementarities (Milgrom and Roberts, 1990; Vives, 1990).[9]

2.1.2. Stability and rationalizability

In a supermodular game with continuous payoffs, simultaneous response best-reply dynamics approach the set $[\underline{s}, \overline{s}]$ and converge monotonically downward (resp. upward) to an equilibrium when starting at any point $s \in S^+$ (resp. $s \in S^-$; Vives, 1990). The results extend to a large class of adaptive dynamics, of which best-reply dynamics are a particular case. Furthermore, the extremal equilibria $\overline{s}$ and $\underline{s}$ correspond to the largest and smallest strategy profiles, respectively, that survive the iterated elimination of strictly dominated strategies (Milgrom and Roberts, 1990). A corollary is that if the equilibrium is unique, then it is globally stable and the game is dominance-solvable.[10]

2.1.3. Comparative statics

Parametrize player i's payoff function by $t \in \mathbb{R}$ and write $u_i(s_i, s_{-i}; t)$. If $u_i(s_i, s_{-i}; t)$ has increasing differences in (s_i, t) (i.e., $\partial^2 u_i(s_i, s_{-i}; t)/(\partial s_{ih}\,\partial t) \geq 0$ for every $h \in \{1, 2, \dots, m\}$), then (i) an increase in t causes the largest and smallest equilibrium points to increase, and (ii) starting from any equilibrium (before the increase in t), best-reply dynamics lead to a (weakly) larger equilibrium after the increase in t (the result extends to adaptive dynamics, such as fictitious play and gradient dynamics).[11]

It is worth noting that continuous equilibrium selections that do not increase monotonically with t predict unstable equilibria (Echenique, 2002). This is a version of Samuelson's Correspondence Principle, based on which unambiguous comparative statics are obtained when we look at locally stable equilibria (applied to smooth equilibrium conditions of one-dimensional economic models). In games with strategic complementarities, we have a multidimensional global version of the principle.

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[4] Milgrom and Shannon (1994) relax supermodularity and increasing differences defining an "ordinal" supermodular game. Supermodularity is relaxed to the weaker concept of quasi-supermodularity and increasing differences to a single-crossing property. Such properties (in contrast to supermodularity and increasing differences) have no differential characterization and need not be preserved under addition or partial maximization operations. Quah and Strulovici (2009) introduce a new property, the interval dominance order, which is weaker than the single-crossing property and, if satisfied, guarantees monotone comparative statics. Che et al. (2021) further relax the interval dominance order, introducing weak dominance and weak interval dominance, under which an individual choice problem exhibits weak monotone comparative statics, where "weak" refers to comparative statics with respect to the weak, instead of the strong, set order. They also provide a new fixed-point theorem and use it to establish existence and weak monotone comparative statics of Nash equilibria in a broader class of games with strategic complementarities than identified previously.

[5] The results to follow hold often more generally with the assumption of the game being (strictly) supermodular replaced by the assumption that it is a game of (strict) strategic complementarities (GSC), as defined in Vives (1985). A game is GSC if the best-reply correspondence of any player has extremal elements that are increasing in the strategies of rivals. It is strict GSC if, in addition, every selection of every player's best-reply correspondence is increasing in the rivals' strategies.

[6] Echenique and Edlin (2004) show that any Nash equilibrium of a strictly supermodular game where at least two players' strategies are properly mixed (i.e., not pure) is unstable for a broad class of learning dynamics. This serves as a rationale for restricting attention to pure equilibria, as we do here.

[7] Etessami et al. (2020) study the computational complexity of finding equilibria of supermodular games.

[8] This implies that a symmetric supermodular game has a unique equilibrium if and only if it has a unique (in the class of symmetric equilibria) symmetric equilibrium.

[9] Rota-Graziosi (2019) show that tax coordination is Pareto improving in a tax competition game with positive spillovers.

[10] Sobel (2019) shows that the main results on supermodular games extend to a slightly broader class of games, called interval-dominance supermodular games. He also provides (i) bounds on the set of strategies that survive the iterated elimination of *weakly* dominated strategies and (ii) conditions under which the order of elimination does not matter.

[11] See Milgrom and Shannon (1994) for extensions. Barthel and Hoffmann (2019) study iterated dominance, stability, and best-reply dynamics in games of strategic heterogeneity, where some players may have monotone increasing best replies while allowing for others to have monotone decreasing best replies.

2.1.4. The scope of the theory

The scope of the theory is relatively narrow if we take the view that the order of the strategy spaces is part of the description of the game (a "natural" order in the strategy spaces). Then, there are many games that are not of strategic complementarities. Indeed, there are many games where the best responses are non-monotone. However, if we take the view that the order of the strategy sets of the players is a modeling choice of the researcher (e.g., reversing the order in one strategy space in a duopoly with strategic substitutes, as we will see below), the answer is different. If we allow the construction of this order ex post, knowing the equilibria of the game, then most games are of strategic complementarities (although this insight is not very useful in practice). The point is that complementarities alone, in the weak stated sense, do not have much predictive power unless they are coupled with additional structure (Echenique, 2004a).

Testing for strategic complementarities. Several methods of testing for strategic complementarities have been developed. In static games with ordered action spaces, equilibrium play implies a covariance restriction—robust to equilibrium multiplicity—between each player's action and a player's strategic index, which captures patterns of strategic substitutabilities or complementarities (Aradillas-López and Gandhi, 2016). de Paula and Tang (2012) and Kline (2016)—in the context of incomplete and complete information games, respectively—develop methods for testing whether the players' actions are strategic complements or substitutes. Lazzati et al. (2023) develop a method for testing for pure strategy Nash equilibrium play in games with monotone best responses, including both games of strategic complements and games of strategic substitutes, and recovering a player's (ordinal) preferences over their strategy space (conditional on covariates and the actions of other players). Leveraging monotone comparative statics, tests of the existence of complementarities between explanatory and dependent variables can be performed more generally in a large class of economic models with continuous dependent variables, encompassing models of individual decision-making and one-dimensional equilibrium models (Echenique and Komunjer, 2009).

A range of papers have checked for the presence of strategic complementarities. In the classic example of price competition, Amiti et al. (2019) indeed find strong evidence of strategic complementarities in pricing among firms in the Belgian manufacturing sector together with substantial heterogeneity across firms. Small firms exhibit no strategic complementarities, while large firms exhibit strong ones. In finance, Chen et al. (2010) identify strategic complementarities in mutual fund investment by noting that redemptions impose higher costs on illiquid funds. Hertzberg et al. (2011) use a credit registry expansion in Argentina as a natural experiment to identify complementarities.

2.2. Extensions

2.2.1. Two-player games of strategic substitutes

In two-player games (i.e., $n = 2$), if we replace property (ii) and assume that instead $u_i(s_i, s_j)$ has decreasing differences in (s_i, s_j); that is, $\partial^2 u_i(s_i, s_j)/\partial s_{ih}\partial s_{jk} \le 0$ for every $s \in S$, $i \neq j$, and every $h, k \in \{1, \dots, m\}$, then the game is one of strategic substitutes. However, if we redefine player 2's strategy to be $\tilde{s}_2 := -s_2$, then the transformed game (where player 1's strategy is s_1 and player 2's strategy is $\tilde{s}_2$) is smooth supermodular and all the results apply (Vives, 1990). A typical example where this trick can be used is a Cournot duopoly.

2.2.2. Symmetric games in totally ordered spaces

Alternative—but also using lattice-theoretic methods—equilibrium results rely on Tarski's (1955) intersection point (between a quasi-increasing and a quasi-decreasing function) theorem.[12] For example, in symmetric games with one-dimensional strategy spaces ($m = 1$), if $f : [0,1] \to [0,1]$ given by $f(x) := \Psi_i(x, x, \dots, x)$ is quasi-increasing, then it must intersect the (both quasi-decreasing and quasi-increasing) identity function.[13] The intersection point is a symmetric equilibrium. This argument has been rediscovered and employed multiple times in the economics literature (e.g., in a class of symmetric Cournot games: McManus, 1962, 1964; Roberts and Sonnenschein, 1976; Milgrom and Roberts, 1994).[14]

2.2.3. Aggregative games

Some of the obtained results can be extended beyond supermodular games to games where best-reply correspondences are, instead, decreasing, but where additional conditions are satisfied. One such class of games is aggregative games, where the payoff of a player depends on his strategy and an aggregator or vector of aggregators (for example, an additive separable function) of the strategies of all the players. In the simplest case, every player's strategy space is one-dimensional ($m = 1$). Each player i's payoff depends on s only through i's strategy and the sum of all players' strategies, that is, $u_i(s) = \tilde{u}_i(s_i, \sum_{j\in N} s_j)$ for every $s \in S$ for some function $\tilde{u}_i$ like in a Cournot game.[15] If the best-reply correspondence Ψ_i (which can be written as a function of s_i and $\sum_{j\neq i} s_j$) of every player i is upper-hemicontinuous and strongly decreasing in $\sum_{j\neq i} s_j$ (i.e., every selection of it is decreasing), then an equilibrium exists. If, in addition, the slope of Ψ_i with respect to $\sum_{j\neq i} s_j$ is higher than -1 for every player i, then the equilibrium is unique (Kukushkin, 1994; Vives, 1999, Theorems 2.7, 2.8). Azar and Vives (2021) use these results to show existence and uniqueness of equilibrium in a general equilibrium oligopoly model.[16]

3. Industrial organization

In this section, we briefly review some recent applications in the theory of industrial organization relating to pricing, multimarket oligopoly, entry, and innovation.[17]

3.1. Price competition

Bertrand competition with differentiated (substitute) goods is a classic example of a supermodular game, although not all such Bertrand games are supermodular (e.g., see Vives, 2018). Strategic complementarities may also arise among buyers (e.g., due to positive network externalities). Consider, for instance, a Bertrand duopoly where the value of the good of either firm to any buyer increases in the number of the buyer's neighbors who buy the same good. When externalities are linear in that number, there are equilibria where one of the firms monopolizes the market. One of these equilibria features bipartition pricing, which offers discounts to some buyers and charges markups to others (Aoyagi, 2018).[18]

[12] A real-valued function is quasi-increasing (resp. quasi-decreasing) if it does not have downward (resp. upward) jumps. See Section 2.3.1 in Vives (1999).

[13] $f(x)$ is the best reply of a player when every other player plays x, which by symmetry does not depend on the player's identity. For simplicity, we take $f(x)$ to be single-valued, but the result also applies with f being a selection of $\Psi_i(x, x, \dots, x)$ (e.g., see Amir and Lambson, 2000; Vives and Vravosinos, 2024).

[14] See Vives (1999) for a discussion.

[15] This was used by Bamón and Fraysse (1985) and Novshek (1985) to show the existence of a Cournot equilibrium.

[16] For more comprehensive treatments, applications, and extensions of the theory of aggregative games (including games with multiple or non-linear aggregates), see Corchón (1994), Acemoglu and Jensen (2013), and Jensen (2018).

[17] For other topics and more extensive discussions, see Vives (1999, 2018) and Amir (2018).

[18] Particularly, (in a two-stage model) the buyers' subgame is one of strategic complements and, thus, typically has multiple equilibria. Namely, an equilibrium (of the subgame) that is the least favorable to firm 1 exists. Similarly, an equilibrium (of the subgame) that is the least favorable to firm 2 exists. To analyze equilibrium behavior, one should check whether firm 1 can profitably deviate (in its pricing strategy) when it expects the least favorable equilibrium for firm 1 to be played in the buyers' subgame.

3.2. Multimarket oligopoly

In a multiproduct logit demand oligopoly model, price best responses are increasing, and there is a unique Bertrand equilibrium. This obtains even though payoffs are not quasi-concave but single-peaked and neither supermodular nor log-supermodular in own prices (Spady, 1984; Vives, 1999). However, strategic complementarity across prices of different firms holds. A more general result is obtained by Nocke and Schutz (2018), who show that although such a game is generally not aggregative, the tools can be used under a quasi-linear demand system—which nests constant elasticity of substitution and multinomial logit demands—satisfying the Independence of Irrelevant Alternatives property. Under such a demand system, (i) each firm's profit depends on the rivals' prices only through an industry-level aggregator, and (ii) a firm-level sufficient statistic summarizes each firm's optimal price vector. These properties allow the authors to derive sufficient conditions for equilibrium existence and monotone comparative static results for extremal equilibria. This work allows the authors to study the unilateral effects of horizontal mergers in a market where multi-product firms with differentiated goods compete in prices.

3.3. Entry

Amir et al. (2014) build on Amir and Lambson (2000), who use lattice-theoretic tools, to study free entry in a Cournot market, extending the Mankiw and Whinston's (1986) finding on the tendency for excessive entry under business-stealing competition (i.e., when individual quantity decreases with the number of firms).[19] When the free-entry Cournot model is extended to account for overlapping ownership (i.e., the case where competing firms have some common shareholders or directly hold stakes in each other), the tendency for excessive entry may be reversed. Pre-entry overlapping ownership may induce insufficient entry, while the excessive entry result is maintained with post-entry overlapping ownership (Vives and Vravosinos, 2024).

Using also lattice-theoretic methods, Mrázová and Neary (2018) study selection effects with heterogeneous firms deciding whether and how to enter a market. They show that "first-order" selection effects (in terms of firms entering or not) are robust, while "second-order" ones (in terms of the entry mode, exporting, or foreign direct investment (FDI) conditional on entry) are not as robust. They also derive micro-foundations for supermodularity to hold in a range of standard models and show when supermodularity may fail with FDI.

3.4. Innovation

Lattice-theoretic methods have proved useful in studying innovation issues. Vives (2008) studies the effects of oligopolistic competition on process innovation and product introduction and uses them to characterize Cournot competition with multiple equilibria and Bertrand equilibrium with product differentiation. Building on Vives (2008), Spulber (2013) studies a two-stage model of invention and innovation. First, inventors enter the market by engaging in uncertain R&D projects to create production technologies. Then, the market for inventions opens, and inventors compete in royalties to sell their inventions to downstream producers. Producers observe the inventions and royalties offered by inventors and make simultaneous technology adoption, variety, and product pricing decisions. The product market competition stage is a symmetric log-supermodular game, and thus, has a unique and symmetric equilibrium.

[19] Anderson et al. (2020) develop tools to analyze aggregative games for asymmetric oligopoly in the short and long run. They derive neutrality results (where the relevant aggregate stays the same) for parameter changes with free entry.

An alternative way to study innovation is through patent race games. Under mild assumptions, a patent race game is strictly log-supermodular and the (per-firm) research intensity is increasing in the number of firms in extremal equilibria (Vives, 2005). However, this is not necessarily true under strong increasing returns to scale in the innovation technology. For example, in the context of winner-takes-all logit contests with symmetric players, under limited (resp. strong increasing) returns to scale in the innovation technology, the total research intensity is increasing (resp. decreasing) in the number of firms (Gama and Rietzke, 2019).

4. Dynamic games

Ordinal complementarity conditions on payoffs or the single-crossing property are very restrictive in extensive-form games (Echenique, 2004b). However, if we do not insist on supermodularity, the broader class of extensive-form games of strategic complements (i.e., games with increasing best replies, as defined by Vives (1985) in the case of static games) is significantly richer (Feng and Sabarwal, 2020). In this section, we survey some examples of applications to dynamic games: overlapping generations, Markov games, and mean-field games.

4.1. Cooperation with overlapping generations

Acemoglu and Jackson (2015) study the evolution of cooperation as a social norm. Each generation of agents interacts in a coordination game (which gives rise to strategic complementarities) with the previous and the following generation. To characterize equilibria in their model, where there are infinitely many players/generations, they employ an extension of Tarski's (1955) fixed-point theorem due to Straccia et al. (2009) to show that a game of strategic complements (with a possibly infinite number of agents) admits an equilibrium and the set of equilibria form a complete lattice. Agents interpret private information about the past and social norms arise as stable patterns of behavior. In sufficiently backward-looking societies, history completely determines equilibrium behavior, which can feature either high or low cooperation. In more forward-looking societies, play starting with high (resp. low) cooperation reverts towards lower (resp. higher) cooperation. Prominent agents, whose actions are observed by all future generations can—thanks to their greater visibility—overturn social norms of low cooperation.

4.2. Markov games

A Markov strategy in a dynamic game depends only on state variables that summarize the direct effect of the past on the current payoff. Denote by $u_i(s,\omega)$ the current payoff of player i, where s is the current action profile vector and ω the state, which evolves according to the law of motion $\omega = f(s^-,\omega^-)$, where s^- and ω^- are the lagged action profile vector and lagged state, respectively. A Markov perfect equilibrium (MPE) is a subgame-perfect equilibrium in Markov strategies. We say that there are "contemporaneous" strategic complementarities (SC) when the value function $V_i(\omega)$ at an MPE has increasing differences in (ω_i,ω_{-i}). There are "intertemporal" SC (resp. strategic substitutabilities, SS) when an increase in a player's state variable today causes an increase (resp., decrease) in the state variable of her rival tomorrow.

Finite-horizon multistage games. In two-stage games, let ω be the action profile in the first stage and s be the action profile in the second stage. Contemporaneous SC obtain if (i) at the second stage, for any action profile s in the first stage, $u_i(s,\omega)$ displays SC for every player i and (ii) the SC property is preserved when payoffs are folded back at the first stage in a subgame-perfect equilibrium. Assume that $u_i(s,\omega)$ displays increasing differences (or is supermodular) in any pair of variables, and let $s^*(\omega)$ be an extremal equilibrium in the second stage, as a function of the first-stage action profile ω (it is well-defined since the second-stage game is supermodular for every ω). $V_i(\omega) \equiv u_i(s^*(\omega), \omega)$, the first-period reduced-form payoff for player i, is supermodular in ω if for every player i and every $j \neq i$, (i) $u_i(s, \omega)$ is increasing and convex in each component of s_j, and (ii) each component of $s_j^*(\omega)$ is supermodular in ω (Vives, 2009). The result can be generalized to Markov finite-horizon multistage games with observable actions, where the payoff to each player displays increasing differences in any two variables.

Infinite-horizon games. In a continuous-time, infinite-horizon, differential, symmetric, differentiated duopoly with quadratic payoffs and adjustment costs, contemporaneous SC or SS are inherited from static SC or SS. However, intertemporal SC or SS arise depending on whether the produced quantity or the price bears the adjustment cost: (i) if production is costly to adjust, then intertemporal SS arise, while if the price is costly to adjust, then there are intertemporal SC (Jun and Vives, 2004). Particularly, under price competition with production adjustment costs, static SC are transformed into intertemporal SS.

Existence of MPE in stochastic games. Given that the existence of MPE in deterministic dynamic games has been shown only in particular models under parametric assumptions and general results in stochastic games remain elusive, lattice-theoretic methods can be helpful when there is enough monotonicity in the problem under study.

In discrete-time, infinite-horizon, stochastic games with complementarities and under certain assumptions, the multiperiod problem can be collapsed into a reduced-form static game (with continuation value functions increasing in the state variable), which can then be shown to be supermodular. A (stationary) MPE can then be found with value functions increasing in the state (Curtat, 1996).[20] Relaxing several of the assumptions (but also strengthening some conditions on transition probabilities) in Curtat (1996), Balbus et al. (2014) show existence of a largest and a smallest stationary Markov-Nash equilibrium, which need not be monotone. Also, the extremal (equilibrium) strategies display intertemporal SC. Existence of Markov equilibria also obtains in mean-field games, which we will now discuss in more detail.

4.3. Mean-field games

Mean-field game (MFG) theory is a recent area of research, developed by Lasry and Lions (2006a,b, 2007) and Huang et al. (2006, 2007). A MFG is a—possibly stochastic and/or dynamic (in continuous or discrete time)—non-atomic, anonymous game.[21] The MFG approach is particularly powerful in studying dynamic, stochastic games. Schmeidler (1973) modeled non-cooperative games with a continuum of players endowed with a non-atomic measure. The MFG approach avoids representing the continuum of players—as Mas-Colell (1984) had already done for the case of static, non-atomic, anonymous games.[22]

[20] Namely, allowing for multidimensional action spaces and a multidimensional state evolving according to a transition probability as a function of the current state and action profile, Curtat (1996) assumes that (i) payoffs are smooth and display per-period complementarities and positive spillovers (i.e., the payoff to a player is increasing in the actions of rivals and the state), (ii) the transition distribution function is smooth, displays complementarities, and is stochastically increasing in actions and states, and (iii) the payoff to a player and the transition distribution function satisfy a strict dominant diagonal condition.

[21] The term "mean-field" comes from the mean-field models in physics, which analyze the behavior of a very large number of interacting particles. As stated by Lasry and Lions (2007), "the main difference certainly lies in the possibility for each "player/particle" to choose its best strategy".

[22] See Carmona and Delarue (2018) and Achdou et al. (2019) for an extensive treatment of mean-field games. Anonymous sequential games with a continuum of players where individual players have insignificant influences on the other players had already been considered by Jovanovic and Rosenthal (1988).

Static, non-atomic, anonymous games. Since the work of Schmeidler (1973) and Mas-Colell (1984), a few papers have studied supermodular static games with a continuum of players, which possess similar properties as the games with a finite number of players (e.g., see Yang and Qi, 2013; Balbus et al., 2015).

Finite-horizon MFGs. In supermodular, finite-horizon MFGs with Itô-type dynamics, the equilibrium set has a lattice structure, whose minimal and maximal elements can be obtained through a simple learning procedure based on iterations of the best-response map (Dianetti et al., 2021). This is also true for a more general class of MFGs (Dianetti et al., 2023).

Infinite-horizon MFGs. Similarly, a largest and a smallest mean-field equilibrium exist in stochastic, infinite-horizon, discrete-time MFGs of strategic complementarities. Myopic learning dynamics converge to these equilibria, which are characterized by monotone comparative statics (Adlakha and Johari, 2013). Building on this paper, Lee et al. (2021) introduce the notion of trembling-hand perfection to the context of MFGs. They show the existence of a trembling-hand-perfect mean-field equilibrium in stationary MFGs with strategic complementarities and study algorithms (e.g., reinforcement learning) for its computation. For the class of infinite-horizon, discrete-time MFGs with strategic complementarities and no aggregate risk, Balbus et al. (2022) introduce the Markov Stationary Nash Distributional Equilibrium, prove its existence, and analyze the comparative statics of equilibrium paths and the steady-state invariant distributions.

Applications. The MFG approach has been used in the study of monetary policy (Alvarez et al., 2023). In a sticky-price economy with complementarities in firms' pricing strategies, strategic complementarities (i) increase the magnitude of the impulse response function (IRF) of output following a shock and (ii) can give rise to a hump-shaped IRF. These hold in the unique equilibrium of the model, which however vanishes if complementarities are too large, in which case the IRF diverges and no equilibrium exists.

5. Games of incomplete information

Games of incomplete information have proven to be a fruitful ground for applying lattice methods since the latter allow for very general payoff functions and action spaces. In this section, we check the usefulness of the approach in delivering existence and equilibrium characterization results, as well as studying (i) coordination failures, (ii) the impact of information frictions on multiple equilibria, and (iii) complementarities in data and information acquisition.

5.1. Existence and characterization of equilibria

A starting observation is that supermodularity of the underlying family of games defined with the ex-post payoffs for given realizations of the types of the players is inherited by the Bayesian game (Vives, 1990, 1999, section 2.7.3). Then, existence of pure-strategy Bayesian equilibria follows.

Van Zandt and Vives (2007) provide a constructive proof of the existence of a greatest and a least Bayesian Nash equilibrium in Bayesian games of strategic complementarities with general action and type spaces. Both equilibria are in strategies that are monotone in type (and there is no need to assume a common prior). In addition to strategic complementarities, they assume that each player's payoff has increasing differences in their own action and the profile of types and that each player's posteriors/interim beliefs are increasing in type with respect to first-order stochastic dominance. The assumptions define a monotone supermodular game. They also show that the greatest and least equilibria are higher if there is a first-order stochastic dominant shift in the posteriors/interim beliefs.

Alternative pure-equilibrium existence results do not depend on strategic complementarities but rather on single-crossing properties, which imply that when every rival employs a monotone (i.e., increasing in his type) strategy, a player responds by using a strategy that is also monotone in her type (e.g., see Athey, 2001; McAdams, 2003, 2006). Reny (2011) weakens the needed conditions on interim payoff functions for the monotone best-reply condition to hold.[23] Mensch (2020) extends Athey (2001) and Reny's (2011) results to dynamic games of incomplete information with strategic complementarities, providing sufficient conditions for the existence of equilibria in monotone strategies, which requires the existence of monotone best replies in the first place. Sufficient conditions for the existence of monotone best replies are significantly more restrictive in dynamic games.[24]

5.2. Global games and coordination failures

Global games (Carlsson and van Damme, 1993) are games of incomplete information where each player observes a noisy signal of the state. They have often been used as a tool for equilibrium selection in games of complete information. Namely, as the noise vanishes, iterated elimination of dominated strategies leads to a unique equilibrium under certain conditions.[25] Vives (2005) notes that global games are usually games of strategic complementarities (many are binary-action with a continuum of players).[26] This helps us understand why and how iterated elimination of dominated strategies works and under what conditions equilibrium selection is successful. Dominance solvability, for example, in the global game of Morris and Shin (2003) obtains because the underlying game is one of strategic complementarities.[27] Furthermore, the key to uniqueness is to find conditions in the deep parameters of the model that make these complementarities not too strong. In this situation, a player faces a lot of uncertainty about the actions of the other players.

While the global games approach suggests a specific perturbation of complete information games to deliver a unique equilibrium, many settings do not fit into this framework—for example, due to heterogeneous beliefs (i.e., non-common prior type spaces). For games with incomplete information and strategic complementarities, Mathevet (2014) derives an upper bound on the distance between any two profiles of rationalizable strategies, generalizing the arguments by Vives (2005) on the connection between global games and games of strategic complementarities and subsuming Carlsson and van Damme's (1993) uniqueness result. This bound depends on the type-sensitivity of each player, which is related to how informative a player thinks his type is.[28] A type-sensitive player acts as if he is not affected much by the strategic complementarities. As the type-sensitivity of every player increases, strategic complementarities are mitigated and the bound on the distance between rationalizable strategies decreases. If all players are highly type-sensitive—which in global games happens as noise vanishes, then the bound becomes zero.

Alternative notions of robustness to incomplete information have also been proposed and applied to supermodular games. In binary-action supermodular games (with general payoffs, beyond two-player or symmetric games, and with general information perturbations), it generically holds that an action profile is robust to incomplete information in the sense of Kajii and Morris (1997) if and only if the game admits a monotone potential that is maximized by that action profile (Oyama and Takahashi, 2020).[29]

The theory has been fruitful in building a bridge between the theory of a self-fulfilling crisis (e.g., Diamond and Dybvig, 1983) and the theory that links a crisis to fundamentals (e.g., Gorton, 1985). Binary-action games of strategic complementarities with incomplete information which are monotone supermodular games have provided the appropriate framework to build the bridge, resulting in investors using threshold strategies (given that equilibria are monotone in type). Vives (2014) presents such a canonical game, which encompasses bank runs (Rochet and Vives, 2004)[30] and currency attacks (Morris and Shin, 1998). This approach distinguishes liquidity from solvency risk and has relevant implications for Lender of Last Resort policy and prudential regulation. It links financial fragility to the degree of strategic complementarities between the actions of agents.

5.3. Information frictions, complementarities, and multiple equilibria

Complementarities are at the root of multiple equilibria in the presence of information frictions. For example, when traders obtain private information about the payoff and the supply of a stock, the supply information acquisition channel increases the possibility of coordination among traders' self-fulfilling expectations in equilibrium, leading to equilibrium multiplicity in the financial market (Ganguli and Yang, 2009). Independent of this multiplicity, the extra dimension of supply information may also make traders' decisions to acquire information complementary. Manzano and Vives (2011) study a general, static, noisy, rational expectations model where investors have private information about asset payoffs, with common and private components, and about their own exposure to an aggregate risk factor, and derive conditions for existence and uniqueness (or multiplicity) of equilibria. They show that the equilibrium characterization depends on whether the actions of investors are strategic substitutes or complements. When the private learning channel from prices (arising from the fact that a trader knows his endowment shock when reading the information from the price) is strong (resp. weak) in relation to the usual public learning channel from prices, there are strong (rep. weak) strategic complementarities and potentially multiple (resp. unique) equilibria. It is found that the market exhibits strategic substitutability in information acquisition at extremal equilibria.

Cespa and Vives (2015) study a dynamic trading model with short-term traders and find that informed investors engage in "retrospective" learning to reassess inferences (about fundamentals) made during the trading game's early stages. This behavior introduces strategic complementarities in the use of information and can yield two stable equilibria

[23] The conditions require the marginal distribution of types to be atomless and payoffs to be bounded, jointly measurable in actions and types, and continuous in actions for every type.

[24] For example, not even supermodularity of payoffs and independence of types—which guarantee existence of a monotone equilibrium in static games—are sufficient to guarantee monotonicity of best replies in games with at least three periods (Mensch, 2020).

[25] Frankel et al. (2003) generalize Carlsson and van Damme's (1993) result (which applies to two-player, two-action games) by showing that as the noise vanishes, the game has a unique strategy profile that survives iterative dominance. However, the surviving profile may depend on fine details of the noise structure. Thus, they also derive sufficient conditions on payoffs for there to be noise-independent selection.

[26] In fact, they are monotone supermodular in the sense of Van Zandt and Vives (2007).

[27] We start by noting that the game is monotone supermodular and, therefore, that extremal equilibria exist and are in monotone (threshold) strategies. The extremal equilibrium thresholds bound the set of rationalizable strategies, and if the equilibrium is unique, then the game is dominance-solvable.

[28] A player i's type-sensitivity has two dimensions. The first dimension measures how strongly i's beliefs over the state respond to changes in his type. The second dimension measures the extent to which, given an increase in player i's type, player i believes that the other players' types increase more than his own type.

[29] Kajii and Morris (1997) say that a Nash equilibrium of a complete information game is robust to incomplete information if in any incomplete information game where with high probability all players know that their payoff functions are given by those in the original complete information game, the equilibrium action profile continues to be played with high probability in some Bayesian Nash equilibrium.

[30] Sections 1.2.2 and 7.2.1 in Carmona and Delarue (2018) develop a dynamic version of the Rochet and Vives (2004) model casting the analysis as a mean-field game.

that can be ranked in terms of liquidity, volatility, and informational efficiency. Cespa and Vives (2023) consider a dynamic trading model where the lack of market transparency can make liquidity demand by hedgers upward-sloping, inducing strategic complementarity and multiple equilibria (with two stable ones). Then, an initial dearth of liquidity may degenerate into a liquidity rout (as in a "flash crash"). Liquidity is fragile since a small change in parameters may induce a large increase in the cost of trading.

5.4. Complementarities in data and information acquisition

Strategic incentives to acquire information are inherited from the underlying game that agents play. For example, consider a two-stage duopoly model with demand uncertainty where firms (i) can purchase precision according to a convex cost function in the first stage and, in the second stage, (ii) compete in the marketplace upon having received their private signals. Then, in the information acquisition stage, the strategies are strategic substitutes (resp. complements) if and only if in the second stage, the duopoly game is submodular (resp. supermodular; see Vives (1988) for the submodular case and Exercise 8.15 in Vives (1999)). Hellwig and Veldkamp (2009) provide a general analysis of the phenomenon.

Strategic complementarities can also arise in models of costly processing of the informational content of asset prices. When other traders process this information, the prices become more informative, which in turn makes processing the information more worthwhile. Mondria et al. (2022) propose a model where at the trading stage, investors are boundedly rational and their costly interpretation of prices injects noise (say sentiment) into the price, generating a source of endogenous noise trading. In an overall equilibrium, investors optimally choose sophistication levels to reduce the injected noise by balancing the benefit of beating the market against the cost of acquiring sophistication. There may be strategic complementarity in sophistication acquisition, leading to multiple equilibria. If the sophistication-acquisition cost is high, then no investor will acquire additional sophistication. If the sophistication-acquisition cost is low, then all investors will coordinate on a high sophistication level to tame the sentiment distortion. It only pays to invest in reducing the sentiment error when the price is informative, and this happens precisely when other traders are also investing to tame it.

Although in a static world, uncertainty encourages information acquisition by raising the value of information, in a dynamic world, uncertainty can depress information acquisition through a dynamic complementarity channel. Cai (2019) studies how information acquisition affects the propagation or stabilization of shocks in a dynamic financial market. Increased uncertainty induces future investors to trade more cautiously, which renders future stock prices less informative and reduces the value of information today. Due to this dynamic complementarity, transitory uncertainty shocks can have long-lasting effects.

The incentives for information acquisition can also be affected when there are multiple types of information and one of them features strategic complementarities. Farboodi and Veldkamp (2020) address the concern that "big data" may induce traders to extract others' information, rather than produce information themselves. They develop a model where investors choose how much to learn about fundamentals (e.g., future dividends) or about other investors' demands (searching for "dumb money"). They show that as other investors learn more about demand, the value of demand data (relative to fundamental data) increases. Thus, there are strategic complementarities in demand information acquisition: investors shift their data analysis from fundamentals to demand when others do more demand analysis. Yet, the authors conclude that, in the long run, as the data processing technology becomes increasingly advanced, both types of data—fundamental and demand—continue to be processed. This balance is preserved thanks to two competing forces: data resolve investment risk, but future data create risk by making future prices more sensitive to future news and, thus, more difficult to forecast today.

6. Mechanism design and information design

Supermodular games have opened new avenues for research in the design of both mechanisms and information structures. We look at those applications in turn.

6.1. Mechanism design

Mathevet (2010) shows that in quasi-linear environments, if a social choice function is implementable by a mechanism that induces a game of bounded strategic substitutes, then the mechanism can be transformed to induce a game of strategic complements while still implementing the social choice function. If the social choice function satisfies some efficiency criterion, then the transformed mechanism can also balance the budget. Using these results, he derives sufficient conditions for a social choice function to be implementable by a supermodular mechanism (i.e., a mechanism that induces a game of strategic complements) in unique equilibrium.

Unique implementation in supermodular games appears in many settings. Some papers following Segal's (1999, 2003) work on contracting with externalities in multi-agent settings have connections to the fact that when the induced game is supermodular, unique implementation (in equilibrium) coincides with (unique) implementation in rationalizable strategies. For example, consider the case of binary team effort provision. Each agent $i \in N$ chooses effort $s_i \in \{0, 1\}$, and the designer wants to incentivize every agent to exert effort in a unique equilibrium through bilateral contracts. Denote by t_i the payment made to agent i for exerting effort and by $u_i(s) := \pi_i(s) + t_i s_i$ agent i's payoff. If $\pi_i(1, s_{-i}) - \pi_i(0, s_{-i})$ is increasing in s_{-i} for every i, then the induced game is one of strategic complements. Then, the designer uniquely and optimally (i.e., at minimum cost) implements effort by every agent by ranking them and then rewarding each agent enough to make her indifferent between exerting effort or not when she believes every agent before her (in the ranking) to exert effort with probability 1 and every agent after her to exert effort with probability 0. In this way, the designer makes it a dominant strategy for the first player (in the ranking) to exert effort. For the second player, it is dominant to exert effort given that the first player does, and so on.[31]

One implication of this construction is that the optimal mechanism is discriminatory (e.g., see Winter, 2004). Namely, even if agents are symmetric (e.g., in terms of their cost and effectiveness of effort), they are rewarded differently. Agents higher in the ranking—who need to be made willing to exert effort when only players before them do—are rewarded more than agents lower in the ranking.[32]

The connection between rationalizability and equilibrium uniqueness in supermodular games has proved important in studying unique implementation in several settings. Halac et al. (2020) study unique implementation by a firm that raises capital from multiple investors to fund a project by offering payments contingent on project success

[31] Strictly put, unique implementation is achieved if we add an arbitrarily small $\varepsilon > 0$ to every player's payment (so that indifferences are broken). If we do not insist on unique implementation, the designer optimally implements effort provision by every agent as *an* equilibrium by setting, for every player i, $t_i^* = \pi_i(1, s_{-i}) - \pi_i(0, s_{-i})$ for $s_{-i} = (1, 1, \dots, 1)$. When $\pi_i(1, s_{-i}) - \pi_i(0, s_{-i})$ is increasing in s_{-i}, this construction does not make every agent exerting effort the unique equilibrium, so unique implementation comes at an extra cost. On the other hand, if $\pi_i(1, s_{-i}) - \pi_i(0, s_{-i})$ is decreasing in s_{-i}, then the induced game is one of strategic substitutes. In that case, setting $t_i^* = \pi_i(1, s_{-i}) - \pi_i(0, s_{-i})$ for $s_{-i} = (1, 1, \dots, 1)$ actually makes every agent exerting effort the unique (and in dominant strategies) equilibrium, so implementation coincides with unique implementation.

[32] Halac et al. (2021), however, show that with private contracts, discrimination is strictly suboptimal. In their model, the optimal incentive scheme that implements work as a unique equilibrium informs each agent only of a ranking distribution and her own bonus.

(where the probability of success is increasing in the amount of capital raised). The optimal scheme induces a supermodular game among the agents (so unique implementation in Nash equilibria also yields unique implementation in rationalizable strategies) and treats investors differently based on size. Under some assumptions on the shape of the probability of success function, larger investors receive higher net returns than smaller investors. Halac et al. (2023) study incentivization of a group of agents to work through a monitoring structure and a scheme of performance-contingent rewards. The monitoring structure partitions the set of agents into monitoring teams, each delivering a signal of joint performance. The induced game is supermodular for any monitoring structure and incentive scheme, so the requirement of unique implementation in Nash equilibria is equivalent to requiring a unique rationalizable outcome. They show that optimal monitoring teams are homogeneous: equally sized and with agents allocated in an anti-assortative fashion. Under a certain constraint on the set of monitoring structures that are available to the principal, agents with higher cost of effort receive lower rents and tend to be monitored more closely than agents with lower cost of effort.

6.2. Information design

The connection between equilibrium uniqueness and dominance-solvability in supermodular games plays a role also in information design.[33] In the context of (two-person) team production where the success probability is a supermodular function of the agents' effort choices, Moriya and Yamashita (2020) derive the optimal information allocation that implements the desired effort levels as the unique Bayesian equilibrium—which, due to supermodularity, coincides with the lowest-effort equilibrium and the (unique) outcome that survives the iterative elimination of strictly dominated strategies. Particularly, they show that in some cases, it is optimal to asymmetrically inform agents even though they may be ex-ante symmetric.

A ranking scheme like the one discussed above (in the mechanism design example on team effort provision) also arises in information design. Namely, in binary-action supermodular games, implementation of an outcome as the smallest equilibrium of the induced game requires a sequential obedience condition, which says that there is a stochastic ordering of players under which players are willing to play the high action as long as those before them (in the ranking) play the high action (Morris et al., 2023). Given this result, Morris et al. (2023) characterize the optimal outcome induced by a pessimistic information designer who (anticipates that the smallest equilibrium will be played and) prefers the high action.[34]

7. Concluding remarks

The lattice-theoretic approach in economics has proven its potency and utility across various fields. Its impact is tangible, with widespread adoption in economic methodologies, being progressively incorporated into the standard toolbox, including empirical studies. This paper underscores the ongoing development of cutting-edge theoretical research, such as mean-field games and information design, highlighting the continued relevance and evolution of this approach.

[33] Bergemann and Morris (2019) review the information design literature, including the role of strategic complementarities and connections to the literature on robust predictions in games of incomplete information.

[34] Sato (2023) studies robust implementation against adversarial equilibrium selection in sequential information design when the players and the designer have a supermodular payoff function with dominant states and an outside option. It is shown that the optimal partially implementable outcome is fully implementable in sequential information design. In sequential information design, the designer chooses the extensive form and the information structure (see also Mathevet et al., 2020).

CRediT authorship contribution statement

Xavier Vives: Formal analysis, Writing – original draft, Writing – review & editing. **Orestis Vravosinos:** Formal analysis, Writing – original draft, Writing – review & editing.

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No data was used for the research described in the article.

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